Some Examples of Optimization Problems

Quadratic optimization problems can take a while to get used to, but the textbook doesn't have many examples. So here are some more.

First off, what is an optimization problem? Optimization is the process of making a quantity as large or small as possible. You'll do this a lot in Math 124 using calculus, and in fact the first few steps of our method are exactly the same. Here's the breakdown:

Step 0. Draw a picture. Label any quantities in the picture that you think might be important. Use numbers for the ones you know, and variables for the ones you don't.

This step isn't always necessary, but it's definitely a good idea if your problem involves geometry and a picture isn't provided.

Step 1. Write a formula for the quantity you want to optimize.

If you don't know how, then you probably need to draw a better picture, or label more things in your picture.

Step 2. (If your formula involves only one variable, you can skip this part and move on to step 3.) Use the other information from the problem to rewrite your formula so that it only uses one variable.

Usually, this means writing down an equation relating the variables, solving that equation for one of those variables, and then plugging it back into the formula from step 1.

Step 3. If your formula is now a quadratic function, awesome! Find the vertex and make a (very simple) sketch of the parabola.

If your formula is not a quadratic function, you might need calculus instead. But sometimes it's still possible to solve with precalculus, as in the case of distance-minimization problems: if you want to minimize the square root of a quadratic, that's the same as making the quadratic as small as possible.

Step 4. Using your sketch from step 3, find the minimum or maximum value of the quantity you want to optimize.

You might have to use common sense or other constraints in the problem to restrict the domain you're looking at. For example, if you have a downward-pointing parabola that tells you the area of a fence, then you should ignore any part of the graph where the area is negative, because that's impossible.

Remember to correctly interpret what the problem is asking for! Suppose you're trying to maximize profit, and at the end of step 2 you have a function for profit $f(x) = -2x^2 + 5x$, where x is the price you set.

If the problem is asking for the price, then you want the x-value which maximizes $f(x)$. But if the problem is asking for the profit, then you want to find the actual value of $f(x)$ instead. Or maybe it's asking for something else entirely! Read carefully as you finish each problem.

Let's start with a profit-maximization problem.

> **Problem #1:** Alma is selling apple juice. It costs her 20¢ to make each cup. If she charges 60¢ per cup, then 70 people will buy a cup. If she charges \$2.00 per cup, then only 35 people will buy a cup.
>
> Assume the number of people who buy apple juice is a linear function of the price. What should she charge to maximize her profit?

Step 0. We're skipping this step, because there's no geometry involved and no picture to draw.

Step 1. We need some variable names: let $\$x$ be the price, and let y be the number of people who buy apple juice. Then her profit (in dollars) is $P = xy - 0.2y$, because she makes $\$x$ for each cup she sells, but she also loses 20¢.

Step 2. We have two variables, so we need some way to relate y and x to make the problem use only one variable. The problem tells us that y is a linear function of x, so let's use that information. We have the two points $(0.6, 70)$ and $(2, 35)$, and we want a straight line between them. The two-point formula says

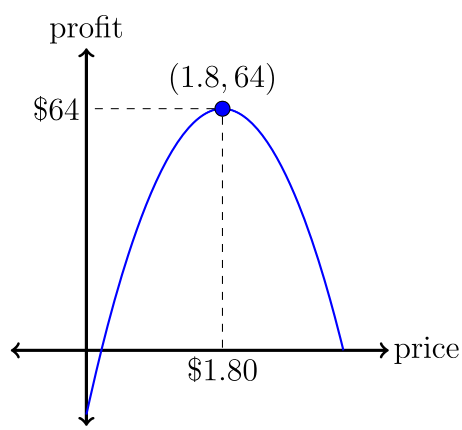
$$y = \frac{35 - 70}{2 - 0.6}(x - 0.6) + 70,$$

which simplifies to $y = -25x + 85$.

We can plug that into our original formula to get $P = x(-25x + 85) - 0.2(-25x + 85)$, which simplifies to $P = -25x^2 + 90x - 17$.

Step 3. That's a quadratic! The vertex is $h = \dfrac{-90}{2(-25)} = 1.8$, and $k = -17 - \dfrac{90^2}{4(-25)} = 64$.

The parabola points down, so it looks like this:



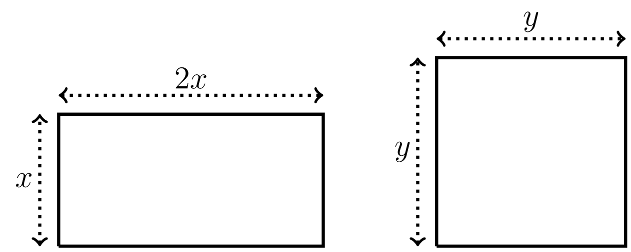
Step 4. Based on the sketch above, the maximum profit happens when the price is \$1.80.

(If the problem asked what the maximum profit actually is, we would say \$64. If it asked for the minimum profit, note that the $-\$17$ on the vertical axis is actually possible: that's where Alma doesn't charge anything, but 85 people come and buy a cup for free, so she eats the \$17 that it costs to make the juice. On the other hand, values to the right of $x = 3.4$ don't make sense, because that's when you start getting a negative number of customers.)

Next, a problem about wire shapes:

> **Problem #2:** You have 50 cm of wire, and you have to use part of this wire to make a rectangle that's twice as long as it is wide, and the rest of the wire (if there is any left) to make a square.
>
> What should the dimensions of the shapes be if you want the total area to be as small as possible? What if you want the total area to be as large as possible?

Step 0. Here's a picture:



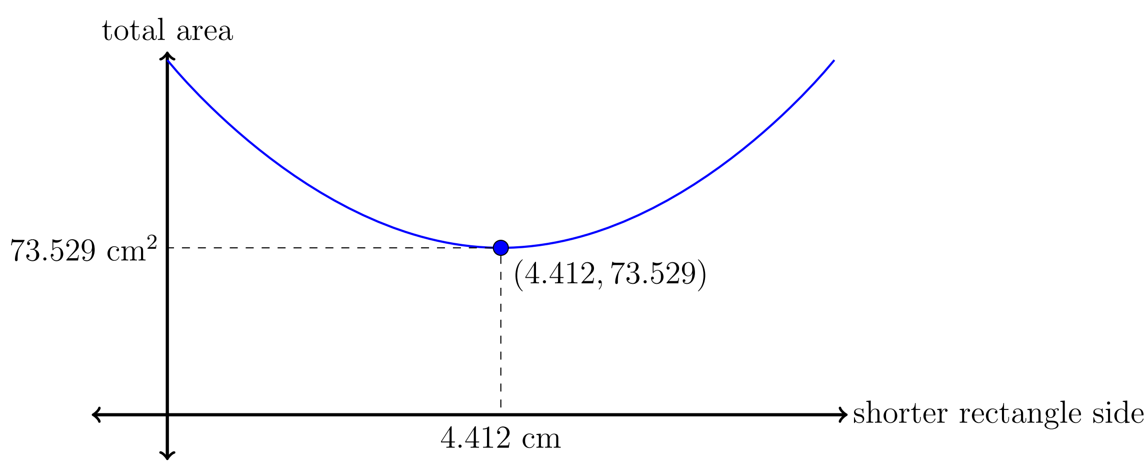
Step 1. The rectangle's area is $x \cdot 2x = 2x^2$, and the square's area is $y \cdot y = y^2$, so the total area is $A = 2x^2 + y^2$.

Step 2. We have two variables. The other information in the problem is that the total amount of wire (that is, the total perimeter of both shapes) is 50 cm. The perimeter of the rectangle is $6x$ and the total perimeter of the square is $4y$, so $6x + 4y = 50$. Solve this for y to get $y = -1.5x + 12.5$, and plug that into the equation from step 1 to get $A = 2x^2 + (-1.5x + 12.5)^2$. This simplifies to:

$$A = 4.25x^2 - 37.5x + 156.25$$

Step 3. That's a quadratic! Vertex: $h = \dfrac{37.5}{2(4.25)} \approx 4.412$, and $k = 156.25 - \dfrac{(-37.5)^2}{4(4.25)} \approx 73.529$.

The parabola points up, so it looks like this:

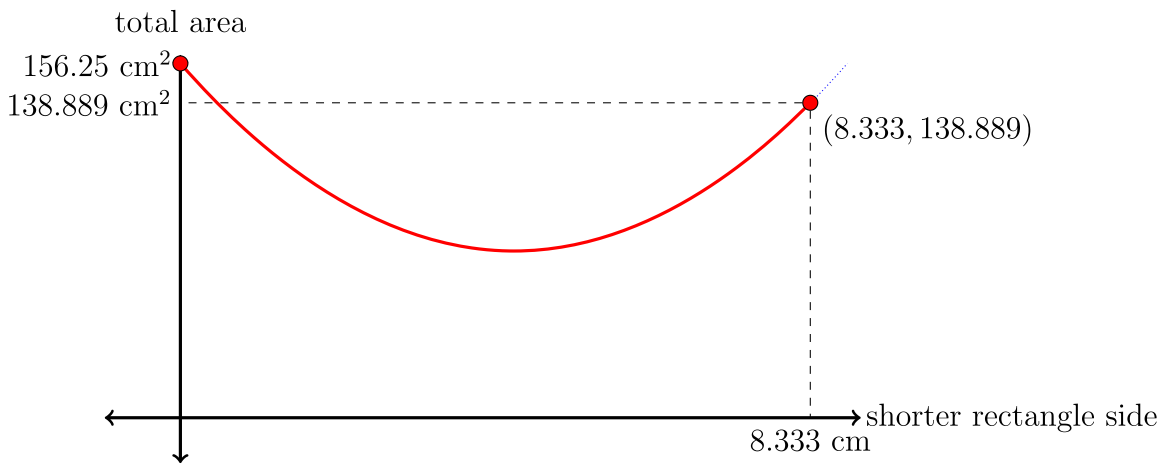


Step 4. It's easy to see that the minimum area occurs when $x = 4.412$. In that case, the short side of the rectangle is 4.412 cm, and the long side is therefore 8.824 cm. Using the earlier equation $y = -1.5x + 12.5$, we see that square is 5.882 cm long on each side.

The maximum is another story. The parabola points upward, but what's the domain?

Well, the smallest possible value of x is 0: that's where you don't use any wire for the rectangle, and you use it all for the square. The largest value of x is where you use all 50 cm for the perimeter of the rectangle, so $6x = 50$, so $x \approx 8.333$ cm.

Here's the graph of the parabola again, but this time the domain is shown in red, along with its boundary points:



So the maximum area occurs when $x = 0$ and we use all 50 cm to make a 12.5 cm $\times$ 12.5 cm square.