Math 126 D - Spring 2015
Midterm Exam Number One
April 23, 2015

Name: ______________________________ Student ID no. : ________________

Signature: ____________________________________ Section: ____________

1	12	
2	12	
3	12	
4	12	
5	12	
Total	60	

- This exam consists of FIVE problems on SIX pages, including this cover sheet.
- Show all work for full credit. Show no work for zero credit.
- You do not need to simplify your answers.
- If you use a trial-and-error or guess-and-check method when a more rigorous method is available, you will not receive full credit.
- Write all of your work on the exam itself. If you use the back of the page, please indicate that you have done so!
- You may use a *scientific*, *non-programmable*, *non-graphing* calculator.
- You may use one hand-written double-sided 8.5″ by 11″ page of notes.
- You have 50 minutes to complete the exam.

∘

1. Consider the two planes $3x - y + 6z = 5$ and $3x - 2y - 3z = 2$.

(a) **[7 points]** Find the line of intersection of the given planes.
Express your answer in parametric form.

We need two things:

1) A point on the line. Solve:

$3x - y + 6z = 5$

$3x - 2y - 3z = 2$

$z = 0$ ← arbitrary

$\downarrow$

$y = 3, \ x = \frac{8}{3}$

$\left(\frac{8}{3}, 3, 0\right)$

2) The direction vector, which is perpendicular to both normal vectors. Cross product!

$\langle 3, -1, 6 \rangle$

$\times \ \langle 3, -2, -3 \rangle$

$= \langle 15, 27, -3 \rangle$

Put 'em together:

$$\boxed{\begin{aligned} x &= \tfrac{8}{3} + 15t \\ y &= 3 + 27t \\ z &= 0 - 3t \end{aligned}}$$

(b) **[5 points]** Find the (acute) angle between the two planes.
Express your answer in radians.

$$\underbrace{\langle 3, -1, 6 \rangle \cdot \langle 3, -2, -3 \rangle}_{-7} = \underbrace{|\langle 3, -1, 6 \rangle|}_{\sqrt{46}} \cdot \underbrace{|\langle 3, -2, -3 \rangle|}_{\sqrt{22}} \cos(\theta)$$

$$-7 = \sqrt{46} \cdot \sqrt{22} \ \cos(\theta)$$

$$\theta = \cos^{-1}\left(\frac{-7}{\sqrt{44}\sqrt{22}}\right) \approx 1.79266 \text{ radians.}$$

Oops, that's obtuse!

Answer: $\pi - \cos^{-1}\left(\frac{-7}{\sqrt{46}\sqrt{22}}\right) \approx \boxed{1.34894}$

2. Here's a polar curve: $r = \cos^2(\theta) + 1$

(a) **[4 points]** Find the Cartesian coordinates of the point on the curve at $\theta = \dfrac{4\pi}{3}$.

$$r = \cos^2\left(\frac{4\pi}{3}\right) + 1 = \frac{5}{4}$$

$$x = r\cos(\theta) = \frac{5}{4}\cos\left(\frac{4\pi}{3}\right) = \frac{-5}{8}$$

$$y = r\sin(\theta) = \frac{5}{4}\sin\left(\frac{4\pi}{3}\right) = \frac{-5\sqrt{3}}{8}$$

$$\boxed{\left(\frac{-5}{8}, \frac{-5\sqrt{3}}{8}\right)}$$

(b) **[6 points]** Find the tangent line to the curve at the point from part (a).
(Your answer should only involve variables x and y. You **do not** need to simplify!)

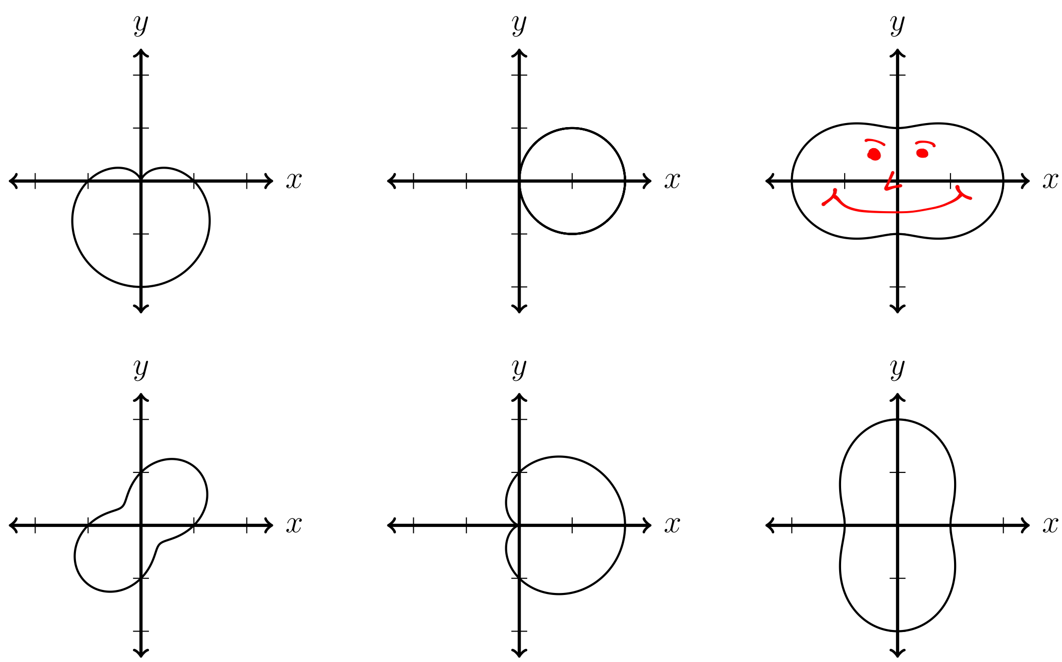
We have the point, so we just need the slope.

This will require $\dfrac{dr}{d\theta} = -2\cos(\theta)\sin(\theta) = -2\cos\left(\frac{4\pi}{3}\right)\sin\left(\frac{4\pi}{3}\right) = \dfrac{-\sqrt{3}}{2}$

$$\text{Slope} = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\left(\frac{-\sqrt{3}}{2}\right)\left(\frac{-\sqrt{3}}{2}\right) + \left(\frac{5}{4}\right)\left(\frac{-1}{2}\right)}{\left(\frac{-\sqrt{3}}{2}\right)\left(\frac{-1}{2}\right) - \left(\frac{5}{4}\right)\left(\frac{-\sqrt{3}}{2}\right)} = \frac{\sqrt{3}}{21}$$

$$\boxed{y = \frac{\sqrt{3}}{21}\left(x + \frac{5}{8}\right) - \frac{5\sqrt{3}}{8}}$$

(c) **[2 points]** Kindly examine the six plots below.
(You don't need to show work on this problem.)



One of these is the curve $r = \cos^2(\theta) + 1$. Draw a smiley face inside the correct plot.

3. Consider the quadric surface defined by $x^2 - y^2 + 10y - z^2 - 2z = 27$.

(a) **[4 points]** Rewrite the surface in standard form.

$$x^2 - (y^2 - 10y) - (z^2 + 2z) = 27$$

$$x^2 - (y^2 - 10y + 25) - (z^2 + 2z + 1) = 27 - 25 - 1$$

$$\boxed{x^2 - (y-5)^2 - (z+1)^2 = 1}$$

(b) **[2 points]** Identify the surface.

Hyperboloid of 2 sheets (oriented along the x-axis).

(c) **[6 points]** Find the intersection of this surface with the line $\frac{x+13}{-10} = \frac{y-14}{7} = \frac{z+4}{-5}$.

In parametric form: $x = -13 - 10t$

$y = 14 + 7t$

$z = -4 - 5t$

Plug into $x^2 - (y-5)^2 - (z+1)^2 = 1$

$$(-13 - 10t)^2 - (14 + 7t - 5)^2 - (-4 - 5t + 1)^2 = 1$$

expand

$$(169 + 260t + 100t^2) - (81 + 126t + 49t^2) - (9 + 30t + 25t^2) = 1$$

simplify

$$26t^2 + 104t + 78 = 0 \xrightarrow[\div 26]{} t^2 + 4t + 3 = 0 \xrightarrow{\text{solve}} t = -1 \text{ or } -3$$

Points: $x = -13 - 10t$

$y = 14 + 7t$

$z = -4 - 5t$

$\boxed{(-3, 7, 1) \ \& \ (17, -7, 11)}$

4. **[12 points]** Consider the space curve defined by the vector function $\mathbf{r} = (3t - t^3)\mathbf{i} + 3t^2\mathbf{j}$.

An ant walks along the curve from the origin to $(18, 27)$ at a constant speed of 5 units per second. How long does it take to get there?

(This problem is worth 12 points. If you get the answer in 30 seconds then maybe you're doing it wrong.)

Need to know distance traveled along curve. Arc length!

$x(t) = 3t - t^3 \rightarrow x'(t) = 3 - 3t^2$

$y(t) = 3t^2 \qquad y'(t) = 6t$

$(0,0)$ is at $t=0$ on this curve.

What about $(18, 27)$? $27 = y(t) = 3t^2$, so $t^2 = 9 \rightarrow t = \pm 3$

But $x(3) = -18$, $x(-3) = 18$, so we want $t = -3$.

So we need the arc length btwn $t = -3$ and $t = 0$:

order doesn't really matter; just take abs. value for dist. afterwards.

$$\int_{-3}^{0} \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = \int_{-3}^{0} \sqrt{(3-3t^2)^2 + (6t)^2}\,dt$$

$$= \int_{-3}^{0} \sqrt{9 - 18t^2 + 9t^4 + 36t^2}\,dt = \int_{-3}^{0} \sqrt{9 + 18t^2 + 9t^4}\,dt$$

$$= \int_{-3}^{0} \sqrt{(3+3t^2)^2}\,dt = \int_{-3}^{0} (3 + 3t^2)\,dt = \left[3t + t^3\right]_{-3}^{0} = 36$$

So the ant takes $\dfrac{36 \text{ units}}{5 \text{ units/sec}} = \boxed{7.2 \text{ seconds}}$

5. Congratulations, you've reached the last page of the exam! Here are four free points:

$$A = (5, 3, 5) \qquad B = (-2, 6, 3) \qquad C = (2, 2, 0) \qquad D = (1, 3, 1)$$

(a) **[6 points]** Are the lines $\overline{AB}$ and $\overline{CD}$ parallel, intersecting, or skew?

Parametric for line AB:

Vector $\vec{AB} = \langle -7, 3, -2 \rangle$

So: $x = 5 - 7t$

$y = 3 + 3t$

$z = 5 - 2t$

Parametric for CD:

Vector $\vec{CD} = \langle -1, 1, 1 \rangle$

$x = 2 - s$

$y = 2 + s$

$z = s$

$\frac{-7}{-1} \neq \frac{3}{1}$, not parallel

set equal!

$5 - 7t = 2 - s$ $\xrightarrow{\text{sum:}}$ $8 - 4t = 4$

$3 + 3t = 2 + s$

$t = 1,\ s = 4$

$5 - 2t = s$

$5 - 2 = 4$?

Nope, not intersecting.

So, skew!

(b) **[6 points]** Find the area of the triangle ABC.

Area of triangle $= \frac{1}{2}$ area of parallelogram

$= \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right|$

$= \frac{1}{2} \left| \langle -7, 3, -2 \rangle \times \langle -3, -1, -5 \rangle \right|$

$= \frac{1}{2} \left| \langle -17, -29, 16 \rangle \right|$

$= \frac{1}{2} \sqrt{289 + 841 + 256}$

$= \frac{1}{2} \sqrt{1386} \approx \boxed{18.615}$