REMARKS ON THE UPPER BOUND THEOREM

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Abstract. Two new relatives of the Upper Bound Theorem are established. The first is a variant of the Strong Upper Bound Theorem for even-dimensional simplicial spheres. The second is an Upper Bound Theorem for odd-dimensional simplicial manifolds admitting a free involution.

1. Introduction

This note contains two simple extensions of the Upper Bound Theorem that appear to be overlooked in the literature so far. The first one is a variant of the Strong Upper Bound Theorem for even-dimensional simplicial spheres (that is, simplicial complexes whose geometric realization is a sphere) and the second one is the Upper Bound Theorem for odd-dimensional simplicial manifolds admitting a free involution. In this section we state the results deferring most of definitions until the next section.

The Upper Bound Theorem (abbreviated UBT), conjectured by Motzkin in 1957, asserts that among all d-dimensional polytopes on n vertices the cyclic polytope, $C_d(n)$, maximizes the number of i-dimensional faces, denoted f_i, for every $i = 1, 2, \ldots, d-1$. The UBT is known to hold for all Eulerian simplicial complexes with a sufficiently large number of vertices [8], all polytopes [10], all simplicial spheres [13], and several classes of simplicial manifolds [12] and odd-dimensional pseudomanifolds with isolated singularities [5].

An extension of the UBT (referred to as the Strong Upper Bound Theorem, abbreviated SUBT), was verified by Kalai for the boundary complexes of simplicial polytopes and their full-dimensional subcomplexes [6]. It asserts that for such a complex K, if $f_{d-1}(K) \geq f_{d-1}(C_d(n))$, where $d-1$ is the dimension of K, then $f_i(K) \geq f_i(C_d(n))$ for all i. Our first result is the following extension of the SUBT.

Theorem 1. *Let K be a $2k$-dimensional Gorenstein* simplicial complex (e.g., a simplicial sphere) such that $f_{2k}(K) \geq f_{2k}(C_{2k+1}(n))$. Then $f_i(K) \geq f_i(C_{2k+1}(n))$ for every $0 \leq i \leq 2k-1$.*

Our second result concerns centrally symmetric simplicial manifolds. We say that a simplicial complex K is *centrally symmetric* if there exists an involution $\phi : K \to K$, that fixes none of the faces of K. A centrally symmetric simplicial complex K is called *nearly k-neighborly* if every set of k of its vertices that does not contain two antipodal vertices (i.e., v and ϕv) forms a face in K. (This is analogous to the notion of k-neighborliness: a simplicial complex on n vertices is k-neighborly if every set of k of its vertices forms a face.) While every cyclic d-polytope is $\lfloor d/2 \rfloor$-neighborly (and many other constructions of $\lfloor d/2 \rfloor$-neighborly d-polytopes are known), the neighborliness of centrally symmetric polytopes is quite

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restricted (see [11], [4]). On the other hand, existence of nearly $\lfloor d/2 \rfloor$-neighborly $(d-1)$-dimensional centrally symmetric spheres on $2m$ vertices for all d and m is plausible, but still open. (See [9, Section 4.1] for a recent development in this area.)

Adin [1] and Stanley proved independently that the number of i-faces of a centrally symmetric $(d-1)$-sphere on $2m$ vertices is bounded above by the number of i-faces of a nearly $\lfloor d/2 \rfloor$-neighborly centrally symmetric $(d-1)$-sphere on $2m$ vertices, assuming such a sphere exists. Our next result – Theorem 3 – shows that the same assertion holds for all odd-dimensional centrally symmetric manifolds.

Our proofs use certain sets of linear combinations of face numbers of a simplicial complex, which were introduced in [2] and in [5], respectively. The proofs and the relevant background are provided in the following two sections.

2. The Strong Upper Bound Theorem

This section is devoted to the proof of Theorem 1. It will require the following definitions. For a $(d-1)$-dimensional simplicial complex K, its f-vector, denoted $f(K)$, is the vector $(f_{-1}, f_0, f_1, \ldots, f_{d-1})$ where f_i counts the number of i-dimensional faces. In particular, $f_{-1} = 1$ and f_0 is the number of vertices of K. The h-vector of K, denoted $h(K)$, is the vector $(h_0, h_1, \ldots, h_d)$ where

$$h_i(K) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}(K), \qquad i = 0, 1, \ldots, d. \tag{1}$$

(In particular, $h_0 = 1$ and $h_1 = f_0 - d$ for all $(d-1)$-dimensional complexes. For (the boundary of) the cyclic d-polytope $h_i(C_d(n)) = \binom{n-d-1+i}{i}$ for all $i \leq \lfloor d/2 \rfloor$.) Equivalently,

$$f_{j-1}(\Delta) = \sum_{i=0}^{j} \binom{d-i}{d-j} h_i(\Delta), \qquad j = 0, 1, \ldots, d. \tag{2}$$

We will also use the following linear combinations of h-numbers (introduced in [2]):

$$\widehat{h}_i(K) := \sum_{j=0}^{i} h_j, \qquad i = 0, 1, \ldots, d.$$

Thus for the d-dimensional cyclic polytope

$$\widehat{h}_i(C_d(n)) = \sum_{j=0}^{i} \binom{n-d-1+j}{j} = \binom{n-d+i}{i} \quad \text{for all } i = 0, 1, \ldots, \lfloor d/2 \rfloor. \tag{3}$$

Finally, we recall the definition of an M-sequence. Given two integers $l, i > 0$ write

$$l = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \ldots + \binom{n_j}{j}, \qquad \text{where } n_i > n_{i-1} > \cdots > n_j \geq j \geq 1.$$

Exactly one such expansion exists for every (l, i). Define

$$\partial^i(l) := \binom{n_i - 1}{i-1} + \binom{n_{i-1} - 1}{i-2} + \ldots + \binom{n_j - 1}{j-1}, \quad \partial^i(0) = 0.$$

A sequence of non-negative integers $(l_0, l_1, \ldots)$ is called an *M-sequence* if $l_0 = 1$ and $\partial^i(l_i) \leq l_{i-1}$ for $i \geq 2$. Stanley [16, Cor. II.3.2] showed that the h-vector of a Cohen-Macaulay complex is an M-sequence. Moreover, the g-theorem proved by Stanley (necessity) [14] and Billera – Lee (sufficiency) [3] asserts that $(h_0, h_1, \ldots, h_d)$ is the

h-vector of a d-dimensional simplicial polytope if and only if it is symmetric (that is, $h_i = h_{d-i}$ for all $0 \le i \le d$), unimodular, and $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is an M-sequence.

The equations $h_i = h_{d-i}$ for the h-numbers of a d-polytope are known as the Dehn-Sommerville relations. These relations hold for a much wider class of simplicial complexes, namely for all Eulerian complexes [7]. We recall that a simplicial complex K is *pure* if all maximal faces of K has the same dimension. A pure simplicial complex K is *Eulerian* if for every face of K (including the empty face) the Euler characteristic of its link is equal to the Euler characteristic of the sphere of the same dimension, that is, $\chi(\mathrm{lk}\, F) = 1 + (-1)^{\dim(\mathrm{lk}\, F)}$. (In particular, all simplicial spheres are Eulerian.) A complex that is both Eulerian and Cohen-Macaulay is called a *Gorenstein** complex.

We are now ready to prove the SUBT.

Proof of Theorem 1. Let K be a $2k$-dimensional Cohen-Macaulay complex. Then by Stanley's result [16, Corollary II.3.2] the sequence $(h_0(K), h_1(K), \ldots, h_k(K))$ is an M-sequence. Hence it follows from (the sufficiency part of) the g-theorem [3] that there exists a simplicial polytope P such that

$$h(P) = \left(\widehat{h}_0(K), \widehat{h}_1(K), \ldots, \widehat{h}_k(K), \widehat{h}_k(K), \ldots, \widehat{h}_1(K), \widehat{h}_0(K)\right).$$

Since the boundary complex of a simplicial polytope is a Cohen-Macaulay complex, the aforementioned result of Stanley implies the following fact.

Lemma 1. *If K is a $2k$-dimensional Cohen-Macaulay complex, then the vector $\left(\widehat{h}_0(K), \widehat{h}_1(K), \ldots, \widehat{h}_k(K)\right)$ is an M-sequence.*

Theorem 1 is then a consequence of the following more general result.

Theorem 2. *Let K be a $2k$-dimensional Eulerian simplicial complex such that the vector $\left(\widehat{h}_0(K), \widehat{h}_1(K), \ldots, \widehat{h}_k(K)\right)$ is an M-sequence. If $f_{2k}(K) \ge f_{2k}(C_{2k+1}(n))$, then $f_i(K) \ge f_i(C_{2k+1}(n))$ for every $i = 0, 1, \ldots, 2k-1$.*

Proof: Since K is an Eulerian complex, $h_i(K) = h_{2k+1-i}(K)$, $0 \le i \le 2k+1$. Hence the f-numbers of K can be written as linear combinations of $h_0, h_1, \ldots, h_k$, and therefore also as linear combinations of $\widehat{h}_0(K), \widehat{h}_1(K), \ldots, \widehat{h}_k(K)$. Moreover, it follows easily from Eq. (2) (see [2]) that the latter combinations are non-negative. Thus to prove the theorem, it suffices to show that

$$\widehat{h}_i(K) \ge \widehat{h}_i(C_{2k+1}(n)) \qquad \text{for every } i = 1, 2, \ldots, k. \tag{4}$$

We verify (4) by induction on i. For $i = k$ it follows from the condition $f_{2k}(K) \ge f_{2k}(C_{2k+1}(n))$ and Eq. (2), which in the case of a $2k$-dimensional Eulerian simplicial complex Δ and $j = 2k+1$ can be rewritten as

$$f_{2k}(\Delta) = \sum_{j=0}^{2k+1} h_j(\Delta) = 2\sum_{j=0}^{k} h_j(\Delta) = 2\widehat{h}_k(\Delta).$$

Also, if (4) holds for a certain $2 \le i \le k$, then, since $(\widehat{h}_0(K), \widehat{h}_1(K), \ldots, \widehat{h}_k(K))$ is an M-sequence, we obtain

$$\widehat{h}_{i-1}(K) \ge \partial^i\left(\widehat{h}_i(K)\right) \ge \partial^i\left(\widehat{h}_i(C_{2k+1}(n))\right)$$
$$= \partial^i\left[\tbinom{n-2k-1+i}{i}\right] = \tbinom{n-2k-2+i}{i-1} = \widehat{h}_{i-1}(C_{2k+1}(n)),$$

and the result follows. □

3. Centrally symmetric manifolds

In this section we extend Adin-Stanley's UBT for centrally symmetric spheres to the class of odd-dimensional centrally symmetric manifolds. If K is a centrally symmetric complex on $2m$ vertices, then K is a subcomplex of the boundary complex of the m-dimensional cross-polytope C_m^Δ. In fact we show the following.

Theorem 3. *Let K be a $(2k-1)$-dimensional simplicial manifold. If K is a subcomplex of C_m^Δ (e.g., K is a centrally symmetric manifold on $2m$ vertices) then $f_i(K)$ (for all $0 \leq i \leq 2k-1$) is bounded above by the number of i-faces of a nearly k-neighborly centrally symmetric $(2k-1)$-sphere on $2m$ vertices, assuming such a sphere exists.*

Note that all such spheres have exactly the same f-vector (equivalently, h-vector). Indeed, let S be a centrally symmetric $(2k-1)$-dimensional sphere on $2m$ vertices. If S is nearly k-neighborly, then its $(k-1)$-dimensional skeleton, $\mathrm{Skel}_{k-1}(S)$, coincides with that of C_m^Δ, and so $f_{i-1}(S) = f_{i-1}(C_m^\Delta)$ for all $i \leq k$. Hence, as follows from Eq. (1) and the Dehn-Sommerville relations,

$$h_j(S) = h_{2k-j}(S) = h_j(\mathrm{Skel}_{2k-1}(C_m^\Delta)) \quad \text{for all } 0 \leq j \leq k.$$

In the proof of Theorem 3 we utilize the notion of a *short simplicial h-vector*, $\widetilde{h}(K)$, introduced in [5]. For a pure $(d-1)$-dimensional complex K on the vertex set V define

$$\widetilde{h}(K) = (\widetilde{h}_0, \widetilde{h}_1, \ldots \widetilde{h}_{d-1}), \text{ where } \quad \widetilde{h}_j(K) = \sum_{v \in V} h_j(\mathrm{lk}\, v).$$

We summarize several properties of $\widetilde{h}$ in the following lemma. Recall that a complex K is *Buchsbaum* if it is pure and the link of every vertex v of K, $\mathrm{lk}\, v$, is Cohen-Macaulay. A complex K is semi-Eulerian if it is pure and for every vertex v of K $\mathrm{lk}\, v$ is Eulerian. Thus simplicial manifolds are both Buchsbaum and semi-Eulerian.

Lemma 2. *Let K be a $(d-1)$-dimensional simplicial complex.*

(1) *If K is semi-Eulerian, then $f_i(K) = \sum_{j=0}^{\lfloor (d-1)/2 \rfloor} a_j^i \widetilde{h}(K)$ for all $0 \leq i \leq d-1$, where the coefficients a_j^i are independent of K and are nonnegative.*
(2) *If K' is a full-dimensional subcomplex of K, and if both K and K' are Buchsbaum complexes, then $\widetilde{h}_j(K') \leq \widetilde{h}_j(K)$ for all $0 \leq j \leq d-1$.*

Proof:
1. If K is semi-Eulerian, then we infer from the Dehn-Sommerville relations for the links of vertices that $\widetilde{h}(K)$ is symmetric: $\widetilde{h}_j(K) = \widetilde{h}_{d-1-j}(K)$ for all $0 \leq j \leq d-1$, and the result follows from [5, Lemma 1] asserting that the f-numbers of K are non-negative linear combinations of $\widetilde{h}_j(K)$, $0 \leq j \leq d-1$.
2. If v is a vertex of both K' and K, then $\mathrm{lk}_{K'} v$ – the link of v in K' – is a full-dimensional subcomplex of $\mathrm{lk}_K v$. Moreover, both of these complexes are Cohen-Macaulay. Hence $h_j(\mathrm{lk}_{K'} v) \leq h_j(\mathrm{lk}_K v)$ for all $0 \leq j \leq d-1$ as follows from the result due to Adin [1] and Stanley [15]. If v is the vertex of K but not of K', then $0 \leq h_j(\mathrm{lk}_K v)$ for all $0 \leq j \leq d-1$ since $\mathrm{lk}_K v$ is Cohen-Macaulay. Summing up these inequalities over all vertices of K yields the result. □

We are now ready to verify Theorem 3.

Proof of Theorem 3. Let K be a simplicial manifold satisfying the conditions of the theorem. Then K is a full-dimensional subcomplex of $\Gamma := \mathrm{Skel}_{2k-1}(C_m^\Delta)$, and hence by part 2 of of Lemma 3

$$\widetilde{h}_j(K) \leq \widetilde{h}_j(\Gamma) \quad \text{for all } 0 \leq j \leq 2k-1.$$

On the other hand, if S is a nearly k-neighborly centrally symmetric $(2k-1)$-sphere on $2m$ vertices, then $\mathrm{Skel}_{k-1}(S) = \mathrm{Skel}_{k-1}(C_m^\Delta) = \mathrm{Skel}_{k-1}(\Gamma)$, and so

$$\widetilde{h}_j(\Gamma) = \widetilde{h}_j(S) \quad \text{for all } 0 \leq j \leq k-1.$$

Thus $\widetilde{h}_j(K) \leq \widetilde{h}_j(S)$ for $0 \leq j \leq k-1$, which together with part 1 of Lemma 3 implies $f_i(K) \leq f_i(S)$ for all $0 \leq i \leq 2k-1$. □

We remark that the same argument using the simplex instead of the cross-polytope provides yet another proof of the UBT for ordinary odd-dimensional simplicial manifolds.

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