A Really Long Problem's Solution

I wasn't kidding: this is a long problem. It can, however, be broken down into smaller pieces. Each character loosely represents a part of the problem that can be solved independently before moving on to the next step. In this writeup of the solution, these waypoints—the moments in the problem where you move on from one character to the next—are written in bold, with a short description of the relevant information that we'll need for the next piece of the puzzle.

Without further ado, here is the solution.

Amy's linear speed is $v = 6$ meters per second, and the radius is $r = 16$ meters, so her angular speed is $\omega = v/r = 6/16$ radians per second. She needs to travel $\theta = 3\pi/2$ radians until she reaches the northernmost point, so the necessary time is $t = \theta/\omega = 4\pi$.

Amy takes 4π seconds to run to the northernmost point.

Basil starts with 32 inches of wire, and unspools an additional 2 inches per second for 4π seconds.

Basil gives $32 + 8\pi$ inches of wire to Clara.

Clara has $32 + 8\pi$ inches of wire, and this will become the sum of the perimeter of the square plus the area of the circle. If the circle has radius r and the square has side length s, then $32 + 8\pi = 2\pi r + 4s$. Solving this for s yields

$$s = (32 + 8\pi - 2\pi r)/4.$$

The total area of the circle and wire is $\pi r^2 + s^2$. Substituting s from the previous paragraph gives us

$$A = \pi r^2 + \left(\frac{32 + 8\pi - 2\pi r}{4}\right)^2.$$

This can be written as

$$A = \left(\pi + \frac{1}{4}\pi^2\right) r^2 + \left(-8\pi - 2\pi^2\right) r + 64 + 4\pi^2 + 32\pi.$$

This is an upward-pointing parabola, and the minimum value occurs when

$$r = h = \frac{-b}{2a} = \frac{-(-8\pi - 2\pi^2)}{2(\pi + \pi^2/4)} = 4.$$

Clara gives a wire circle of radius 4 to Desmond.

Desmond's bike has a front sprocket with radius 8 inches, a rear sprocket with radius 4 inches (taken from Clara), and a rear wheel of radius 15 inches. Furthermore, Ernest will be pedaling the front sprocket at an angular speed of 6 radians per second.

We can keep track of all of this information with the following table. v is measured in inches per second, ω is measured in radians per second, and r is measured in inches:

	v	ω	r
Front Sprocket		6	8
Rear Sprocket			4
Rear Wheel			15

Because the rear and front sprockets are connected by a belt, they should have the same linear speed. Likewise, the rear sprocket and rear wheel should have the same angular speed. These pairs of boxes are connected in the above table to indicate that they should contain the same numbers.

We can now go about filling in all the blank boxes of the table by using two rules:

- When we know two out of three numbers in a row, we can find the third one by using the equation $v = \omega r$.
- When we fill in a cell that's connected to another cell, we can fill in the other cell with the same value.

Using those two rules lets us quickly fill in the rest of the table:

	v	ω	r
Front Sprocket	48	6	8
Rear Sprocket	48	12	4
Rear Wheel	180	12	15

As the table suggests, the rear wheel of the bike will rotate at a linear speed of 180 inches per second. Since the next part of the problem is in feet, we can helpfully restate this as $180/12 = 15$ feet per second.

Ernest pedals the bike at 15 feet per second.

Francine is $\sqrt{96^2 + 72^2} = 120$ feet away from Ernest, so it takes him $120/15 = 8$ seconds to reach her. (This won't be important until Ida's section of the problem.)

Ernest reaches Francine in 8 seconds.

George stands at $(81, 33)$. The line from Ernest to Francine is given by the equation $y = (96 - 0)/(72 - 0)(x - 0) + 0$, which simplifies to $y = (4/3)x$. To determine when Ernest is closest to George, we look at the perpendicular line which goes through the point $(81, 33)$ and has slope $-3/4$. This line is $y = (-3/4)(x - 81) + 33$.

Ernest is closest to George at the point where these lines intersect, so we solve

$$\frac{4}{3}x = \frac{-3}{4}(x - 81) + 33$$

to get $x = 45$, $y = 60$. This is $\sqrt{45^2 + 60^2} = 75$ feet away from Ernest's starting location, so it takes him $75/15 = 5$ seconds to get there.

Ernest is closest to George after 5 seconds of biking.

Let $t = 0$ be the time when Ernest starts biking. George's pie is $250°$ when $t = 0$ and $200°$ when Ernest is closest to him at $t = 5$. Suppose $T = A_0 b^t$ gives the temperature (T) of the pie t seconds after Ernest begins biking. Then $A_0 = 250$ is the initial temperature, and $200 = 250b^5$. Solving this yields $b = \sqrt[5]{4/5}$, which means $T = 250(\sqrt[5]{4/5})^t$.

We'll want to know when the pie is $128°$, so we set $128 = 250(\sqrt[5]{4/5})^t$. Dividing by 250 and taking the natural log of both sides gives $\ln(128/250) = \ln((\sqrt[5]{4/5})^t)$, which simplifies to

$$\ln(128) - \ln(250) = t/5(ln(4) - \ln(5)),$$

and solving for t yields $t = 15$ seconds.

George's pie takes 15 seconds to cool to $128°$ after Ernest starts biking.

Next, we want to find a linear-to-linear rational function for the height of Hector's beanstalk t seconds Ernest starts biking, of the form

$$f(x) = \frac{at + b}{t + d}.$$

We know that $f(0) = 10$, $f(15) = 64$ (because the beanstalk would be 64 meters high when Ernest's pie is $128°$), and that $a = 100$ (since the beanstalk's height approaches 100 meters in the long run). So we've got three equations:

$$10 = \frac{a(0) + b}{0 + d}$$

$$64 = \frac{a(15) + b}{15 + d}$$

$$a = 100$$

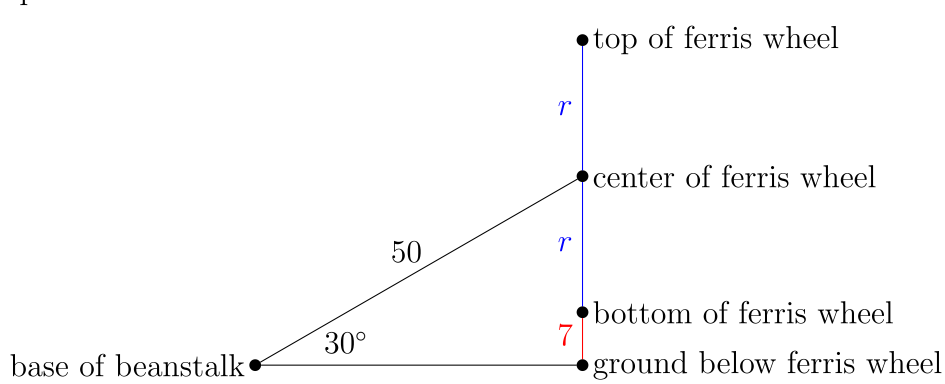
The first equation simplifies to $b = 10d$. Plugging this and the third equation into the second and simplifying yields $1500+10d = 960+64d$. So $54d = 540$, so $d = 10$ and $b = 10(10) = 100$. Therefore our equation is

$$f(t) = \frac{100t + 100}{t + 10}.$$

Ida chops down the beanstalk after when Ernest reaches Francine (at $t = 8$), so the beanstalk will be $f(8) = (100(8) + 100)/(8 + 10) = 50$ meters tall.

When Ida cuts down Hector's beanstalk, it's 50 meters tall.

Here's a picture of the beanstalk as it touches the ferris wheel:



By the above picture, we have $\sin(30°) = (r+7)/50$, so the radius is $r = 50\sin(30°) - 7 = 18$, and the ferris wheel is centered $r + 7 = 25$ meters above the ground.

James's ferris wheel has a radius of 18 meters, and is centered 25 meters off of the ground.

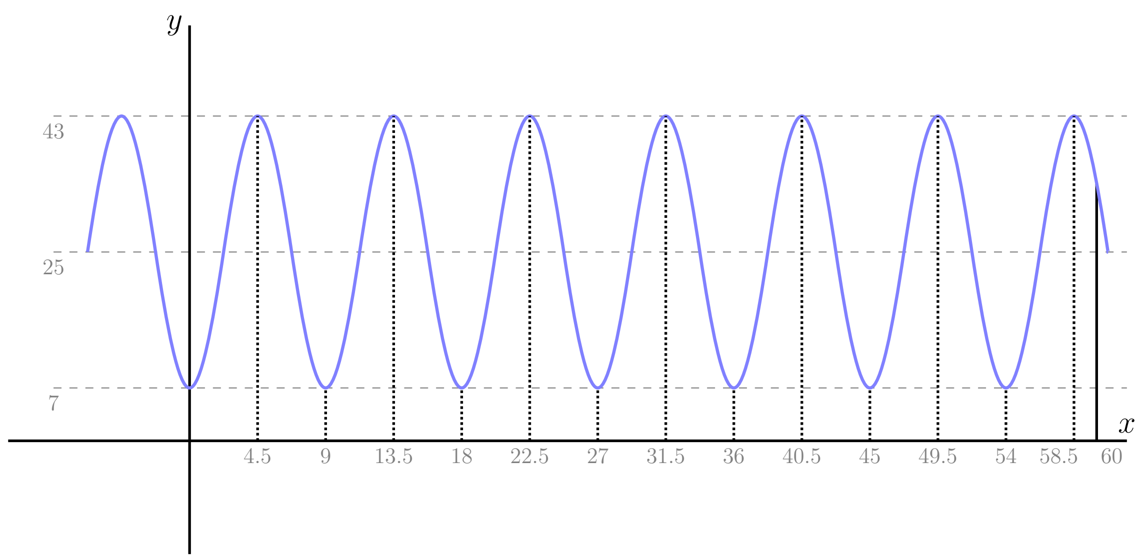
We're finally able to get an equation for James's height as he revolves around the wheel. It's a sinusoidal function, with:

- Amplitude $A = 18$, since the amplitude is half the vertical distance between the top and bottom of the wheel. This is the same as the radius of the circle.
- Period $B = 9$, since it takes James 9 minutes to make a full circle.
- Phase shift $C = 2.25$, since it takes James $9/4 = 2.25$ minutes to make it halfway from the bottom (where he starts) to the middle.
- Mean $D = 25$, since the circle is centered 25 meters above the ground.

So after x minutes, James's height above the ground is given by the function

$$f(x) = 18\sin\left(\frac{2\pi}{9}(x - 2.25)\right) + 25.$$

Here's a graph of this function for the first 60 minutes:



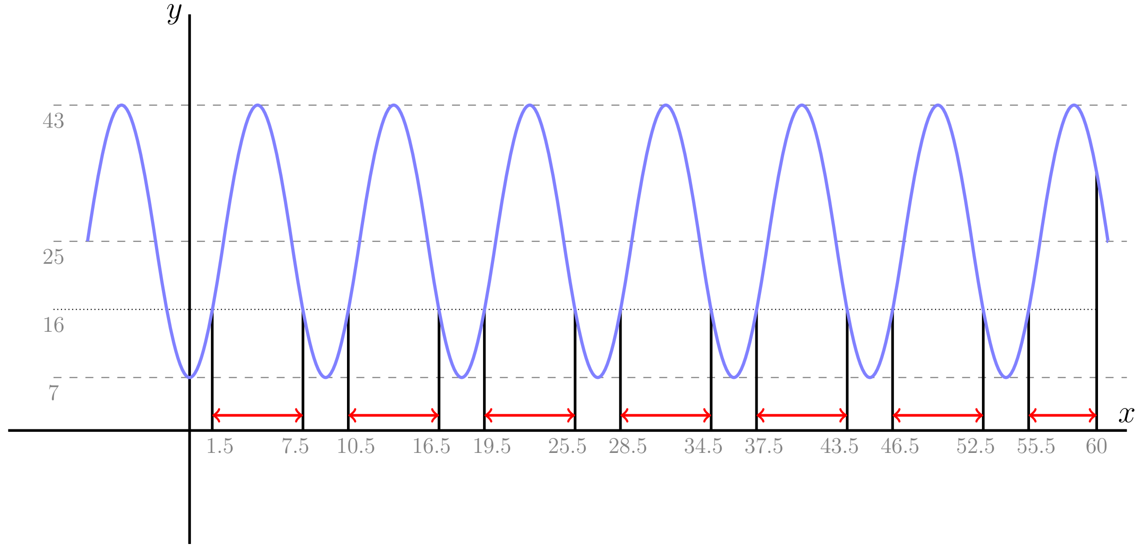
We'll start by finding the principal solution to the equation

$$16 = 18\sin\left(\frac{2\pi}{9}(x - 2.25)\right) + 25$$

to get $x = 2.25 + \frac{9}{2\pi}\arcsin(-1/2) = 1.5$.

To find the symmetry solution, we note that the closest peak is at $x = 4.5$. The principal solution is three units left of the peak, so the symmetry solution should be three units to the right of the peak at $x = 7.5$.

We can find more solutions by adding multiples of 9 to the principal and symmetry solutions. These are depicted along the x-axis in the graph below:



The intervals during which James is greater than 16 meters off the ground are indicated by the red arrows. Notice that the first six such intervals are all six seconds long, so the total time is given by

$$\text{Total Time } = 6 \times 6 + (60 - 55.5) = 40.5 \text{ minutes.}$$

That time is the solution to the problem:

In the first hour, the total length of time that James spends at an elevation greater than 16 meters off the ground is 40 minutes and 30 seconds.