DFEP #1: Friday, October 3rd.

Suppose you know the following:

$$\lim_{x\to 2} f(x) = 5$$

$$\lim_{x\to 2} g(x) = 3$$

$$\lim_{x\to 3} f(x) = 2$$

Compute each of the following limits if you can. If a limit does not exist or if there isn't enough information, explain.

(a) $\lim_{x\to 2} \big(4f(x) + 3g(x)\big)$

(b) $\lim_{x\to 3^+} \dfrac{f(x)+7}{x-3}$

(c) $\lim_{x\to 2} f(g(x))$

(d) $\lim_{x\to 2} \dfrac{\sqrt{f(x)+4}-3}{f(x)-5}$

DFEP #1 Solution:

(a) Use the sum law: $4 \cdot 5 + 3 \cdot 3 = 29$

(b) The numerator approaches 9, but the denominator goes to 0 (from above). ∞

(c) You might be tempted to write something like this:

$$\lim_{x\to 2} f(g(x)) = \lim_{x\to 3} f(x) = 2$$

Seems reasonable, right? Since $g(x)$ is approaching 3, the numbers you're plugging into $f(x)$ approach 3.

But, nope. That wasn't one of the rules we learned, and in fact the limit might be anything at all!

All we know about $g(x)$ is that $\lim_{x\to 2} g(x) = 3$. One possible solution is that $g(x) = 3$. In that case, we get

$$\lim_{x\to 2} f(g(x)) = \lim_{x\to 2} f(3) = f(3)$$

We don't actually know what $f(3)$ is, because we don't know whether $f(x)$ is continuous! So the limit could be anything.

(d) Either the limit DNE, or the limit is 1/6. Multiplying by the conjugate we have:

$$\lim_{x\to 2} \frac{\sqrt{f(x)+4}-3}{f(x)-5} \cdot \frac{\sqrt{f(x)+4}+3}{\sqrt{f(x)+4}+3} = \lim_{x\to 2} \frac{f(x)+4-9}{(f(x)-5)\left(\sqrt{f(x)+4}+3\right)}$$

$$= \lim_{x\to 2} \left(\frac{f(x)-5}{f(x)-5} \cdot \frac{1}{\sqrt{f(x)+4}+3} \right)$$

At this point we'd like to cancel $f(x) - 5$ from the numerator and denominator, but there's one problem: our ability to cancel terms inside a limit requires that whatever we cancel isn't zero, at least for values of x near the limit. If $f(x) = 5$ is constant, then this limit does not exist, because it's always 0/0. On the other hand, if we can cancel, then the limit simplifies to 1/6. So, we need more information.

DFEP #2: Monday, October 6th.

(a) Find values of a, b, and c so that the following function is continuous at $x = 7$:

$$f(x) = \begin{cases} \dfrac{x^2-8x+7}{ax^2-5ax-14a} & \text{if } x < 7 \\ b & \text{if } x = 7 \\ \dfrac{\sqrt{x+2}-c}{x^2-8x+7} & \text{if } x > 7 \end{cases}$$

(b) Is the function you found continuous everywhere? Explain.

DFEP #2 Solution:

(a) In order for the function to be continuous at $x = 7$, we need that

$$\lim_{x\to 7^+} f(x) = \lim_{x\to 7^-} f(x) = f(7).$$

Let's start with the limit from above.

$$\lim_{x\to 7^+} f(x) = \lim_{x\to 7^+} \frac{\sqrt{x+2} - c}{x^2 - 8x + 7}$$

Since the denominator approaches zero as x approaches 7, the only way this limit is a finite number is if the numerator also approaches zero, which means that $c = 3$. We can then multiply by the conjugate and simplify to find that the limit is $1/36$. That means $f(7) = b = 1/36$ as well. Finally, we can factor out $1/a$ from the left-hand limit, factor, and simplify, to find that the left-hand limit is $2/(3a)$. Setting this equal to $1/36$ yields $a = 24$.

(b) No, this function is not continuous everywhere! There's a vertical asymptote at $x = -2$.

DFEP #3: Wednesday, October 8th.

Compute each limit. If the limit is infinite, say so. If the limit doesn't exist, explain why.

(a) $\displaystyle\lim_{x\to 0} \frac{x^3 - 8}{\cos^2(x)}$

(b) $\displaystyle\lim_{x\to\infty} \left(\tan^{-1}(x)\frac{2x^2 + 3x - 2}{9x^2 + 2x + 1}\right)$

(c) $\displaystyle\lim_{x\to\infty} \sqrt{x^2 + 8x} - \sqrt{x^2 + 3}$

(d) $\displaystyle\lim_{x\to\infty} \tan\left(\frac{\pi x + 3}{\sqrt{9x^2 + 6}}\right)$

DFEP #3 Solution:

(a) $\displaystyle\lim_{x\to 0}\frac{x^3-8}{\cos^2(x)}=\frac{0^3-8}{\cos^2(0)}=-8$

(b) $\displaystyle\lim_{x\to\infty}\left(\tan^{-1}(x)\frac{2x^2+3x-2}{9x^2+2x+1}\right)=\lim_{x\to\infty}\tan^{-1}(x)\cdot\lim_{x\to\infty}\frac{2x^2+3x-2}{9x^2+2x+1}=\frac{\pi}{2}\cdot\frac{2}{9}=\frac{\pi}{9}$

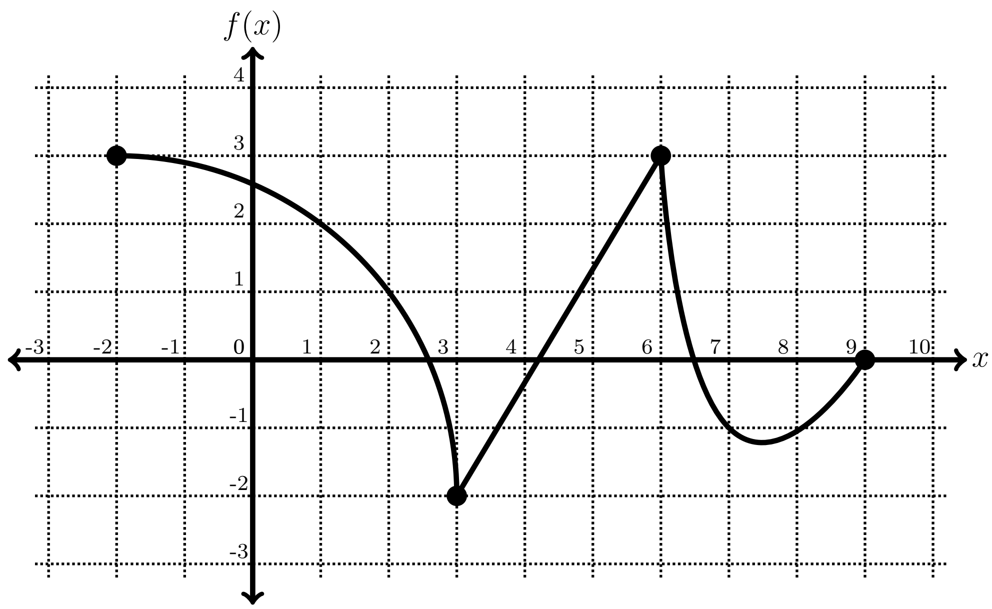
(c) $\displaystyle\lim_{x\to\infty}\sqrt{x^2+8x}-\sqrt{x^2+3}=\lim_{x\to\infty}\frac{\sqrt{x^2+8x}-\sqrt{x^2+3}}{1}\cdot\frac{\sqrt{x^2+8x}+\sqrt{x^2+3}}{\sqrt{x^2+8x}+\sqrt{x^2+3}}$

$\displaystyle=\lim_{x\to\infty}\frac{8x-3}{\sqrt{x^2+8x}+\sqrt{x^2+3}}=\lim_{x\to\infty}\frac{8-\frac{3}{x}}{\sqrt{1+\frac{8}{x}}+\sqrt{1+\frac{3}{x^2}}}=\frac{8}{\sqrt{1}+\sqrt{1}}=4$

(d) It's not too hard to see that $\displaystyle\lim_{x\to\infty}\frac{\pi x+3}{\sqrt{9x^2+6}}=\frac{\pi}{3}$. Since $\tan(x)$ is continuous everywhere on its domain, that means the limit is $\tan(\pi/3)=\sqrt{3}$.

DFEP #4: Friday, October 10th.

The function f is plotted below. Use its graph to answer the questions. (The portion on the interval $[-2,3]$ is a quarter-circular arc.)



(a) Compute $\displaystyle\lim_{x\to 8}\frac{f(x)-8}{(x-8)^2}$.

(b) Compute $\displaystyle\lim_{h\to 0^-}\frac{f(6+h)-3}{h}$.

(c) Compute $\displaystyle\lim_{x\to 1}\frac{f(x)-2}{x-1}$.

(d) Prove that there exists a c in $[-2,9]$ such that $f(c)=-\cos(\pi c)$.

DFEP #4 Solution:

(a) The numerator approaches to -9 while the denominator 0 from above, so the limit is $-\infty$.

(b) This is the derivative "from the left" at $x = 6$. To the left, the function is linear with slope $5/3$, so the limit is $5/3$.

(c) This is $f'(1)$. We can find the slope of the tangent line by recalling that a tangent to a circle is perpendicular to the radius, which in this case has slope $4/3$. So the derivative is $-3/4$.

(d) Consider the function $g(x) = f(c) + \cos(\pi c)$. $g(x)$ is continuous since it's the sum of two continuous functions. Furthermore, $g(-2) = 4$ while $g(9) = -1$, so there's a c in $[-2, 9]$ such that $g(c) = 0$, which means that $f(c) = -\cos(\pi c)$.

DFEP #5: Monday, October 13th.

Consider the following function:

$$f(x) = \frac{8}{1 - x}$$

Find all tangent lines to $f(x)$ with x-intercept 5.

DFEP #5 Solution:

If $f(x) = 8/(1-x)$, then $f'(x) = 8/(1-x)^2$, as we can learn either by computing the limit or by using the quotient rule. Let $(x, 8/(1-x))$ be the point of tangency of a tangent line. If that line passes through the point $(5,0)$, then it has slope $(8/(1-x))/(x-5)$. On the other hand, its slope is given by the derivative, so:

$$\frac{8}{(1-x)(x-5)} = \frac{8}{(1-x)^2},$$

so $x - 5 = 1 - x$, so $x = 3$. Therefore we want the line through the point $(3, -4)$ with slope 2, whose equation is $y = 2(x-3) - 4$.

DFEP #6: Wednesday, October 15th.

Compute the derivative of each function with respect to x. Use whatever techniques you like.

(a) $f(x) = 5x + e^3$.

(b) $f(x) = \sqrt{x} + \dfrac{12}{\sqrt[6]{x^5}}$

(c) $f(x) = (2e^x + 17x)\left(\dfrac{21}{x^4} + 12x^2\right)$

(d) $f(x) = \dfrac{4x^\pi + 1}{x^2 e^x}$

(e) $f(x) = 2 + \sqrt{9 - (x-1)^2}$

(Yes, there's a thing called the chain rule. You can solve this without it!)

DFEP #6 Solution:

(a) If $f(x) = 5x + e^3$, then $f'(x) = 5$.

(b) If $f(x) = \sqrt{x} + \dfrac{12}{\sqrt[6]{x^5}} = x^{1/2} + 12x^{-5/6}$, then $f'(x) = (1/2)x^{-1/2} + 12(-5/6)x^{-11/6}$.

(c) We'll use the product rule, writing $f(x) = g(x)h(x)$. Here $g(x) = 2e^x + 17x$, and $h(x) = 21x^{-4} + 12x^2$.
So $g'(x) = 2e^x + 17$, and $h'(x) = -84x^{-5} + 24x$.
So the derivative is $(2e^x + 17)(21x^{-4} + 12x^2) + (-84x^{-5} + 24x)(2e^x + 17x)$.

(d) We'll use the quotient rule, writing $f(x) = g(x)/h(x)$. Here $g(x) = 4x^\pi + 1$, and $h(x) = x^2e^x$.
That means $g'(x) = 4\pi x^{\pi-1}$, and $h'(x) = 2xe^x + e^x x^2 = e^x(2x + x^2)$ (by the product rule.)
So by the quotient rule, we get the derivative as

$$f'(x) = \frac{(4\pi x^{\pi-1})(x^2e^x) - (e^x(2x + x^2))(4x^\pi + 1)}{(x^2e^x)^2}$$

(e) Hey, $y = 2 + \sqrt{9 - (x-1)^2}$ is an upper semicircle of radius 3 centered at $(1, 2)$. So for a point $(x, 2 + \sqrt{9 - (x-1)^2})$ on the semicircle, the tangent line is perpendicular to the radius, so its slope is

$$\frac{(-1)(x-1)}{\left(2 + \sqrt{9 - (x-1)^2}\right) - 2}.$$

DFEP #7: Friday, October 17th.

Consider the following curve:

$$y = \frac{\sqrt{x^2 + 7} - 4}{x - 3}$$

(a) Find all horizontal asymptotes of this curve. If there are none, explain.

(b) Find all vertical asymptotes of this curve. If there are none, explain.

DFEP #7 Solution:

(a) We'll consider the limits of the function as x goes to infinity and negative infinity. (Note: the first three steps here aren't really necessary for part (a), but they make part (b) easier.)

$$\begin{aligned}
&\lim_{x\to\infty} \frac{\sqrt{x^2+7}-4}{x-3} \cdot \frac{\sqrt{x^2+7}+4}{\sqrt{x^2+7}+4} \\
&= \lim_{x\to\infty} \frac{x^2-9}{(x-3)(\sqrt{x^2+7}+4)} \\
&= \lim_{x\to\infty} \frac{(x-3)(x+3)}{(x-3)(\sqrt{x^2+7}+4)} \\
&= \lim_{x\to\infty} \frac{x+3}{\sqrt{x^2+7}+4} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{1+\frac{3}{x}}{\sqrt{\frac{1}{x^2}}\sqrt{x^2+7}+\frac{4}{x}} \\
&= \lim_{x\to\infty} \frac{1+\frac{3}{x}}{\sqrt{1+\frac{7}{x^2}}+\frac{4}{x}} = \frac{1}{1} = 1
\end{aligned}$$

The limit as x goes to negative infinity is similar, except for one key difference:

$$\frac{1}{x} = -\sqrt{\frac{1}{x^2}}.$$

So ultimately the limit will simplify to

$$\lim_{x\to\infty} \frac{1+\frac{3}{x}}{-\sqrt{1+\frac{7}{x^2}}+\frac{4}{x}} = \frac{1}{-1} = -1$$

So $y=1$ and $y=-1$ are the horizontal asymptotes.

(b) To find the vertical asymptotes, we look for values a such that $\lim_{x\to a} f(x) = \pm\infty$. In this case, we just need to check where the denominator is zero, so $x=3$. But $\lim_{x\to 3} \frac{\sqrt{x^2+7}-4}{x-3} = 3/4$, so there are no vertical asymptotes.

DFEP #8: Wednesday, October 22nd.

Compute the derivative of each function.

(a) $f(x) = \sqrt{\sec(x)}$

(b) $f(x) = \sin(3^{\tan(x)})$

(c) $f(x) = \cos(1 + \cos(1 + \cos(1 + \cos(x))))$

DFEP #8 Solution:

(a) $f(x) = \sqrt{\sec(x)}$. Here the outer function is $\sqrt{x}$, and the inner function is $\sec(x)$, so we have

$$f'(x) = \frac{1}{2\sqrt{\sec(x)}}\sec(x)\tan(x)$$

(b) $f(x) = \sin(3^{\tan(x)})$. We've got three nested functions this time: $\sin(x)$ on the outside, then 3^x, then $\tan(x)$. According to the 2 Chainz Rule™:

$$f'(x) = \cos(3^{\tan(x)}) \cdot 3^{\tan(x)} \ln(3) \cdot \sec^2(x)$$

(c) $f(x) = \cos(1 + \cos(1 + \cos(1 + \cos(x))))$. Chain, chain, chain:

$$\begin{aligned} f'(x) = &-\sin(1 + \cos(1 + \cos(1 + \cos(x)))) \\ &\cdot (-\sin(1 + \cos(1 + \cos(x)))) \\ &\cdot (-\sin(1 + \cos(x))) \\ &\cdot (-\sin(x)) \end{aligned}$$

DFEP #9: Friday, October 24th.

Consider the following parametric curve:

$$x(t) = t\sin(\pi t)$$

$$y(t) = 3^t + t^2$$

Find the equation of the tangent line to this curve at the point $(0, 36)$.

DFEP #9 Solution:

We want the tangent line to the curve

$$x(t) = t\sin(\pi t) \qquad y(t) = 3^t + t^2$$

at the point $(0, 36)$, which means we just need the slope and we can use the point-slope formula. The slope of the tangent line is

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\ln(3)3^t + 2t}{\sin(\pi t) + \pi t\cos(\pi t)},$$

which would be all well and good if we knew what t was, but we don't! Instead, we have a point on the curve. We need to find t so that $x(t) = t\sin(\pi t) = 0$ and $y(t) = 3^t + t^2 = 36$.

To solve the first equation, we have $t = 0$ or $\sin(\pi t) = 0$, which just means that t is an integer.

So now we look for integer solutions to $3^t + t^2 = 36$, which would ordinarily be pretty difficult to solve algebraically, except in this case we can just try plugging in some small numbers and, hey, $t = 3$ works. So:

$$\frac{dy}{dx} = \frac{\ln(3)3^t + 2t}{\sin(\pi t) + \pi t\cos(\pi t)} = \frac{\ln(3)3^3 + 6}{\sin(3\pi) + 3\pi\cos(3\pi)} = \frac{27\ln(3) + 6}{-3\pi}.$$

Finally, the line with that slope going through the point $(0, 36)$ is

$$y = \frac{27\ln(3) + 6}{-3\pi}(x - 0) + 36.$$

DFEP #10: Monday, October 27th.

Consider the curve given by the following equation:

$$x^3 + y^3 = (xy + \sqrt{y})^2$$

Find the equation for the tangent line to this curve at the point $(2, 1)$.

DFEP #10 Solution:

We take the equation

$$x^3 + y^3 = (xy + \sqrt{y})^2$$

and differentiate both sides with respect to x. This yields

$$3x^2 + 3y^2y' = 2(xy + \sqrt{y})(y + xy' + \frac{1}{2\sqrt{y}}y')$$

which expands to

$$3x^2 + 3y^2y' = 2(xy^2 + x^2yy' + x\sqrt{y}y'/2 + \sqrt{y}y + x\sqrt{y}y' + y'/2)$$

which simplifies to

$$3x^2 + 3y^2y' = 2xy^2 + 2x^2yy' + x\sqrt{y}y' + 2\sqrt{y}y + 2x\sqrt{y}y' + y'$$

So we get

$$y' = \frac{3x^2 - 2xy^2 - 2\sqrt{y}y}{2x^2y + x\sqrt{y} + 2x\sqrt{y} + 1 - 3y^2}$$

Plugging in $x = 2$ and $y = 1$ yields $y' = 1/2$, so the tangent line has equation

$$y = \frac{1}{2}(x - 2) + 1 = x/2$$

DFEP #11: Wednesday, October 29th.

Find the equation of a tangent line to the curve $xy + y^2 = 2x$ with x-intercept -2.

DFEP #11 Solution:

We'll need to use implicit differentiation on $xy + y^2 = 2x$, and we get $y' = \dfrac{2-y}{x+2y}$.

Suppose (a, b) is the point of tangency on this curve. Then on the one hand, the slope of the tangent line is the slope from (a, b) to $(-2, 0)$, and on the other hand it's y' with $x = a$, $y = b$, so:

$$\frac{b-0}{a+2} = \frac{2-b}{a+2b}$$

We also know that (a, b) is on the curve, so:

$$ab + b^2 = 2b$$

Solving these two equations yields (eventually) $a = b = 1$, so the point of tangency is $(1, 1)$, and the line from $(1, 1)$ to $(-2, 0)$ is

$$y = \frac{1}{3}(x-1) + 1$$

DFEP #12: Friday, October 31st.

More derivatives! Find $\dfrac{dy}{dx}$:

(a) $y = \arcsin(5x)$

(b) $y = \sqrt{5 + \ln(2x)}$

(c) $y = \cos(x)^{\sin(x)}$

DFEP #12 Solution:

(a) $y = \arcsin(5x)$, so by the chain rule we have $\dfrac{dy}{dx} = \dfrac{5}{\sqrt{1-(5x)^2}}$

(b) $y = \sqrt{5 + \ln(2x)}$, so again, chain rule: $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{5+\ln(2x)}}\left(0 + \dfrac{2}{2x}\right)$

(c) $y = \cos(x)^{\sin(x)}$. Take the natural log of both sides so we have $\ln(y) = \ln\big(\cos(x)^{\sin(x)}\big)$. This can be rewritten as $\ln(y) = \sin(x)\ln(\cos(x))$, and implicit differentiation (along with the product rule) gives:

$$\frac{y'}{y} = \cos(x)\ln(\cos(x)) + \sin(x)\frac{-\sin(x)}{\cos(x)}$$

Multiplying both sides by y and substituting $y = \cos(x)^{\sin(x)}$ yields the answer

$$\frac{dy}{dx} = \cos(x)^{\sin(x)}\left(\cos(x)\ln(\cos(x)) + \sin(x)\frac{-\sin(x)}{\cos(x)}\right)$$

DFEP #13: Monday, November 3rd.

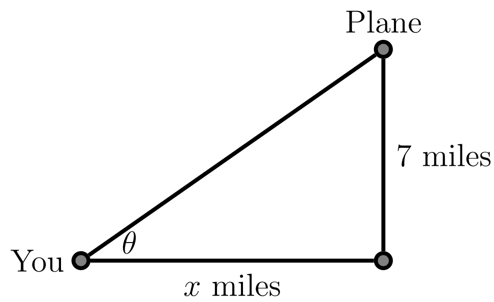
A plane is flying straight across the sky
and passes o'er your head along the way.
The plane's precisely seven miles high.
(I hope you've learned related rates today.)

You watch the plane, in order to succeed,
and use your special estimation power.
With that in mind, you calculate the speed:
it's equal to five hundred miles per hour.

And after that velocity you found,
you quickly choose to track some other data.
The angle of the plane above the ground
from your perspective shall be known as θ.

When θ's ten degrees, here's what you're thinking:
how quickly is the angle θ shrinking?

DFEP #13 Solution:



1. We know $\frac{dx}{dt} = 500$ miles per hour and $\theta = 10^\circ = \pi/18$ radians. We want $\frac{d\theta}{dt}$.
2. We'll use the equation $\cot(\theta) = x/7$.
3. Differentiation yields $-\csc^2(\theta)\frac{d\theta}{dt} = \frac{1}{7}\frac{dx}{dt}$.
4. Plugging in what we know and solving yields $\frac{d\theta}{dt} = \frac{-500}{7\csc^2(\pi/18)} \approx -2.154$ radians per hour, so the angle is shrinking at a rate of approximately 123.4° per hour. (That probably seems like a lot, but keep in mind that the rate is changing and will have slowed down considerably by the time a whole hour has elapsed.)

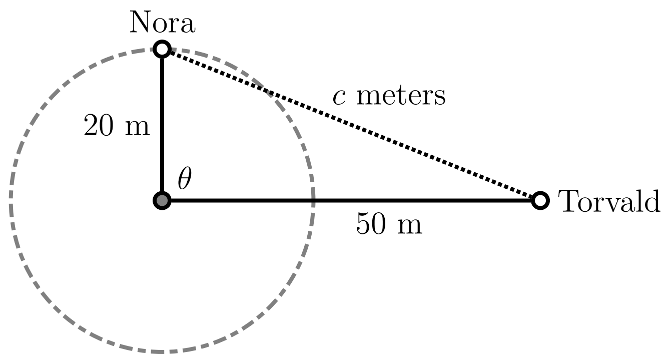
DFEP #14: Wednesday, November 5th.

Nora is running counterclockwise around a circular track of radius 20 meters, and makes one complete lap every 40 seconds.

Torvald stands 30 meters east of the easternmost point of the track.

When Nora is at the northernmost point of the track, at what rate is the distance between them changing?

DFEP #14 Solution:



1. $\dfrac{d\theta}{dt} = \dfrac{2\pi}{40}$, and $\theta = \dfrac{\pi}{2}$. We want to find $\dfrac{dc}{dt}$.

2. $c^2 = 20^2 + 50^2 - 2(20)(50)\cos(\theta)$.

3. $2c\dfrac{dc}{dt} = 0 + 0 + 2000\sin(\theta)\dfrac{d\theta}{dt}$.

4. We'll need to know c, so we plug $\theta = \pi/2$ into the equation from step (2) to get $c = \sqrt{2900}$.

Then we can plug everything we know into the equation from step 3 and solve:

$$\frac{dc}{dt} = \frac{2000\sin(\pi/2)}{2\sqrt{2900}} \cdot \frac{2\pi}{40} \approx 2.917 \text{ meters per second.}$$

DFEP #15: Friday, November 7th.

(a) Find the linear approximation to $f(x) = \sqrt[3]{(x+1)^2}$ at $a = 7$.

(b) Use your answer from part (a) to estimate $\sqrt[3]{7.996^2}$.

DFEP #15 Solution:

(a) We need $f(a)$ and $f'(a)$, where $a = 7$. $f(7) = \sqrt[3]{(7+1)^2} = \sqrt[3]{64} = 4$.
$f'(x) = \frac{1}{3}((x+1)^2)^{-2/3}2(x+1) \cdot 1$, so $f'(7) = 1/3$.
So $f(x) \approx f(a) + f'(a)(x-a) = 4 + (1/3)(x-7)$.

(b) We want $\sqrt[3]{7.996^2} = \sqrt[3]{(6.996+1)^2} = f(6.996) \approx 4 + (1/3)(6.996-7) = 3.998\overline{6}$.
The actual cube root is 3.9986665553..., which is pretty close to what we got.

DFEP #16: Monday, November 10th.

Consider the following curve:

$$10\sec(x)y + 2x^2 = y^3 + 3e^x$$

Use linear approximation at the point $(0, 3)$ to estimate the y-coordinate of the point with x-coordinate 0.01.

DFEP #16 Solution:

We begin with implicit differentiation so we can find y':

$$10\sec(x)\tan(x)y + 10\sec(x)y' + 4x = 3y^2y' + 3e^x$$

$$y' = \frac{3e^x - 10\sec(x)\tan(x)y - 4x}{10\sec(x) - 3y^2}$$

Plugging in $x = 0$, $y = 3$ gives:

$$y' = \frac{3e^0 - 10\sec(0)\tan(0)(3) - 4(0)}{10\sec(0) - 3(3)^2} = \frac{3}{-17}$$

So the tangent line approximation at $(0, 3)$ is:

$$y = \frac{3}{-17}(x - 0) + 3$$

If we plug in $x = 0.01$, we get $y \approx \frac{3}{-17}(.01) + 3$.

DFEP #17: Friday, November 14th.

For each function, find the absolute extrema over the given interval.

(a) $f(x) = \sin(x) - x/2$, over the interval $[0, \pi]$.

(b) $g(x) = e^x\sqrt[3]{x}$, over the interval $[-2, 2]$.

DFEP #17 Solution:

(a) $f(x) = \sin(x) - x/2$ is continuous everywhere, and $[0, \pi]$ is a closed interval. Good.

Next, we take the derivative: $f'(x) = \cos(x) - 1/2$. This is never undefined, and setting it equal to zero gives $\cos(x) = 1/2$, so $x = \pi/3$.

Finally, we plug $x = 0$, $x = \pi/3$, and $x = \pi$ into the original function. $f(0) = 0$, $f(\pi/3) = \sqrt{3}/2 - \pi/6 \approx 0.342$, and $f(\pi) = 0 - \pi/2 \approx -1.571$.

So 0.342 is the absolute maximum, and $-\pi/2$ is the absolute minimum.

(b) First, we note that $f(x) = e^x \sqrt[3]{x}$ is continuous everywhere, and that the given interval is closed.

We'll need the derivative. By the product rule:

$$f'(x) = e^x \sqrt[3]{x} + \frac{e^x}{3\sqrt[3]{x^2}} = e^x \left(\sqrt[3]{x} + \frac{1}{3\sqrt[3]{x^2}} \right),$$

which simplifies further to

$$f'(x) = \frac{e^x(3x+1)}{3\sqrt[3]{x^2}}$$

This is zero when $x = -1/3$, and undefined when $x = 0$, so we need to check four things: $f(-1/3)$, $f(0)$, $f(-2)$, and $f(2)$. $f(2) \approx 9.31$ is the absolute maximum, while $f(-1/3) \approx -0.497$ is the absolute minimum.

DFEP #18: Wednesday, November 19th.

Consider the function $f(x) = \dfrac{e^x}{x+1}$.

(a) Where is $f(x)$ increasing, and where is it decreasing?

(b) Identify all local extrema.

(c) Where is $f(x)$ concave up, and where is it concave down?

(d) Find all points of inflection.

DFEP #18 Solution:

We're going to want to know the first and second derivatives for this problem:

$$f(x) = \frac{e^x}{x+1}$$
$$f'(x) = \frac{(x+1)e^x - e^x}{(x+1)^2} = \frac{xe^x}{(x+1)^2}$$
$$f''(x) = \frac{(x+1)^2(e^x + xe^x) - 2(x+1)xe^x}{(x+1)^4} = \frac{e^x(x^2+1)}{(x+1)^3}$$

(a) $f'(x)$ is zero when $x = 0$, and undefined when $x = -1$. It's negative when x is negative and positive when x is positive, so $f'(x)$ is increasing on $(0, \infty)$ and decreasing on $(-\infty, -1) \cup (-1, 0)$.

(b) The only critical number where $f(x)$ is defined is at $x = 0$, and $f(0) = 1$. Here $f(x)$ switches from decreasing to increasing, so $(0, 1)$ is a local minimum.

(c) $f''(x)$ is never zero, but it's undefined when $x = -1$. When $x + 1$ is negative, $f''(x)$ is negative, and when $x+1$ is positive, $f''(x)$ is positive. So $f(x)$ is concave down on $(-\infty, -1)$ and concave up on $(-1, \infty)$.

(d) $f(x)$ switches concavity at $x = -1$, but it's not continuous there. So there are no points of inflection.

DFEP #19: Friday, November 21st.

Compute each limit.

(a) $\displaystyle\lim_{x\to 0} \frac{\cos(2x) - e^x}{\sin(x)}$

(b) $\displaystyle\lim_{x\to 1^+} \frac{2x^2 + 3x + 1}{x^2 - 1}$

(c) $\displaystyle\lim_{x\to\infty} \left(x^2 + 5x\right)^{\frac{1}{x}}$

DFEP #19 Solution:

(a) $\lim_{x\to 0} \dfrac{\cos(2x) - e^x}{\sin(x)}$ is a limit of the form $\dfrac{0}{0}$, so we can use l'Hôpital's rule:

$$\lim_{x\to 0} \frac{\cos(2x) - e^x}{\sin(x)} = \lim_{x\to 0} \frac{-2\sin(2x) - e^x}{\cos(x)} = \frac{-1}{1} = -1$$

(b) Did you get that the limit is 3.5? Then you are a bad person who uses l'Hôpital's rule carelessly and now look what you've done: the limit is actually ∞! No, no, it's okay. I'm not mad, I'm just disappointed.

Make sure that your limit is of the form $\dfrac{0}{0}$ or $\dfrac{\pm\infty}{\pm\infty}$ next time.

(c) Let's say $y = (x^2 + 5x)^{\frac{1}{x}}$. It's hard to find $\lim_{x\to\infty} y$, but if we take the natural log then it's easy to apply l'Hôpital's rule: $\ln(y) = \dfrac{\ln(x^2 + 5x)}{x}$, as x goes to infinity, is a limit of the form $\dfrac{\infty}{\infty}$. So:

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\ln(x^2 + 5x)}{x} = \lim_{x\to\infty} \frac{2x + 5}{x^2 + 5x} = 0$$

But if $\ln(y)$ is approaching 0, then y is approaching 1. So the limit is 1.

DFEP #20: Monday, November 24th.

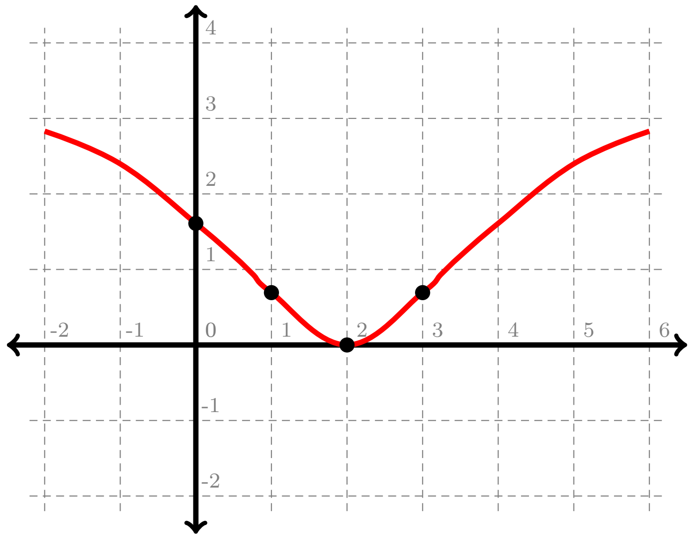
Sketch the following function. Label all intercepts, asymptotes, local extrema, and points of inflection.

$$y = \ln(x^2 - 4x + 5)$$

DFEP #20 Solution: Once again, the function was $y = \ln(x^2 - 4x + 5)$. This has a y-intercept of $(0, \ln(5)) \approx (0, 1.61)$, and an x-intercept of $(2, 0)$.

There's no vertical asymptote since $x^2 - 4x + 5$ is always positive, and there's no horizontal asymptote because the curve tends to infinity as x goes to $\pm\infty$.

The function is decreasing on the interval $(-\infty, 2)$ and increasing on the interval $(2, \infty)$. There's a local minimum at $(2, 0)$. It's concave down on the interval $(-\infty, 1)$ and $(3, \infty)$, and concave up on the interval $(1, 3)$. There are points of inflection at $(1, \ln(2)) \approx (1, 0.69)$ and $(3, \ln(2)) \approx (3, 0.69)$. Graphing all of this gives:



Okay confession: I don't really know how to plot this graph nicely using the software I'm using. Those four circled points should be labeled with their coordinates, if this were actually your answer on the exam.

DFEP #21: Monday, December 1st.

You're producing little cylindrical barrels with no top, so that the volume of the cylinder is 100 cubic centimeters. Find the dimensions of such a barrel which minimize the surface area.