Math 124 Section H, Autumn 2014
Midterm Exam Number One: Solutions

1. (a) We can rewrite the limit as

$$\lim_{x\to 0}\frac{\sin(4x)}{x}+\lim_{x\to 0}\frac{9x^2}{x}$$

The first limit is 4 (one proof: multiply the numerator and denominator by 4, then rewrite as $\lim_{t\to 0} 4\sin(t)/t = 4$.), and the second limit is 0, so the answer is $4+0=4$.

(b) Write as

$$\lim_{x\to 3}\sqrt{x}\frac{x-3}{e^x-e^3}=\sqrt{3}\left(\lim_{x\to 3}\frac{x-3}{e^x-e^3}\right)$$

Hey, that's just $\sqrt{3}$ times the reciprocal of the derivative of e^x at $x=3$. So it's $\sqrt{3}/e^3$.

(c) Let's first find the limit of everything inside the $\tan(\quad)$:

$$\lim_{x\to -\infty}\sqrt{9x^2+\pi x}+3x\cdot\frac{\sqrt{9x^2+\pi x}-3x}{\sqrt{9x^2+\pi x}-3x}=\lim_{x\to-\infty}\frac{\pi x}{\sqrt{9x^2+\pi x}-3x}$$

The next part is tricky; remember that as x approaches $-\infty$, $\sqrt{x^2}=-x$.

$$=\lim_{x\to-\infty}\frac{\pi x}{\sqrt{9x^2+\pi x}-3x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to-\infty}\frac{\pi}{-\sqrt{9+\frac{\pi}{x}}-3}=-\frac{\pi}{6}$$

And because $\tan(x)$ is continuous at $\pi/6$, that means

$$\lim_{x\to-\infty}\tan\left(\sqrt{9x^2+\pi x}+3x\right)=\tan\left(\lim_{x\to-\infty}\sqrt{9x^2+\pi x}+3x\right)=\tan\left(-\frac{\pi}{6}\right)=-\frac{1}{\sqrt{3}}$$

2. Hey, it's an intermediate value theorem problem! What we really want is to find a solution to the equation

$$f(x)=2^x+\sin(\pi x)-x^2-3=0$$

The left side is a continuous function. Furthermore, $f(4)=-3$ and $f(5)=4$, so at some point in $[4,5]$ we have $f(x)=0$, which satisfies $2^x+\sin(\pi x)=x^2+3$.

3. Product rule!

$$f'(x)=-\sin(x)(e^x+3x)+\cos(x)(e^x+3)$$

Again!

$$f''(x)=-\cos(x)(e^x+3x)-2\sin(x)(e^x+3)+\cos(x)(e^x)$$

And finally,

$$f''(\pi)=-\cos(\pi)(e^\pi+3\pi)-2\sin(\pi)(e^\pi+3)+\cos(\pi)e^\pi=3\pi$$

4. (a) The only place we might have a vertical asymptote in this function is when a denominator is zero. That means $x^2 + 2x - 8 = 0$ or $\cos(x) + 5 = 0$. Well, $x^2 + 2x - 8 = 0$ when $x = -4$ or $x = 2$, but only $x = -4$ is in the domain of that piece. And $\cos(x) + 5$ is never zero, so we just have one vertical asymptote at $x = -4$. You can confirm this by taking $\lim_{x \to -4} f(x)$ from either side and, yup, it's infinite.

   (b) To find horizontal asymptotes, we look at $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{5}{x^2 + 2x - 8} = 0$$
$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{e^x}{\cos(x) + 5} = \infty$$

   Okay, so $y = 0$ is the only horizontal asymptote.

   (c) Let's use the hint and imagine that $(a, f(a))$, the correct point of tangency, is in that simple-looking middle piece. I sure hope this works!

   On the one hand, the slope of the tangent line is $f'(a) = 4a - 3$. On the other hand, it's the slope of a line between $(0, -4.5)$ and $(a, 2a^2 - 3a + 8)$. Set those equal:

$$4a - 3 = \frac{2a^2 - 3a + 8 - (-4.5)}{a - 0}$$

   and solve to get $a = \pm 2.5$, so $a = 2.5$ is the x-coordinate we want. So the slope is $f'(2.5) = 4(2.5) - 3 = 7$, and with y-intercept -4.5 we've got the equation $y = 7x - 4.5$.

5. (a) Yes, okay, I get it. (Some variations of this answer were accepted.)

   (b) That's just $f'(3)$, which we can clearly see is 4.

   (c) According to the quotient rule:

$$g'(-1) = \frac{f'(-1)f''(-1) - f(-1)f'''(-1)}{\big(f''(-1)\big)^2}$$

   Then we just need to know that $f'(-1) = 2$ (by reading the graph), $f''(-1) = -1/3$ (because it's the slope of f'), and $f'''(-1) = 0$ (because the derivative of a linear function is constant, and the derivative of a constant is zero). So we get

$$g'(-1) = \frac{2 \cdot (-1/3) - f(-1) \cdot 0}{(-1/3)^2} = -6$$

   (d) Maybe! It could be continuous, and that jump in the derivative might just represent a cusp point—the slope coming from the left is different from the slope coming in from the right. But it also might be discontinuous: throwing in a hole or a jump discontinuity at that cusp point wouldn't affect the graph of the derivative.