Math 126 DFEPs Spring 2015

The deal: every day I'll post an exam-like problem at the start of class, along with the answer to the problem from the previous class. Please attempt these problems!

DFEP #1: Friday, April 10th.

(a) Give the equation of a plane containing the line $\frac{x-2}{4} = \frac{y}{-2} = \frac{z+6}{3}$ and the point $(6, 1, 5)$.

(b) Find the intersection of this plane with the line $\frac{x+1}{-6} = \frac{y-5}{2} = z - 7$.

DFEP #1 Solution:

(a) We want a plane through the line $\dfrac{x-2}{4} = \dfrac{y}{-2} = \dfrac{z+6}{3}$ and the point $(6,1,5)$. Certainly this plane contains the line's direction vector $\langle 4,-2,3\rangle$. It also contains the points $(2,0,-6)$ and $(6,1,5)$, which means it contains the vector $\langle 4,1,11\rangle$. So to find the normal vector, we can take the cross product $\langle 4,-2,3\rangle \times \langle 4,1,11\rangle$ to get $\langle -25,-32,12\rangle$. The plane with normal vector $\langle -25,-32,12\rangle$ through the point $(6,1,5)$ has equation

$$-25x - 32y + 12z = -25(6) - 32(1) + 12(5)$$

or

$$-25x - 32y + 12z = -122$$

(b) Let's write that line in parametric form: $x = -1 - 6t$, $y = 5 + 2t$, $z = 7 + t$. Plugging that into the equation of the plane yields

$$-25(-1-6t) - 32(5+2t) + 12(7+t) = -122$$

which we can solve to get $t = -71/98 \approx -0.7245$, so the point of intersection is $(x,y,z) = (3.347, 3.551, 6.276)$.

DFEP #2: Monday, April 13th.

Suppose $\mathbf{a} = \langle -1,8,4\rangle$. Find a vector $\mathbf{b}$ so that:

- The angle between $\mathbf{a}$ and $\mathbf{b}$ is 60°,
- $\mathbf{b}$ is perpendicular to $\mathbf{k}$, and
- $||\mathbf{b}|| = 4$.

DFEP #2 Solution:

Let's say that $\mathbf{b} = \langle x, y, z \rangle$. We know that $\mathbf{b} \cdot \mathbf{k} = 0$, so $z = 0$.

We also know that $\mathbf{a} \cdot \mathbf{b} = -x + 8y + 4z = -x + 8y$. But on the other hand, $\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| \cdot ||\mathbf{b}|| \cos(60°)$. Since $||\mathbf{a}|| = 9$ and $||\mathbf{b}|| = 4$, that means $-x + 8y = 18$, or $x = 8y - 18$.

Finally, since $||\mathbf{b}|| = 4$, we know that $x^2 + y^2 = 16$, so $(8y - 18)^2 + y^2 = 16$, which simplifies to $65y^2 - 288y + 308 = 0$.

Solving that tells us that $y = \dfrac{288 \pm \sqrt{288^2 - 4 \cdot 65 \cdot 308}}{130} \approx 2.627$ or 1.804.

And since $x = 8y - 18$, that means we have two possible answers:

$$\mathbf{b} = \langle 3.016, 2.627, 0 \rangle \qquad \text{or} \qquad \mathbf{b} = \langle -3.570, 1.804, 0 \rangle$$

DFEP #3: Wednesday, April 15th:

Consider the vector function $\mathbf{r} = \langle t + 1, 2^t, 3t + 2t^2 \rangle$.

(a) Does the curve defined by $\mathbf{r}$ intersect the following line? If so, where?

$$\frac{x - 15}{2} = y - 10 = 8 - z$$

(b) Suppose $\mathbf{r}$ intersects the surface $5x^2 + Cy^2 + 2z^2 = 1$ in the yz-plane. Solve for the constant C.

(c) Describe the surface from part (b). Your answer should be a short phrase.

DFEP #3 Solution:

(a) We want to find the intersection of the vector functions $\langle t+1, 2^t, 3t+2t^2\rangle$ and $\langle 15+2s, 10+s, 8-s\rangle$. So we set their components equal:

$$t+1 = 15+2s \qquad 2^t = 10+s \qquad 3t+2t^2 = 8-s$$

Yikes, let's ignore that second equation for now. Solving the first and third gives a quadratic $4t^2+7t-30=0$, which factors as $(4t+15)(t-2)=0$. So we have either $t=2, s=-6$ or $t=-15/4, s=-71/8$. Plugging those into the second equation, we have $t=2, s=-6$ as the only solution.

So where's the point? Plug t or s into the corresponding vector function to get $(3,4,14)$ as the intersection.

(b) Okay, $\mathbf{r} = \langle t+1, 2^t, 3t+2t^2\rangle$ intersects the yz-plane when $x=0$, so $t=-1$, which is at the point $(0, \frac{1}{2}, -1)$. Since this intersects the curve $5x^2+Cy^2+2z^2=1$, we have $C\left(\frac{1}{2}\right)^2+2=1$, so $C=-4$.

(c) The curve $5x^2-4y^2+2z^2=1$ is a hyperboloid of one sheet, centered around the y-axis.

DFEP #4: Friday, April 17th.

Consider the curve defined by the vector function $\mathbf{r} = \langle t+6, t^3, e^{t^2-6t+8}\rangle$.

(a) Find all points where the curve intersects the plane $z=1$.

(b) Find the (acute) angle between the curve and plane at each point from part (a).

DFEP #4 Solution:

(a) The curve defined by $\langle t+6, t^3, e^{t^2-6t+8}\rangle$ intersects $z = 1$ when its z-component is 1, which means that $e^{t^2-6t+8} = 1$. Therefore $t^2 - 6t + 8 = 0$, so $t = 2$ or $t = 4$.

To find the points of intersection, we plug $t = 2$ and $t = 4$ back into the vector function to get $(8, 8, 1)$ and $(10, 64, 1)$.

(b) We'll need to know the tangent vectors for the points from part (a). The derivative $\mathbf{r}'(t) = \langle 1, 3t^2, (2t-6)e^{t^2-6t+8}\rangle$.

At $t = 2$, this is the vector $\langle 1, 12, -2\rangle$, and at $t = 4$ it's $\langle 1, 48, 2\rangle$.

To find the angle between the curve and the plane, we'll start by finding the angles between the normal vector and the tangent vector:

$\langle 1, 12, -2\rangle \cdot \langle 0, 0, 1\rangle = ||\langle 1, 12, -2\rangle|| \cdot 1\cos(\theta)$, so $\theta = \cos^{-1}(-2/\sqrt{149}) \approx 99.43°$. But, wait, that's the angle between the curve and the normal vector. The angle between the curve and the plane is 90° less, or 9.43°.

A similar calculator for the other point gives 2.39°.

DFEP #5: Monday, April 20th.

Consider the polar curve $r = 2\cos(\theta) - 5$. (Give all answers in polar coordinates.)

1. Find all intersections of this curve with the line $x = 3$.
2. Find all points on the curve where the tangent line is horizontal.

DFEP #5 Solution:

(a) Okay, so we have the curve $r = 2\cos(\theta) - 5$, and we want to know where $x = 3$. But $x = r\cos(\theta)$, so $r\cos(\theta) = 3$, which means $(2\cos(\theta) - 5)\cos(\theta) = 3$.

Expanding that gives $2\cos^2(\theta) - 5\cos(\theta) - 3 = 0$, which factors into $(2\cos(\theta) + 1)(\cos(\theta) - 3) = 0$. Since $\cos(\theta)$ can never equal 3, this means $\cos(\theta) = -1/2$, so $\theta = 2\pi/3$ or $4\pi/3$.

At those values of θ, we get $r = 2\cos(\theta) - 5 = -6$. So $(x, y) = (r\cos(\theta), r\sin(\theta)) = (3, 3\sqrt{3})$ or $(3, -3\sqrt{3})$.

(b) We want to know when the tangent line is horizontal. That tells us:

$$\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta) = 0.$$

But $\frac{dr}{d\theta} = -2\sin(\theta)$, so we want to solve:

$$-2\sin^2(\theta) + 2\cos^2(\theta) - 5\cos(\theta) = 0$$

which simplifies to

$$4\cos^2(\theta) - 5\cos(\theta) - 2 = 0$$

which means

$$\cos(\theta) = \frac{5 \pm \sqrt{57}}{8}$$

$$\theta = \arccos\left(\frac{5 - \sqrt{57}}{8}\right) \approx 1.895$$

Okay, uh, maybe the algebra here is a little grosser than you'll see on an exam.

DFEP #6: Friday, April 24th.

Give an equation for the normal plane to the following curve at the point $\left(27, 5, \dfrac{1}{26}\right)$:

$$x = 2^t - t \qquad y = t^2 - 4t \qquad z = \frac{1}{1+t^2}$$

DFEP #6 Solution:

We want the normal plane to $\left\langle 2^t - t, t^2 - 4t, \dfrac{1}{1+t^2} \right\rangle$ at $\left(27, 5, \dfrac{1}{26}\right)$, which is at $t = 5$. So we just need to know the tangent vector at $t = 5$, and that will give us the normal vector to the plane.

That tangent vector is $\left\langle \ln(2)2^t - 1, 2t - 4, \dfrac{-2t}{(1+t^2)^2} \right\rangle$, which at $t = 5$ is:

$$\left\langle 32\ln(2) - 1, 6, \frac{-5}{338} \right\rangle$$

So the normal plane is:

$$(32\ln(2) - 1)x + 6y - \frac{5}{388}z = (32\ln(2) - 1) \cdot 27 + 30 - \frac{5}{388 \cdot 26}$$

DFEP #7: Monday, April 27th.

Consider the vector function $\mathbf{r}(t) = \langle \arctan(t), 3t^2 - 4t + 1, \ln(t^2) \rangle$.

Compute the curvature of $\mathbf{r}(t)$ when $t = 1$.

DFEP #7 Solution:

Recall that we can find κ by computing $\dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.

$\mathbf{r}(t) = \langle \arctan(t), 3t^2 - 4t + 1, \ln(t^2) \rangle$.

$\mathbf{r}'(t) = \left\langle \dfrac{1}{1+t^2}, 6t - 4, \dfrac{2}{t} \right\rangle$.

$\mathbf{r}''(t) = \left\langle \dfrac{-2t}{(1+t^2)^2}, 6, \dfrac{-2}{t} \right\rangle$.

We want the curvature at $t = 1$, so $\mathbf{r}'(1) = \langle 0.5, 2, 2 \rangle$, and $\mathbf{r}''(1) = \langle -0.5, 6, -2 \rangle$.

$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle -16, 0, 4 \rangle$, so $|\mathbf{r}'(1) \times \mathbf{r}''(1)| = \sqrt{272}$, and $|\mathbf{r}'(1)| = \sqrt{8.25}$.

So $\kappa = \dfrac{\sqrt{272}}{(\sqrt{8.25})^3} \approx 0.696$.

DFEP #8: Wednesday, April 29th.

The position of a bee over time on the interval $[0, \infty)$ is given by the vector function $\mathbf{r}(t) = \langle \cos(\pi t), t^4 - 4t^3 + 4t^2, \sqrt{t} \rangle$. Compute the tangential and normal acceleration of the bee after $t = 4$ seconds.

DFEP #8 Solution:

We are given the position vector $\mathbf{r} = \langle \cos(\pi t), t^4 - 4t^3 + 4t^2, \sqrt{t} \rangle$ and we want tangential and normal acceleration after $t = 4$ seconds.

First, we need $\mathbf{r}'(t) = \langle -\pi \sin(\pi t), 4t^3 - 12t^2 + 8t, 1/(2\sqrt{t}) \rangle$ (so $\mathbf{r}'(4) = \langle 0, 96, 1/4 \rangle$) as well as $\mathbf{r}''(t) = \langle -\pi^2 \cos(\pi t), 12t^2 - 24t + 8, -1/(4\sqrt{t^3}) \rangle$ (so $\mathbf{r}''(4) = \langle -\pi^2, 104, -1/32 \rangle$).

The usual formulas tell us a_T and a_N:

$$a_T = \frac{r'(4) \cdot r''(4)}{|r'(4)|} = \frac{9983.99219}{\sqrt{96^2 + (1/4)^2}} \approx 103.9996$$

and

$$a_N = \frac{|r'(4) \times r''(4)|}{|r'(4)|} = \frac{|\langle -29, -\pi^2/4, 96\pi^2 \rangle|}{\sqrt{96^2 + (1/4)^2}} \approx 9.8742$$

DFEP #9: Friday, May 1st.

The force exerted on a 5 kg ball after t seconds, in Newtons, is given by the vector function $\mathbf{F}(t) = \langle 5\cos(t)\sin(t), 10e^{5t}, 45t^2 \rangle$.

The initial velocity (in meters per second) and position (in meters) of the ball are the by the vectors $\mathbf{v}(0) = \langle 3, -2, 6 \rangle$ and $\mathbf{r}(0) = \langle 4, 1, 0 \rangle$.

Compute the position of the ball $\mathbf{r}(t)$ (in meters) after t seconds.

DFEP #9 Solution:

First, we compute the acceleration by dividing the force $\mathbf{F}(t)$ by the mass 5 to get

$$\mathbf{a}(t) = \langle \cos(t)\sin(t), 2e^{5t}, 9t^2 \rangle$$

Integrating once gives

$$\mathbf{v}(t) = \left\langle \sin^2(t) + C_1, \frac{2}{5}e^{5t} + C_2, 3t^3 + C_3 \right\rangle$$

and since $\mathbf{v}(0) = \langle 3, -2, 6 \rangle$ we can solve for the constants to get

$$\mathbf{v}(t) = \left\langle \sin^2(t) + 3, \frac{2}{5}e^{5t} - \frac{12}{5}, 3t^3 + 6 \right\rangle.$$

Integrate again (using the half-angle formula to integrate $\sin^2(t)$) and we have

$$\mathbf{r}(t) = \left\langle \frac{1}{2}t - \frac{1}{4}\sin(2t) + 3t + C_4, \frac{2}{25}e^{5t} - \frac{12}{5}t + C_5, \frac{3}{4}t^4 + 6t + C_6 \right\rangle$$

and, one more time, we can use $\mathbf{r}(0) = \langle 4, 1, 0 \rangle$ to solve for the constants:

$$\mathbf{r}(t) = \left\langle \frac{1}{2}t - \frac{1}{4}\sin(2t) + 3t + 4, \frac{2}{25}e^{5t} - \frac{12}{5}t + \frac{23}{25}, \frac{3}{4}t^4 + 6t \right\rangle$$

DFEP #10: Monday, May 4th.

Compute the all the partial derivatives (one for each variable) of the given functions:

(a) $f(x, y) = x^2y^3 - xy + 5x^3$

(b) $g(x, y) = \dfrac{x^2 + 1}{xy + y^2}$

(c) $h(x, y, z) = (2 + \arctan(x + y^2))^z$

DFEP #10 Solution:

I don't really have anything to say about this one. Here are some derivatives.

(a) $f_x(x,y) = 2xy^3 - y + 13x^2$
$f_y(x,y) = 3x^2y^2 - x$

(b) $$g_x(x,y) = \frac{2x(xy+y^2) - y(x^2+1)}{(xy+y^2)^2}$$

$$g_y(x,y) = \frac{-(x^2+1)(x+2y)}{(xy+y^2)^2}$$

(c) $$h_x(x,y,z) = \frac{z(2+\arctan(x+y^2))^{z-1}}{1+(x+y^2)^2}$$

$$h_y(x,y,z) = \frac{2yz(2+\arctan(x+y^2))^{z-1}}{1+(x+y^2)^2}$$

$$h_z(x,y,z) = (2+\arctan(x+y^2))^z \ln(2+\arctan(x+y^2))$$

DFEP #11: Wednesday, May 6th.

Consider the surface $z = x^3e^y - 8\cos(y) + 4x\sin(y)$.

Let P be the point where this surface intersects the x-axis.

Find the equation for the plane tangent to the surface at the point P.

DFEP #11 Solution:

We want the tangent plane to $z = x^3 e^y - 8\cos(y) + 4x\sin(y)$ at the point where it intersects the x-axis.

At that point, the y- and z-coordinates are zero, so we have $0 = x^3 - 8$, so $x = 2$. So the point is $(2, 0, 0)$.

What's the normal vector? We need the partial derivatives:

$$\frac{\partial z}{\partial x} = 3x^2 e^y + 4\sin(y) = 12$$

$$\frac{\partial z}{\partial y} = x^3 e^y + 8\sin(y) + 4x\cos(y) = 16$$

So we get the plane $z = 12(x - 2) + 16y$.

DFEP #12: Friday, May 8th.

Find all critical points of the function $f(x, y) = x + 3y - e^x - y^3$, and classify them as local minima, local maxima, or saddle points.

DFEP #12 Solution:

We need the critical points of $f(x, y) = x + 3y - e^x - y^3$, so we want to solve the equations:

$$f_x(x, y) = 1 - e^x = 0$$

$$f_y(x, y) = 3 - 3y^2 = 0$$

Which has two solutions: $(0, 1)$ and $(0, -1)$. Let's check $D(x, y)$ at each point:

The second derivatives are $f_{xx}(x, y) = -e^x$, $f_{yy}(x, y) = -6y$, and $f_{xy}(x, y) = 0$.

So $D(0, 1) = 6$ and $D(0, -1) = -6$. Since $f_{xx}(x, y) < 0$ for all (x, y), that means $(0, 1)$ is a local maximum and $(0, -1)$ is a saddlepoint.

DFEP #13: Monday, May 11th.

Compute the average value of $f(x, y) = y \sin(2y) \cos(xy)$ over the region $[0, 2] \times [0, \pi/4]$.

DFEP #13 Solution:

We want to compute

$$\int_0^2 \int_0^{\pi/4} y \sin(2y) \cos(xy)\, dy\, dx$$

Oh, wait, that seems maybe impossible. Let's flip it around:

$$\int_0^{\pi/4} \int_0^2 y \sin(2y) \cos(xy)\, dx\, dy$$

That's easier: $y \sin(2y)$ is a constant, and the antiderivative of $\cos(xy)$ with respect to x is $\sin(xy)/y$. So we get:

$$\int_0^{\pi/4} \left(\sin(2y) \sin(xy) \Big]_0^2 \right) dy$$

That's just

$$\int_0^{\pi/4} \sin^2(2y)\, dy = \int_0^{\pi/4} \frac{1}{2}(1 - \cos(4y))\, dy$$

which comes out to $\pi/8$. And since we want the average value over a rectangle of area $\pi/2$, we divide this by $\pi/2$ to get $1/4$.

DFEP #14: Wednesday, May 13th.

Compute the double integral:

$$\int_0^{e^9} \int_{\sqrt{\ln(y)}}^3 2xye^{x^2}\, dx\, dy$$

DFEP #14 Solution:

Okay, this is pretty easy as is:

$$\int_0^{e^9} \int_{\sqrt{\ln(y)}}^3 2xye^{x^2}\, dx\, dy = \int_0^{e^9} \left(ye^{x^2}\right)\Big]_{\sqrt{\ln(y)}}^3 dy = \int_0^{e^9} (ye^9 - y^2)\, dy$$

which we can evaluate as

$$\frac{e^9}{2}y^2 - \frac{1}{3}y^3\Big]_0^{e^9} = \frac{e^{27}}{6}$$

But you should totally try reversing the order of integration anyway, for practice. You'll get:

$$\int_0^3 \int_0^{e^{x^2}} 2xye^{x^2}\, dy\, dx$$

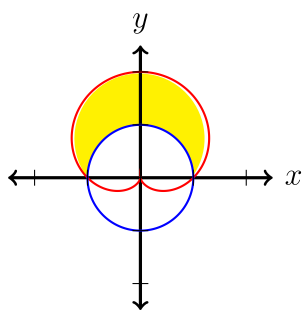
which also comes out to $\dfrac{e^{27}}{6}$.

DFEP #15: Friday, May 15th.

Compute the area inside the cardioid $r = \sin(\theta) + 1$ but outside the circle $x^2 + y^2 = 1$.

DFEP #15 Solution:

Here's a picture:



We want to integrate 1 over this domain. θ runs from 0 to π, and for any given θ, r runs from 1 to $1+\sin(\theta)$. So we want:

$$\int_0^\pi \int_1^{1+\sin(\theta)} r\,dr\,d\theta = \int_0^\pi \left(\frac{1}{2}r^2\right)\Big]_1^{1+\sin(\theta)} d\theta = \frac{1}{2}\int_0^\pi (\sin^2(\theta) + 2\sin(\theta))\,d\theta$$

This becomes

$$\frac{1}{2}\int_0^\pi \left(\frac{1}{2}(1-\cos(2\theta)) + 2\sin(\theta)\right) d\theta = \frac{1}{2}\left(\frac{x}{2} - \frac{1}{4}\sin(2\theta) - 2\cos(\theta)\right)\Big]_0^\pi$$

which simplifies to $\dfrac{\pi}{4} + 2$.

DFEP #16: Wednesday, May 27th.

Consider the function $f(x) = \ln(2x - 5)$.

(a) Find the second Taylor polynomial $T_2(x)$ for $f(x)$ centered at $b = 3$.

(b) Use your answer from part (a) to approximate $\ln(1.04)$.

(c) Give a reasonable error bound on your answer from part (b).

DFEP #16 Solution:

(a) We want $T_2(x)$ centered at $b = 3$ for the function $\ln(2x-5)$. The first few derivatives are:

$$f(x) = \ln(2x-5) \qquad f'(x) = \frac{2}{2x-5} \qquad f''(x) = \frac{-4}{(2x-5)^2}$$

Plugging in $x = 3$ we find $f(3) = 0$, $f'(3) = 2$, and $f''(3) = -4$. So:

$$T_2(x) = 0 + 2(x-3) + \frac{1}{2}(-4)(x-3)^2 = 2(x-3) - 2(x-3)^2$$

(b) We want to approximate $\ln(1.04)$. Well, that's $\ln(2(3.02)-5)$, so it's $f(3.02)$. We can approximate it as

$$T_2(3.02) = 2(3.02-3) - 2(3.02-3)^2 = .04 - 2(.0004) = .0392$$

(c) To find an error bound, we'll need to know $f'''(x) = \dfrac{16}{(2x-5)^3}$.

On the interval $[3, 3.02]$, this is largest when the denominator is smallest, so the maximum is at $x = 3$ and we get $M = 16$. So the error is bounded by:

$$|T_2(3.02) - f(3.02)| \leq \frac{1}{6}(16)|3.02-3|^3 \approx .00002133$$

DFEP #17: Friday, May 29th.

Let $T_n(x)$ be the nth Taylor polynomial for $f(x) = \sin(4x)$ centered at $b = 0$.

Use Taylor's inequality to find an interval $I = [-a, a]$ so that the error $|T_n(x) - f(x)|$ on the interval I less than or equal to 0.01. Your answer will depend on n.

DFEP #17 Solution:

If $f(x) = \sin(4x)$, then $f'(x) = 4\cos(4x)$, $f''(x) = -16\sin(4x)$, $f'''(x) = -64\cos(4x)$, and in general $f^{(n)}(x) = \pm 4^n \sin(x)$ or $\pm 4^n \cos(x)$.

We could spend a while worrying about whether the nth derivative is positive or negative and whether it's $\sin(4x)$ or $\cos(4x)$, but remember that our goal is to find an error bound, so we only need the maximum of $|f^{(n+1)}(x)|$. Since both $|\sin(4x)|$ and $|\cos(4x)|$ have maximum values of 1, we end up with $M = 4^{n+1}$.

So, on the interval $[-a, a]$, the nth Taylor polynomial has error bound

$$|T_n(x) - f(x)| \leq \frac{1}{(n+1)!} 4^{n+1} a^{n+1}$$

If we want this to be less than or equal to 0.01, then we set

$$\frac{1}{(n+1)!} 4^{n+1} a^{n+1} \leq 0.01$$

and solve to get

$$a \leq \sqrt[n+1]{\frac{(n+1)!}{100 \cdot 4^{n+1}}}$$

DFEP #18: Monday, June 1st.

Let $f(x) = \sin(2x^3)$.

(a) Find the Taylor series for $f(x)$ centered at $b = 0$. Write your answer in Σ-notation.

(b) Compute $f^{(45)}(0)$.