SYMMETRIC ITERATED BETTI NUMBERS

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Abstract. We define a set of invariants of a homogeneous ideal I in a polynomial ring called the symmetric iterated Betti numbers of I. We prove that for I_Γ, the Stanley-Reisner ideal of a simplicial complex Γ, these numbers are the symmetric counterparts of the exterior iterated Betti numbers of Γ introduced by Duval and Rose, and that the extremal Betti numbers of I_Γ are precisely the extremal (symmetric or exterior) iterated Betti numbers of Γ. We show that the symmetric iterated Betti numbers of an ideal I coincide with those of a particular reverse lexicographic generic initial ideal $\mathrm{Gin}(I)$ of I, and interpret these invariants in terms of the associated primes and standard pairs of $\mathrm{Gin}(I)$. We close with results and conjectures about the relationship between symmetric and exterior iterated Betti numbers of a simplicial complex.

1. Introduction

The goal of this paper is to define and study a set of invariants of a homogeneous ideal in a polynomial ring, called the *symmetric iterated Betti numbers* of the ideal. For a simplicial complex Γ, the symmetric iterated Betti numbers of the Stanley-Reisner ideal of Γ (also referred to as the symmetric iterated Betti numbers of Γ) are preserved by *symmetric algebraic shifting.*

We discuss two versions of algebraic shifting (both introduced by Kalai [6], [16]) which given a simplicial complex Γ with vertex set $[n] := \{1, 2, \dots, n\}$ provide new simplicial complexes with the same vertex set. We denote these versions by $\Delta(\Gamma)$ for the *symmetric shifting* of Γ (see Definition 2.1) and by $\Delta^e(\Gamma)$ for the *exterior shifting* of Γ (see Definition 7.1). We write $\Delta^{(e)}(\Gamma)$ when we refer to both complexes. For both of these operations it is known that:

- (P1) $\Delta^{(e)}(\Gamma)$ are *shifted*, that is, for every $F \in \Delta^{(e)}(\Gamma)$, if $j < i \in F$, then $(F \setminus \{i\}) \cup \{j\} \in \Delta^{(e)}(\Gamma)$.
- (P2) If Γ is shifted, then $\Delta^{(e)}(\Gamma) = \Gamma$.
- (P3) Γ and $\Delta^{(e)}(\Gamma)$ have the same f-vector, that is, they have the same number of i-dimensional faces for every i.
- (P4) If Γ' is a subcomplex of Γ, then $\Delta^{(e)}(\Gamma') \subset \Delta^{(e)}(\Gamma)$.

Both versions were studied extensively from the algebraic point of view in a series of recent papers by Aramova, Herzog, Hibi and others (surveyed in [13]).

Consider the polynomial ring $S = \mathbf{k}[y_1, \dots, y_n]$ where $\mathbf{k}$ is a field of characteristic zero. Let $\mathbb{N}$ denote the set of non-negative integers. If $A \subseteq [n]$ then write $y^A = \prod_{a\in A} y_a$. Denote by $\mathbb{N}^{[n]}$ the monomials of S by identifying a function $f : [n] \to \mathbb{N}$ in $\mathbb{N}^{[n]}$ with the monomial $\prod_{i\in[n]} y_i^{f(i)}$ and consider $\mathbb{N}^{[n]}$ as a multiplicative monoid. Thus $\{0,1\}^{[n]} = y^{2^{[n]}}$ is the set of squarefree monomials. If $\Gamma \subseteq 2^{[n]}$

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is a simplicial complex then the *Stanley-Reisner* ideal of Γ [20, Def. II.1.1] is the squarefree monomial ideal

$$I_\Gamma := \langle y^{2^{[n]}-\Gamma} \rangle \subset S.$$

The (bi-graded) *Betti numbers* of a homogeneous ideal $I \subset S$ are the invariants $\beta_{i,j}(I)$ that appear in the minimal free resolution of I as an S-module.

$$\to \ldots \bigoplus_j S(-j)^{\beta_{i,j}(I)} \to \ldots \to \bigoplus_j S(-j)^{\beta_{1,j}(I)} \to \bigoplus_j S(-j)^{\beta_{0,j}(I)} \to I \to 0$$

Here $S(-j)$ denotes S with grading shifted by j. We say that $\beta_{i,i+j}(I)$, is *extremal* if $0 \neq \beta_{i,i+j}(I) = \sum_{i' \geq i, j' \geq j} \beta_{i',i'+j'}(I)$. (This is equivalent to having $0 \neq \beta_{i,i+j}(I)$ and $0 = \beta_{i',i'+j'}(I)$ for every $i' \geq i$ and $j' \geq j$, $(i',j') \neq (i,j)$.)

Since $\Delta^{(e)}(\Gamma)$ are shifted complexes, their combinatorial structures are simpler than that of Γ. Nonetheless, operations $\Delta^{(e)}$ preserve many combinatorial and topological properties.

1. $\Delta^{(e)}$ preserve topological Betti numbers (see [6, Thm. 3.1], [1, Prop. 8.3] for exterior shifting and [13, Cor. 8.25] for symmetric shifting). Moreover, exterior algebraic shifting preserves the *exterior iterated Betti numbers* of a simplicial complex. (There are two versions of exterior iterated Betti numbers — one due to Kalai [17, Cor. 3.4] and another due to Duval and Rose [9]. Both sets of numbers are preserved under exterior shifting.)
2. $\Delta^{(e)}$ preserve Cohen-Macaulayness: a simplicial complex Γ is Cohen-Macaulay if and only if $\Delta^{(e)}(\Gamma)$ is Cohen-Macaulay, which happens if and only if $\Delta^{(e)}(\Gamma)$ is pure (see [17, Thm. 5.3], [1, Prop. 8.4] for exterior shifting and [16, Thm. 6.4] for symmetric shifting).
3. $\Delta^{(e)}$ preserve extremal Betti numbers: $\beta_{i,i+j}(I_\Gamma)$ is an extremal Betti number of I_Γ if and only if $\beta_{i,i+j}(I_{\Delta^{(e)}(\Gamma)})$ is extremal for $I_{\Delta^{(e)}(\Gamma)}$, in which case $\beta_{i,i+j}(I_\Gamma) = \beta_{i,i+j}(I_{\Delta^{(e)}(\Gamma)})$ (see [4] for symmetric shifting and [1, Thm. 9.7] for both versions.)

Property 3 is a far-reaching generalization of Property 2, while Property 1 played a crucial role in Kalai's proof of Property 2 for exterior shifting. This suggests that there might be a connection between the iterated Betti numbers of a simplicial complex Γ on the one hand and the extremal Betti numbers of the ideal I_Γ on the other. This is one of the connections we establish in this paper.

Consider the action of $GL(S_1)$ on S and choose $u \in GL(S_1)$ to be generic. Denote by $\mathbf{m} = \langle S_1 \rangle$ the irrelevant ideal of S. If I is a homogeneous ideal in S then define the S-modules $M_i(I)$ as follows: $M_0(I) := S/uI$ and $M_i(I) := M_{i-1}(I)/\left(y_i M_{i-1}(I) + H^0(M_{i-1}(I))\right)$ for $1 \leq i \leq n$, where $H^0(-)$ denotes the 0-th local cohomology with respect to the irrelevant ideal $\mathbf{m}$. We now come to the central definition of this paper.

Definition 1.1. The symmetric iterated Betti numbers of a homogeneous ideal I in S are

$$b_{i,r}(I) := \dim H^0(M_i(I))_r \quad \text{for } 0 \leq i, r \leq n,$$

where $H^0(-)_r$ stands for the r-th component of the 0-th local cohomology.

If Γ is a simplicial complex with vertex set $[n]$, define the symmetric iterated Betti numbers of Γ to be $b_{i,r}(\Gamma) := b_{i,r}(I_\Gamma)$, $0 \leq i, r \leq n$.

Our first result gives a combinatorial interpretation of the symmetric iterated Betti numbers of a simplicial complex Γ and shows that they are invariant under symmetric algebraic shifting. Let $\max(\Gamma)$ denote the set of facets (maximal faces) of Γ. Write $\dim(\Gamma) = \max\{|F| - 1 : F \in \Gamma\}$.

Theorem 4.1. *Let Γ be a simplicial complex. Then*

$$b_{i,r}(\Gamma) = \begin{cases} |\{F \in \max(\Delta(\Gamma)) : |F| = i,\ [i-r] \subseteq F,\ i-r+1 \notin F\}| & \text{if } r \leq i \\ 0 & \text{otherwise.} \end{cases}$$

In particular, since $\Delta(\Delta(\Gamma)) = \Delta(\Gamma)$, it follows that the symmetric iterated Betti numbers of Γ are invariant under symmetric shifting.

Theorem 4.1 implies that $b_{i,r}(\Gamma) = 0$ unless $0 \leq r \leq i \leq \dim(\Gamma)+1$. The exterior iterated Betti numbers of Γ, $b^e_{i,r}(\Gamma)$, defined by Duval and Rose have precisely the same combinatorial formula (up to a slight change in indexes), except that in their definition, one replaces $\Delta(\Gamma)$ by $\Delta^e(\Gamma)$ [9, Thm. 4.1]. In fact, the operation of "peeling off" the i-th variable at step i we are using to define the symmetric iterated Betti numbers is analogous to the "deconing" of the shifted complex used in the Duval-Rose definition.

The extremal Betti numbers of an ideal $I = I_\Gamma$ are the extremal iterated Betti numbers (symmetric or exterior) of the simplicial complex Γ in the following sense.

Theorem 5.3 and 7.4. *Let Γ be a simplicial complex. The extremal Betti numbers of I_Γ form a subset of the symmetric as well as of the exterior iterated Betti numbers of Γ. More precisely, $\beta_{j-1,i+j}(I_\Gamma)$ is an extremal Betti number of I_Γ if and only if*

$$b^{(e)}_{n-j',i'}(\Gamma) = 0 \quad \forall (i',j') \neq (i,j),\ i' \geq i,\ j' \geq j,\ \text{and}\ b^{(e)}_{n-j,i}(\Gamma) \neq 0.$$

In such a case $\beta_{j-1,i+j}(I_\Gamma) = b_{n-j,i}(\Gamma) = b^e_{n-j,i}(\Gamma)$.

Let $\mathrm{Gin}\,(I)$ denote the reverse lexicographic *generic initial ideal* of a homogeneous ideal I in S with variables ordered as $y_n \succ y_{n-1} \succ \cdots \succ y_1$. It follows from [4, Cor. 1.7] that the symmetric iterated Betti numbers of I coincide with those of $\mathrm{Gin}\,(I)$. We provide an alternate proof of this fact in Section 4 (see Corollary 4.6). Our next result — Theorem 6.6— interprets the symmetric iterated Betti numbers $b_{i,r}(I)$ in terms of the associated primes of $\mathrm{Gin}\,(I)$.

This paper is organized as follows. In Section 2 we recall the basics of symmetric shifting. Section 3 defines and interprets certain monomial sets that are at the root of all our proofs. In Sections 4–7 we prove the theorems stated above. We conclude in Section 7 with some results and conjectures on the relationship between the exterior and symmetric iterated Betti numbers of a simplicial complex.

2. ALGEBRAIC SHIFTING

In this section we recall the basics of symmetric algebraic shifting. (The description of exterior shifting is deferred to Section 7.) For further details on symmetric and exterior shifting see the survey articles by Herzog [13] and Kalai [18].

Let $\mathbb{N}^\sigma$ denote the set of all finite degree monomials in the variables y_i with $i \in \sigma$ and $\mathbb{N}^\sigma_r$ denote the set of elements of degree r in $\mathbb{N}^\sigma$. In particular, if $[n] = [1,n] = \{1,\ldots,n\}$ then $\mathbb{N}^{[n]}$ is the set of all monomials in S and $\{0,1\}^\sigma$ is the set of all square free finite degree monomials in $\mathbb{N}^\sigma$. In this paper we fix the reverse lexicographic order $\succ$ on $\mathbb{N}^\mathbb{Z}$ with $y_i \succ y_{i-1}$ for all $i \in \mathbb{Z}$ extending the partial ordering by degree. (Thus, for example, $y_n^2 \succ y_{n-1}y_n \succ y_{n-1}^2 \succ y_{n-2}y_n \succ y_{n-2}y_{n-1} \succ \cdots$.) We also define the square free map $\Phi : \mathbb{N}^\mathbb{Z} \to \{0,1\}^\mathbb{Z}$ by

$$\Phi(y_{i_0}y_{i_1}\cdots y_{i_k}) := y_{i_0-k}y_{i_1-(k-1)}\cdots y_{i_j-(k-j)}\cdots y_{i_k},\ \text{where}\ i_0 \leq i_1 \leq \cdots \leq i_k.$$

In other words, Φ is the unique degree and order preserving bijection for which $\Phi(y_0^n) = \prod_{-n<i\leq 0} y_i$.

For each homogeneous ideal $I \subset S$ there exists a Zariski open set $U(I) \subset GL(S_1)$ such that the ideal $\mathrm{In}_{\succ}(uI)$, (the *initial ideal* of uI with respect to the monomial order $\succ$ on S), is independent of the choice of $u \in U(I)$. The ideal $\mathrm{In}_{\succ}(uI)$ is called the *generic initial ideal* of I with respect to $\succ$ and is denoted by $\mathrm{Gin}\,(I) = \mathrm{Gin}_{\succ}(I)$ (see [10, Chapter 15]). If I is a homogeneous ideal in S then one way to explicitly and uniformly construct an element $\alpha \in U(I)$ is to consider the extension $\mathbf{K} = \mathbf{k}(\{\alpha_{i,j}\}_{i,j\in[n]})/\mathbf{k}$ and then for any ideal I in S the element

$$\alpha : S_{\mathbf{K}} = S \otimes_{\mathbf{k}} \mathbf{K} \to S_{\mathbf{K}} \quad \text{given by} \quad \alpha y_i = \sum_{j=1}^{n} \alpha_{i,j} y_j$$

is generic for $\mathbf{K}I$ as an ideal of $S_{\mathbf{K}}$.

For a homogeneous ideal I in S and a generic linear map $u \in U(I)$ define

$$B(I) = \{m \in \mathbb{N}^{[n]} \ : \ m \text{ is not in the linear span of } \{n|m \succ n\} \cup uI\}.$$

Note that $B(I)$ is a basis of the vector space $M_0(I) = S/uI$ and hence $B(I) = \mathbb{N}^{[n]} - \mathrm{Gin}\,(I)$.

Definition 2.1. The symmetric algebraic shifting of a simplicial complex $\Gamma \subseteq 2^{[n]}$ is $\Delta(\Gamma)$ where $y^{\Delta(\Gamma)} = \Phi(B(I_\Gamma)) \cap \mathbb{N}^{[1,\infty]} \subseteq \{0,1\}^{[n]}$.

Note that this means that $I_{\Delta(\Gamma)} = \langle \Phi(\mathbb{N}^{[n]} - B(I_\Gamma))\rangle$.

The fact that $\Delta(\Gamma)$ is a simplicial complex satisfying conditions (P1)–(P4) was proved in [16, Thm. 6.4], [3] by using certain properties of $B(I)$. We list some of them below.

(B1) $B(I)$ is a basis of S/uI, as well as of $S/\mathrm{Gin}\,(I)$.
(B2) $B(I)$ is an order ideal — if $m \in B(I)$ and $m'|m$, then $m' \in B(I)$.
(B3) $B(I)$ is shifted — if $j < i$ and $y_i m \in B(I)$ then $y_j m \in B(I)$.

(B1) was discussed above while (B2) follows from the fact that $\mathrm{Gin}\,(I)$ is an ideal. (B3) is a consequence of the fact that generic initial ideals are Borel fixed [10, Theorem 15.20]. In characteristic 0, this is equivalent to $\mathrm{Gin}\,(I)$ being *strongly stable* [10, Theorem 15.23], which means that if $j < i$ and $y_j m \in \mathrm{Gin}\,(I)$ then $y_i m \in \mathrm{Gin}\,(I)$.

In the case when $I = I_\Gamma$, $B(I_\Gamma)$ has another fundamental property:

(B4) If $m \in B(I_\Gamma) \cap \mathbb{N}_r^{[k,n]}$ and $r \geq k$ then $m\mathbb{N}^{\{k\}} \subseteq B(I_\Gamma)$ as well.

This is due to Kalai [16, Lemma 6.3] and implies that $y_1, \ldots, y_n$ is an *almost regular* $M_0(I_\Gamma)$-sequence (a notion introduced by Aramova and Herzog [1]; it played a crucial role in their proof that extremal Betti numbers are preserved by algebraic shifting).

3. Special monomial subsets

In this section we identify and interpret certain subsets of monomials in the basis $B(I)$ of $M_0(I)$ that are at the root of all our proofs.

Definition 3.1. Let I be a homogeneous ideal in S. For $i \in [0, n]$ define

$$A_i := \left\{m \in \mathbb{N}^{[i+1,n]} \ : \ m\mathbb{N}^{\{i\}} \subseteq B(I),\ m\mathbb{N}^{\{i+1\}} \not\subseteq B(I)\right\}; \quad A_{i,r}(I) := A_i(I) \cap \mathbb{N}_r^{\mathbb{N}}.$$

Several remarks are in order. Since $B(I)$ is shifted (B3), $m\mathbb{N}^{\{i\}} \subseteq B(I)$ iff $m\mathbb{N}^{[i]} \subseteq B(I)$. Since $B(I_\Gamma)$ satisfies (B4),

$$(1) \qquad A_{i,r}(I_\Gamma) = \emptyset \qquad \text{if } r > i \qquad \text{and hence} \qquad A_i(I_\Gamma) = \cup_{r=0}^{i} A_{i,r}(I_\Gamma).$$

Also if $m \in \mathbb{N}_r^{[r,n]}$ then $m \in B(I_\Gamma)$ iff $m\mathbb{N}^{[r]} \subseteq B(I_\Gamma)$. Hence

$$(2) \qquad A_{i,r}(I_\Gamma) = \left\{ m \in \mathbb{N}_r^{[i+1,n]} : y_i^{i-r} \cdot m \in B(I_\Gamma),\ y_{i+1}^{i-r+1} \cdot m \notin B(I_\Gamma) \right\}.$$

In [21], Sturmfels, Trung and Vogel introduced a decomposition of the standard monomials of an arbitrary monomial ideal M, called its *standard pair decomposition*, in order to study the multiplicities of associated primes and degrees of M. We study these quantities for the monomial ideals I_Γ, $I_{\Delta(\Gamma)}$, and $\mathrm{Gin}\,(I)$. In Section 6 we show their relationship to the symmetric iterated Betti numbers of Γ and I, respectively. These results rely on the fact that the sets of monomials $A_i(I)$ defined above index the *standard pairs* of $\mathrm{Gin}\,(I)$. For a monomial $m \in \mathbb{N}^{\mathbb{Z}}$, let $\mathrm{supp}\,(m) := \{i : y_i | m\} \subset \mathbb{Z}$ be called the *support* of m. Thus $\mathrm{supp} : \{0,1\}^\sigma \to 2^\sigma$ is a bijection.

Definition 3.2. [21] Let $M = \langle M \cap \mathbb{N}^{[n]} \rangle \subseteq S$ be a monomial ideal. A **standard monomial** of M is an element of $\mathbb{N}^{[n]} - M$. An **admissible pair** of M is a subset $m\mathbb{N}^\sigma \subseteq \mathbb{N}^{[n]} - M$ with $m \in \mathbb{N}^{[n]-\sigma}$ or equivalently if we take $\mathbb{Z}^\sigma$ to be Laurent monomials then an admissible pair is a subset $m\mathbb{Z}^\sigma \cap \mathbb{N}^{[n]}$ with $m\mathbb{Z}^\sigma \cap M = \emptyset$. A **standard pair** of M is a(n inclusion) maximal admissible pair.

Lemma 3.3. *If $I \subseteq S$ is an ideal then the standard pairs of* $\mathrm{Gin}\,(I)$ *are* $\{a\mathbb{N}^{[i]} : a \in A_i(I)\}$. *(Here* $[0] = \emptyset$*.)*

Proof: We first argue that all standard pairs of $\mathrm{Gin}\,(I)$ are of the form $a\mathbb{N}^{[i]}$ for some $i \in [0,n]$. Suppose $m\mathbb{N}^\sigma$ is an admissible pair of $\mathrm{Gin}\,(I)$ with $k = \max(\sigma)$. Since $B(I)$ is shifted (B3) and $m\mathbb{N}^{\{k\}} \subseteq B(I)$ we obtain that $m\mathbb{N}^{[k]} \subseteq B(I)$ and hence $m\mathbb{N}^\sigma \subseteq m\mathbb{Z}^{[k]} \cap \mathbb{N}^{[n]} \subseteq B(I)$. If $m\mathbb{N}^\sigma$ is standard (maximal) this implies that $m\mathbb{N}^\sigma = m\mathbb{Z}^{[k]} \cap \mathbb{N}^{[n]}$ and thus that $\sigma = [k]$.

If $m\mathbb{N}^{[i]} \subseteq B(I)$ is standard then by the above argument $m\mathbb{N}^{\{i+1\}} \not\subseteq B(I)$ so $m \in A_i(I)$.

Finally, if $m \in A_i(I)$, then $m\mathbb{N}^{\{i\}} \subseteq B(I)$ and hence $m\mathbb{N}^{[i]} \subseteq B(I)$ is admissible. If $m\mathbb{N}^{[i]} \subseteq B(I)$ is not standard then $m\mathbb{N}^{[i]} \subset m'\mathbb{N}^{[i']} \subseteq B(I)$ so $i' > i$ and $m\mathbb{N}^{\{i+1\}} \subseteq B(I)$ contradicting the choice of m. $\square$

We remark that Lemma 3.3 and eqn. (1) imply that if $m\mathbb{N}^{[i]}$ is a standard pair of $\mathrm{Gin}\,(I_\Gamma)$ then the degree of m is at most i.

The standard pairs of monomial ideals of moderate size can be computed using the computer algebra package *Macaulay 2* [12] (see the chapter *Monomial Ideals* in [11] for details). This gives a method for computing the sets $A_i(I)$ for small examples — see Example 3.7 below.

In the case when $I = I_\Gamma$ there is another interpretation of the monomials in $A_i(I_\Gamma)$ that relates them to the shifted complex $\Delta(\Gamma)$, and is useful for the proofs of Theorems 4.1 and 6.6.

Lemma 3.4. *There is a bijection between the sets*

$$A_{i,r}(I_\Gamma) \ \ \textit{and} \ \ \{F \in \max(\Delta(\Gamma)) : |F| = i,\ [i-r] \subseteq F,\ i-r+1 \notin F\}$$

given by Φ *with* $A_{i,r}(I_\Gamma) \ni m \mapsto [i-r] \cup \mathrm{supp}\,(\Phi(m)) = \mathrm{supp}\,(\Phi(my_i^{i-r}))$.

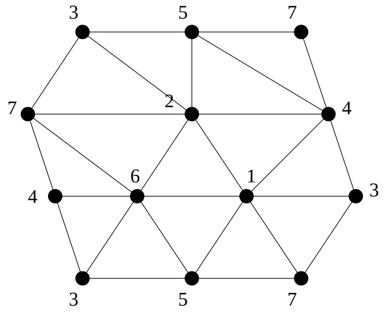

FIGURE 1. The simplicial complex Γ in Example 3.7. Here parallel boundary regions are identified.

Proof: For $r > i$ the assertion follows from the fact that both sets are empty (see (1)). To deal with the case $r \le i$, note that

$$\begin{aligned}
&\{\Phi(m) \,:\, m \in A_{i,r}(I_\Gamma)\} \overset{\text{by (2)}}{=} \\
&\left\{ \begin{array}{ll} \Phi(m) \;: & m \in B(I_\Gamma) \cap \mathbb{N}_r^{[i+1,n]}, \\ & y_i^{i-r} \cdot m \in B(I_\Gamma),\; y_{i+1}^{i-r+1} \cdot m \notin B(I_\Gamma) \end{array} \right\} \overset{\text{Def.2.1}}{=} \\
(3)\quad &\left\{ \begin{array}{ll} G \in \Delta(\Gamma) \;: & |G| = r,\, G \cap [i-r+1] = \emptyset, \\ & G \cup [i-r] \in \Delta(\Gamma),\; G \cup [i-r+1] \notin \Delta(\Gamma) \end{array} \right\} = \\
&\{F \setminus [i-r] \,:\, F \in \max(\Delta(\Gamma)),\; |F| = i,\; [i-r] \subseteq F, i-r+1 \notin F\},
\end{aligned}$$

where in the last equality we used the fact that $\Delta(\Gamma)$ is shifted. Indeed, if $G \cup [i-r+1] \notin \Delta(\Gamma)$, then $G \cup [i-r] \cup \{j\} \notin \Delta(\Gamma)$ for every $j > i-r+1$, $j \notin G$, implying that $G \cup [i-r]$ is a facet of $\Delta(\Gamma)$ for every element G of the set (3). ☐

Corollary 3.5. *The standard pairs of* $\mathrm{Gin}\,(I_\Gamma)$ *are in bijection with the facets of* $\Delta(\Gamma)$*:* $m\mathbb{N}^{[i]}$ *is a standard pair of* $\mathrm{Gin}\,(I_\Gamma)$ *if and only if* $[i-r] \cup \mathrm{supp}\,(\Phi(m))$ *is a facet of* $\Delta(\Gamma)$ *of size* i*.*

In Section 4 we verify that $b_{i,r}(I) = |A_{i,r}(I)|$ for all $i, r \in [0, n]$, which via Lemma 3.4 proves Theorem 4.1. (Hence, in particular, it follows from Theorem 4.1 that $A_i(I_\Gamma) = \emptyset$ for all $i > \dim(\Gamma) + 1$.) Thus the sets $A_{i,r}(I_\Gamma)$ and their cardinalities $b_{i,r}(\Gamma)$ carry important information about Γ, and we record them in the following *triangles*.

Definition 3.6. The b-triangle and monomial b-triangle of a simplicial complex Γ are the lower triangular matrices whose respective (i, r)-th entries are $b_{i,r}(\Gamma)$ and $A_{i,r}(I_\Gamma)$ for $0 \le i \le r \le \dim(\Gamma) + 1$.

Example 3.7. Let Γ be the simplicial complex whose facets are

$$\begin{aligned}\max(\Gamma) = \{&\{1,2,4\}, \{1,2,6\}, \{1,3,4\}, \{1,3,7\}, \{1,5,6\}, \{1,5,7\}, \{2,3,5\}, \\ &\{2,3,7\}, \{2,4,5\}, \{2,6,7\}, \{3,4,6\}, \{3,5,6\}, \{4,5,7\}, \{4,6,7\}\}.\end{aligned}$$

Then it can be checked from Figure 1 that the Stanley-Reisner ideal of Γ in the ring $S := \mathbf{k}[a, b, c, d, e, f, g]$ is:

$$\begin{aligned}I_\Gamma = \langle &efg, cfg, afg, ceg, beg, cdg, bdg, adg, abg, def, bef, bdf, adf, bcf, \\ &acf, cde, ade, ace, abe, bcd, abc\rangle.\end{aligned}$$

Using *Macaulay 2* one can compute that under the reverse lexicographic order $\succ$ with $g \succ f \succ \cdots \succ b \succ a$,

$$\begin{aligned} \mathrm{Gin}\,(I_\Gamma) &= \langle gf^2, f^3, f^2e, g^2f, gfe, fe^2, gfd, f^2d, fed, g^2e, ge^2, e^3, \\ &\quad ged, e^2d, fd^2, g^3, g^2d, gd^2, ed^2, g^2c, gfc, d^4\rangle. \end{aligned}$$

Applying the map Φ to the generators of $\mathrm{Gin}\,(I_\Gamma)$ we get

$$\begin{aligned} I_{\Delta(\Gamma)} &= \langle gea, gfa, ecb, fcb, gcb, edb, fdb, gdb, feb, geb, gfb, edc, \\ &\quad fdc, gdc, fec, gec, gfc, fed, ged, gfd, gfe, dcba\rangle, \end{aligned}$$

which shows that the shifted complex $\Delta(\Gamma)$ has facets:

$$\begin{aligned} \max(\Delta(\Gamma)) &= \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,2,7\}, \\ &\quad \{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,3,7\},\{1,4,5\}, \\ &\quad \{1,4,6\},\{1,4,7\},\{1,5,6\},\{2,3,4\},\{5,7\},\{6,7\}\}. \end{aligned}$$

The one skeleton of $\Delta(\Gamma)$ is (like that of Γ) the complete graph on its 7 vertices. The triangles are obtained by coning 1 with all edges involving the vertices $2,3,4,5,6$ and 7 except for $\{5,7\}$ and $\{6,7\}$ and adding the triangle $\{2,3,4\}$. Thus all the triangles and the edges $\{5,7\}$ and $\{6,7\}$ are facets.

We compute the b-triangle and the monomial b-triangle of Γ by first computing the standard pairs of $\mathrm{Gin}\,(I_\Gamma)$ using Macaulay 2.

b-triangle of Γ					monomial b-triangle of Γ				
	0	1	2	3		0	1	2	3
0	0				0	$\emptyset$			
1	0	0			1	$\emptyset$	$\emptyset$		
2	0	0	2		2	$\emptyset$	$\emptyset$	$\{g^2, gf\}$	
3	1	4	8	1	3	$\{1\}$	$\{g,f,e,d\}$	$\{ge, gd, f^2, fe, fd, e^2, ed, d^2\}$	$\{d^3\}$

The standard pairs of I_Γ, $\mathrm{Gin}\,(I_\Gamma)$ and $I_{\Delta(\Gamma)}$ are shown in the following table. Columns 2, 3, and 4 illustrate Lemma 3.4 and Corollary 3.5.

$\mathrm{StdPairs}(I_\Gamma)$ form: $\mathbb{N}^\sigma$	$\mathrm{StdPairs}(\mathrm{Gin}\,(I_\Gamma))$ form: $m\mathbb{N}^{[i]}$	$\Phi(m)$ $\to \mathrm{supp}\,(\Phi(m))$	$\mathrm{StdPairs}(I_{\Delta(\Gamma)})$ form: $\mathbb{N}^\sigma$
$\mathbb{N}^{\{4,6,7\}}$	$\mathbb{N}^{\{1,2,3\}}$	$1 \to \emptyset$	$\mathbb{N}^{\{1,2,3\}}$
$\mathbb{N}^{\{2,6,7\}}$	$g\mathbb{N}^{\{1,2,3\}}$	$g \to \{7\}$	$\mathbb{N}^{\{1,2,7\}}$
$\mathbb{N}^{\{4,5,7\}}$	$ge\mathbb{N}^{\{1,2,3\}}$	$gd \to \{7,4\}$	$\mathbb{N}^{\{1,4,7\}}$
$\mathbb{N}^{\{1,5,7\}}$	$gd\mathbb{N}^{\{1,2,3\}}$	$gc \to \{7,3\}$	$\mathbb{N}^{\{1,3,7\}}$
$\mathbb{N}^{\{2,3,7\}}$	$f\mathbb{N}^{\{1,2,3\}}$	$f \to \{6\}$	$\mathbb{N}^{\{1,2,6\}}$
$\mathbb{N}^{\{1,3,7\}}$	$f^2\mathbb{N}^{\{1,2,3\}}$	$fe \to \{6,5\}$	$\mathbb{N}^{\{1,5,6\}}$
$\mathbb{N}^{\{3,5,6\}}$	$fe\mathbb{N}^{\{1,2,3\}}$	$fd \to \{6,4\}$	$\mathbb{N}^{\{1,4,6\}}$
$\mathbb{N}^{\{1,5,6\}}$	$fd\mathbb{N}^{\{1,2,3\}}$	$fc \to \{6,3\}$	$\mathbb{N}^{\{1,3,6\}}$
$\mathbb{N}^{\{3,4,6\}}$	$e\mathbb{N}^{\{1,2,3\}}$	$e \to \{5\}$	$\mathbb{N}^{\{1,2,5\}}$
$\mathbb{N}^{\{1,2,6\}}$	$e^2\mathbb{N}^{\{1,2,3\}}$	$ed \to \{5,4\}$	$\mathbb{N}^{\{1,4,5\}}$
$\mathbb{N}^{\{2,4,5\}}$	$ed\mathbb{N}^{\{1,2,3\}}$	$ec \to \{5,3\}$	$\mathbb{N}^{\{1,3,5\}}$
$\mathbb{N}^{\{2,3,5\}}$	$d\mathbb{N}^{\{1,2,3\}}$	$d \to \{4\}$	$\mathbb{N}^{\{1,2,4\}}$
$\mathbb{N}^{\{1,3,4\}}$	$d^2\mathbb{N}^{\{1,2,3\}}$	$dc \to \{4,3\}$	$\mathbb{N}^{\{1,3,4\}}$
$\mathbb{N}^{\{1,2,4\}}$	$d^3\mathbb{N}^{\{1,2,3\}}$	$dcb \to \{4,3,2\}$	$\mathbb{N}^{\{2,3,4\}}$
	$g^2\mathbb{N}^{\{1,2\}}$	$gf \to \{7,6\}$	$\mathbb{N}^{\{6,7\}}$
	$gf\mathbb{N}^{\{1,2\}}$	$ge \to \{7,5\}$	$\mathbb{N}^{\{5,7\}}$

4. Local cohomology

In this section we prove Theorem 4.1, which provides a simple combinatorial formula for the symmetric iterated Betti numbers of a simplicial complex.

Theorem 4.1. *For a simplicial complex* Γ

$$b_{i,r}(\Gamma) = \begin{cases} |\{F \in \max(\Delta(\Gamma)) : |F| = i, [i-r] \subseteq F, i-r+1 \notin F\}| & \text{if } r \leq i \\ 0 & \text{otherwise} \end{cases}$$

The symmetric iterated Betti numbers $b_{i,r}(\Gamma)$ were defined as the dimensions of the vector spaces $H^0(M_i(I_\Gamma))_r$, where for a homogeneous ideal I in S and a generic linear map $u \in U(I)$, $M_0(I) = S/uI$ and $M_i(I) = M_{i-1}(I)/(y_i M_{i-1}(I) + H^0(M_{i-1}(I)))$ for $1 \leq i \leq n$. Thus at step i we "peel off" the i-th variable. This is similar to the "deconing" of the shifted complex $\Delta^e(\Gamma)$ used in the definition of the exterior iterated Betti numbers of Γ by Duval and Rose [9].

In view of Lemma 3.4, it suffices to show that $|A_{i,r}(I)| = \dim H^0(M_i(I))_r$ for all $i, r \geq 0$ in order to prove Theorem 4.1. We establish this in Lemma 4.3 below. However, we first digress briefly to derive and illustrate certain facts needed in the proof of Lemma 4.3.

Recall that if M is an S-module, N is a submodule and I is an ideal in S then $(N : I^\infty)_M = \{m \in M |$ for some $r \in \mathbb{N}, I^r m \subseteq N\}$ and if $I = \langle f \rangle$ it is typical to write $(N : \langle f \rangle^\infty) = (N : f^\infty)$. For an S-module M, the 0-th local cohomology of M with respect to the irrelevant ideal $\mathbf{m} = S_+ = \langle y_1, \dots, y_n \rangle$ is defined as

$$H^0(M) = \{m \in M : \mathbf{m}^k \cdot m = 0 \text{ for some } k\} = (0 : \mathbf{m}^\infty)_M.$$

In particular $H^0(M)$ is graded when M is graded. Hence the equivalent definition of $M_i(I)$ is $M_i(I) = S/J_i(I)$, where

$$J_0(I) := uI \quad \text{and} \quad J_i(I) := y_i S + (J_{i-1}(I) : \mathbf{m}^\infty).$$

Fix an i such that $1 \leq i < n$. Then for all $1 \leq k < i$, $(J_{i-1}(I) : y_k^\infty) = S$ since $y_k \in J_{i-1}(I)$. For a j such that $1 \leq i < j \leq n$, consider the family of automorphisms $g_a \in \mathrm{GL}(S_1)$ such that $g_a(y_i) = ay_i + (1-a)y_j$, $g_a(y_j) = (1-a)y_i + ay_j$ and $g_a(y_r) = y_r$ otherwise, parameterized by all $a \in \mathbf{k}$. By induction $g_a(J_{i-1}(I)) = J_{i-1}(I)$. Thus $g_a(J_{i-1}(I) : y_i^\infty) = (J_{i-1}(I) : (ay_i + (1-a)y_j)^\infty)$, and so the two colon ideals are isomorphic. If $I \subseteq S$ is a fixed ideal and $f \in S$ varies then the ideal $(I : f^\infty)$ depends only on which associated primes of I contain f. Thus if for a family of f's over $\mathbf{k}$ all the colon ideals $(I : f^\infty)$ are isomorphic they must in fact be equal. Hence

$$\forall\, i < j, \text{ we have } (J_{i-1}(I) : y_i^\infty) = (J_{i-1}(I) : y_j^\infty) = (J_{i-1}(I) : \mathbf{m}^\infty).$$

Fix I and write $B = B(I)$, $A_i = A_i(I)$, $J_i = J_i(I)$ and $M_i = M_i(I)$. For the proof of Lemma 4.3 we introduce the sets

$$C_i = (B - \cup_{j \leq i} A_j) \cap \mathbb{N}^{[i+1,n]}.$$

Lemma 4.2. *The sets A_i and C_i have the following properties:*

1. *C_{i-1} is the disjoint union $C_{i-1} = A_i \,\dot\cup\, C_i \,\dot\cup\, y_i C_{i-1}$.*
2. *C_i and $C_i \cup A_i$ are shifted order ideals in $\mathbb{N}^{[i+1,n]}$.*
3. *$\mathbf{k}C_i \cap J_i = \{0\}$.*
4. *$B \subseteq \mathbf{k}C_i + (J_i : y_{i+1}^\infty)$.*

Proof: (1). If $m\mathbb{N}^{[j]}$ is a standard pair of $\mathrm{Gin}\,(I)$ with $j \le i$, then no monomial in $m\mathbb{N}^{[j]}$ lies in C_i. Thus C_i is the set of all monomials in $B \cap \mathbb{N}^{[i+1,n]}$ that lie in standard pairs of the form $*\mathbb{N}^{[j]}$ where $j > i$. In other words, $C_i = \{m \in B \cap \mathbb{N}^{[i+1,n]} : m\mathbb{N}^{\{i+1\}} \subseteq B\}$. This implies that $y_{i+1}C_i \subseteq C_i$ and

$$\begin{aligned} C_{i-1} &= \{m \in B \cap \mathbb{N}^{[i,n]} : m\mathbb{N}^{\{i\}} \subseteq B\} \\ &= \{m \in B \cap \mathbb{N}^{[i+1,n]} : m\mathbb{N}^{\{i+1\}} \subseteq B\} \,\dot\cup \\ &\quad\ \{m \in B \cap \mathbb{N}^{[i+1,n]} : m\mathbb{N}^{\{i+1\}} \not\subseteq B\} \,\dot\cup\, y_iC_{i-1} \\ &= C_i \,\dot\cup\, A_i \,\dot\cup\, y_iC_{i-1} \end{aligned}$$

(2). The definitions of C_i and A_i and the fact that B is shifted imply via an induction that C_i and $C_i \cup A_i$ are shifted order ideals in $\mathbb{N}^{[i+1,n]}$.

(3). We establish this fact by induction on i. Note that $\mathbf{k}C_0 \cap J_0 = \{0\}$ since all elements of $C_0 \subseteq B$ are standard monomials of $\mathrm{Gin}\,(I) = \mathrm{In}_\succ(J_0)$. Assume $\mathbf{k}C_{i-1} \cap J_{i-1} = \{0\}$, but there exists $0 \ne f \in \mathbf{k}C_i \cap J_i$. Since $f = \sum \beta_m m \in \mathbf{k}C_i$, each $m \in C_i \subset \mathbb{N}^{[i+1,n]}$, and so $m \notin \langle y_1, \ldots, y_i\rangle$. Therefore, $f \in J_i = y_iS + (J_{i-1} : y_i^\infty)$ implies that $f \in (J_{i-1} : y_i^\infty)$ — i.e., $fy_i^k \in J_{i-1}$ for some k. Since $\mathbf{k}C_{i-1} \cap J_{i-1} = \{0\}$, $fy_i^k = \sum \beta_m m \cdot y_i^k \notin \mathbf{k}C_{i-1}$ and we infer that at least one of the monomials m is not in C_{i-1} (since $y_iC_{i-1} \subseteq C_{i-1}$). However, $C_i \subseteq C_{i-1}$, and hence this m is also not in C_i, which is a contradiction. Thus $\mathbf{k}C_i \cap J_i = \{0\}$.

(4). Since $B \cap \mathbb{N}^{[i+1,n]} = (C_i \cup \cup_{j \le i} A_i) \cap \mathbb{N}^{[i+1,n]}$ it suffices to show that if $j \le i$ then $A_j \subseteq \mathbf{k}C_i + (J_i : y_{i+1}^\infty)$. For every $a \in \cup_{j\le i} A_j$, there exists some $t > 0$ such that $ay_{i+1}^t \in \mathrm{Gin}\,(I) = \mathrm{In}_\succ(J_0) \subseteq \mathrm{In}_\succ(J_i)$ (since $J_0 \subseteq J_i$). Therefore there exists $f \in J_i$ such that $f = ay_{i+1}^t - \sum \alpha_m m$ where ay_{i+1}^t is the leading term of f with respect to $\succ$, $\alpha_m \in \mathbf{k}$, and the monomials m are standard monomials of $\mathrm{In}_\succ(J_i)$. Hence $m \in B \cap \mathbb{N}^{[i+1,n]}$ (since all standard monomials of $\mathrm{In}_\succ(J_i)$ are in B and $y_1, \ldots, y_i \in \mathrm{In}_\succ(J_i)$). Further, since $ay_{i+1}^t \succ m$ for each m and since $ay_{i+1}^t, m \in \mathbb{N}^{[i+1,n]}$, it follows that $y_{i+1}^t | m$. Thus $a = f/y_{i+1}^t + \sum \alpha_m (m/y_{i+1}^t)$. Finally, since each $m/y_{i+1}^t \prec a$ in the above sum, we are done by induction. □

Lemma 4.3. $|A_{i,r}(I)| = \dim H^0(M_i(I))_r$ *for all* $i, r \ge 0$.

Proof: Note that

$$\begin{aligned} \dim H^0(M_i)_r &= \dim(\{m \in (S/J_i)_r : m \in (J_i : \mathbf{m}^\infty)\}) \\ &= \dim(\{m \in (S/J_i)_r : m \in (J_i : y_{i+1}^\infty)\}) \\ &= \dim(J_i : y_{i+1}^\infty)_r - \dim(J_i)_r. \end{aligned}$$

Hence, to prove the lemma it suffices to show that $\mathbf{k}A_i \oplus J_i = (J_i : y_{i+1}^\infty)$. We do this by establishing the following set of equalities:

$$S \overset{[1]_i}{=} \mathbf{k}C_i \oplus \mathbf{k}A_i \oplus J_i \overset{[2]_i}{=} \mathbf{k}C_i \oplus (J_i : y_{i+1}^\infty) \quad \text{for all } i \ge 0.$$

For $i = 0$ the first equality, $[1]_0$, follows from the facts that $\mathbf{k}B = \mathbf{k}A_0 \oplus \mathbf{k}C_0$ and $\mathbf{k}B \oplus J_0 = S$. We now show that $[1]_i$ implies $[2]_i$ and $[1]_{i+1}$. Assume that $S = \mathbf{k}C_i \oplus \mathbf{k}A_i \oplus J_i$. Since $A_i \subseteq \mathbf{k}C_i + (J_i : y_{i+1}^\infty)$ (Lemma 4.2 (4)) we obtain that $S = \mathbf{k}C_i + (J_i : y_{i+1}^\infty)$, and in order to verify $[2]_i$ we must show that $\mathbf{k}C_i \cap (J_i : y_{i+1}^\infty) = \{0\}$. This follows from the facts that $y_{i+1}C_i \subseteq C_i$ and that $\mathbf{k}C_i \cap J_i = \{0\}$

(Lemma 4.2 (3)). Thus

$$S \overset{[2]_i}{=} \mathbf{k}C_i \oplus (J_i : y_{i+1}^{\infty}) \overset{\text{Lemma 4.2 (1)}}{=}$$
$$\mathbf{k}C_{i+1} \oplus \mathbf{k}A_{i+1} \oplus \mathbf{k}y_{i+1}C_i \oplus (J_i : y_{i+1}^{\infty}) = \mathbf{k}C_{i+1} \oplus \mathbf{k}A_{i+1} \oplus J_{i+1},$$

where in the last step we used Lemma 4.2 (4) along with the definition of J_{i+1}. Equality $[1]_{i+1}$ follows. □

We close this section with several remarks.

Remark 4.4. Lemma 4.3 was verified in the special case of Stanley-Reisner ideals of Buchsbaum complexes in [19].

Remark 4.5. Recall that by property (B1), $B(I)$ is a basis of S/uI as well as of $S/\mathrm{Gin}\,(I)$. Thus the proof of Lemma 4.3 implies also that

$$|A_{i,r}(I)| = \dim H^0(M_i(\mathrm{Gin}\,(I))_r \quad \text{for all} \quad i, r \geq 0,$$

and we recover the following fact (originally due to Bayer, Charalambous, and Popescu [4, Cor. 1.7]).

Corollary 4.6. *Modules $H^0(M_i(I))$ and $H^0(M_i(\mathrm{Gin}\,(I)))$ have the same Hilbert function (for $i = 0, 1, \ldots n$). In other words, the symmetric iterated Betti numbers of I are identical to those of* $\mathrm{Gin}\,(I)$.

Remark 4.7. Similar to the proof of Theorem 4.1, one can show that

$$\dim H^0(S/\langle uI_\Gamma, y_1, \ldots, y_i\rangle)_r =$$
$$|\{G \in \Delta(\Gamma) : |G| = r,\ [i-r+1] \cap G = \emptyset,\ [i-r+1] \cup G \notin \Delta(\Gamma)\}|.$$

Thus for the shifted complex Γ, dimensions of the modules $H^0(S/\langle uI_\Gamma, y_1, \ldots, y_i\rangle)_r$ coincide with Kalai's (exterior) iterated Betti numbers of Γ (see [17, Section 3]). For that reason we refer to the numbers $\bar{b}_{i,r}(\Gamma) := \dim H^0(S/\langle uI_\Gamma, y_1, \ldots, y_i\rangle)_r$ as Kalai's symmetric iterated Betti numbers.

Remark 4.8. Another fact worth mentioning is that $b_{i,i}(\Gamma) = \bar{b}_{i,i}(\Gamma)$ are just reduced (topological) Betti numbers of Γ, that is,

$$b_{i,i}(\Gamma) = \bar{b}_{i,i}(\Gamma) = \beta_{i-1}(\Gamma) \quad \forall\, 0 \leq i \leq \dim(\Gamma)+1, \text{ where } \beta_{i-1}(\Gamma) = \dim \widetilde{H}_{i-1}(\Gamma, \mathbf{k}).$$

This result is a consequence of Theorem 4.1 together with the fact [6] that for a shifted complex K

$$\beta_{i-1}(K) = |\{F \in max(K) \,:\, |F| = i,\ 1 \notin F\}|,$$

and the fact that symmetric shifting preserves topological Betti numbers [13].

Remark 4.9. Finally we note that if Γ is a Buchsbaum complex (i.e., a pure simplicial complex all of whose vertices have Cohen-Macaulay links), then $b_{i,r}(\Gamma) = \binom{i-1}{r-1}\beta_{r-1}(\Gamma)$ for every $0 \leq r \leq i \leq \dim(\Gamma)$, where $\beta_{r-1}(\Gamma)$ are reduced (topological) Betti numbers of Γ. This follows from Lemma 4.3 and [19, Lemma 4.1].

5. Extremal Betti numbers

This section is devoted to the proof of Theorem 5.3, which relates the graded algebraic Betti numbers of I_Γ to the symmetric iterated Betti numbers of Γ. Every homogeneous ideal $I \subset S$ admits a graded free S-resolution of the form

$$\to \dots \bigoplus_j S(-j)^{\beta_{i,j}} \to \dots \to \bigoplus_j S(-j)^{\beta_{1,j}} \to \bigoplus_j S(-j)^{\beta_{0,j}} \to I \to 0,$$

where $S(-j)$ denotes S with grading shifted by j. Moreover, there exists a unique (up to isomorphism) resolution in which all the exponents $\beta_{i,j}$ are simultaneously minimized, called the *minimal graded free S-resolution* of I. The numbers $\beta_{i,j}$ appearing in this minimal free resolution of I are called the *graded Betti numbers* of I. A Betti number of I, $\beta_{i,i+j}(I)$, is *extremal* if $\beta_{i,i+j}(I) \neq 0$, but $\beta_{i',i'+j'}(I) = 0$ for all $i' \geq i$, $j' \geq j$, $(i', j') \neq (i, j)$. This terminology comes from the Betti diagram of I output by the program *Macaulay 2* in which the Betti numbers are arranged in a rectangular array whose columns are indexed by i and rows by j and the (i, j)-th entry is the Betti number $\beta_{i,i+j}$. Thus $\beta_{i,i+j}$ is an extremal Betti number of I if it lies in a south-east corner of the *Macaulay 2* Betti diagram of I.

Let Γ be a simplicial complex on the vertex set $[n]$. The *Alexander dual* of Γ is the simplicial complex

$$\Gamma^* = \{F \subseteq [n] : [n] \setminus F \notin \Gamma\}.$$

The next two results (both due to Bayer, Charalambous and Popescu [4], see [1] also for the second theorem) provide connections between the extremal Betti numbers of the Stanley-Reisner ideals of Γ and Γ^*, and the shifted complex $\Delta(\Gamma)$.

Theorem 5.1. *Let Γ be a simplicial complex and Γ^* be its Alexander dual. The Stanley-Reisner ideals I_Γ and I_{Γ^*} have the same extremal Betti numbers. More precisely, $\beta_{i,i+j}(I_{\Gamma^*})$ is extremal if and only if $\beta_{j-1,i+j}(I_\Gamma)$ is extremal. Also, in such a case $\beta_{i,i+j}(I_{\Gamma^*}) = \beta_{j-1,i+j}(I_\Gamma)$.*

Theorem 5.2. *Extremal Betti numbers are preserved by algebraic shifting: for a simplicial complex Γ, $\beta_{i,i+j}(I_\Gamma)$ is extremal if and only if $\beta_{i,i+j}(I_{\Delta(\Gamma)})$ is extremal. Moreover, in such a case $\beta_{i,i+j}(I_\Gamma) = \beta_{i,i+j}(I_{\Delta(\Gamma)})$.*

We are now in a position to prove the main theorem of this section.

Theorem 5.3. *The extremal Betti numbers of I_Γ are contained among the symmetric iterated Betti numbers of Γ. They are precisely the extremal entries in the b-triangle of Γ: $\beta_{j-1,i+j}(I_\Gamma)$ is an extremal Betti number of I_Γ if and only if*

$$b_{n-j',i'}(\Gamma) = 0 \quad \forall (i', j') \neq (i, j),\ i' \geq i,\ j' \geq j,\ \textit{and}\ b_{n-j,i}(\Gamma) \neq 0.$$

Moreover, in this case, $\beta_{j-1,i+j}(I_\Gamma) = b_{n-j,i}(\Gamma)$.

Example 3.7 continued: The minimal free resolution and Betti diagram of I_Γ (computed by *Macaulay 2*) are given below. Note that the entries in the southeast corners of the Betti diagram of I_Γ (the extremal Betti numbers of I_Γ) are precisely the entries in the north-east corners of the b-triangle of Γ from Section 3.

$$0 \to S^2 \to S^{15} \to S^{42} \to S^{49} \to S^{21} \to S \to 0$$

total:	1	21	49	42	15	2
0 :	1	.	.	.	.	.
1 :	.	.	.	.	.	.
2 :	.	21	49	42	14	2
3 :	.	.	.	.	1	.

The proof of Theorem 5.3 relies on the following lemma, which is a consequence of [14, Thm. 2.1(b)] (see also [15, Prop. 12]) and [8, Cor. 6.2]. For completeness we provide a different self-contained proof.

Lemma 5.4. *The symmetric iterated Betti numbers of* Γ *are related to the graded Betti numbers of the Stanley-Reisner ideal* $I_{\Delta(\Gamma^*)}$ *as follows:*

$$\beta_{i,i+j}(I_{\Delta(\Gamma^*)}) = \sum_r \binom{n-r-j}{i} b_{n-j,n-r-j}(\Gamma).$$

Proof: Let Γ be a simplicial complex and let Γ^* be its Alexander dual. Recall that the Stanley-Reisner ideal $I_{\Delta(\Gamma^*)} \subset S$ is a squarefree monomial ideal whose minimal generators correspond to minimal non-faces of $\Delta(\Gamma^*)$. Let G be the set of minimal generators of $I_{\Delta(\Gamma^*)}$, let $G_j = G \cap \mathbb{N}_j^{[n]}$, and let $\min(g) = \min\{i : y_i | g\}$ for a monomial $g \in G$.

Since $\Delta(\Gamma^*)$ is a shifted complex, it follows that the ideal $I_{\Delta(\Gamma^*)}$ is *squarefree strongly stable*, which means that for a monomial $m \in I_{\Delta(\Gamma^*)}$, if $y_i | m$, $i < j \leq n$, and y_j is not a divisor of m, then $my_j/y_i \in I_{\Delta(\Gamma^*)}$ as well. Hence the graded Betti numbers of $I_{\Delta(\Gamma^*)}$ are given by the following formula [13, Cor. 3.4] (which is the analog of the Eliahou-Kervaire formula for strongly stable ideals):

$$\begin{aligned} \beta_{i,i+j}(I_{\Delta(\Gamma^*)}) &= \sum_{g \in G_j} \binom{n - \min(g) + 1 - j}{i} \\ &= \sum_r \binom{n-r-j}{i} |\{g \in G_j : \min(g) = r+1\}|. \end{aligned} \tag{4}$$

It is well known and is easy to prove that $\Delta(\Gamma^*) = \Delta(\Gamma)^*$. Thus, $g \in G_j$ if and only if $\sigma = \mathrm{supp}\,(g)$ is a minimal non-face of $\Delta(\Gamma^*)$ of size j, which happens if and only if $[n] \setminus \sigma$ is a facet of $\Delta(\Gamma)$ of size $n-j$. Moreover, $r+1 = \min\{i : i \in \sigma\}$ if and only if $[r] \subseteq [n] \setminus \sigma$, but $r+1 \notin [n] \setminus \sigma$. Hence

$$\begin{aligned} & |\{g \in G_j : \min(g) = r+1\}| = \\ & |\{F \in \max(\Delta(\Gamma)) : |F| = n-j,\ [r] \subseteq F,\ \text{but } r+1 \notin F\}| \overset{\text{Thm 4.1}}{=} \\ & b_{n-j,n-r-j}(\Gamma). \end{aligned} \tag{5}$$

Substituting (5) in (4) gives the result. $\square$

Proof of Theorem 5.3: The theorem is an easy consequence of Lemma 5.4. Indeed, since $\binom{n-r-j}{i}$ is positive for $r \leq n-i-j$ and is zero otherwise, it follows from the lemma that

$$\beta_{i',i'+j'}(I_{\Delta(\Gamma^*)}) = 0 \quad \text{iff} \quad b_{n-j',n-j'-r}(\Gamma) = 0 \text{ for all } r \leq n-i'-j'.$$

Thus,

$$\begin{aligned} \beta_{i,i+j}(I_{\Delta(\Gamma^*)}) \neq 0 \text{ is extremal} \quad &\Longleftrightarrow \\ \beta_{i',i'+j'}(I_{\Delta(\Gamma^*)}) = 0 \text{ for all } i' \geq i,\ j' \geq j,\ (i,j) \neq (i',j') \quad &\Longleftrightarrow \\ b_{n-j',i'}(\Gamma) = 0 \text{ for all } i' \geq i,\ j' \geq j,\ (i,j) \neq (i',j'). \end{aligned}$$

Moreover, if this is the case, then all except the first summand in

$$\beta_{i,i+j}(I_{\Delta(\Gamma^*)}) = \binom{i}{i} b_{n-j,i}(\Gamma) + \sum_{r<n-i-j} \binom{n-r-j}{i} b_{n-j,n-r-j}(\Gamma)$$

vanish, implying that $\beta_{i,i+j}(I_{\Delta(\Gamma^*)}) = b_{n-j,i}(\Gamma)$. The result then follows from Theorems 5.1 and 5.2. $\square$

We remark that one can use [1, Cor . 1.2] and certain properties of sets $A_i(I)$ to provide a different proof of the following more general result. We omit the details.

Theorem 5.5. *Let I be a homogeneous ideal. The extremal Betti numbers of I form a subset of the symmetric iterated Betti numbers of I. More precisely, $\beta_{j-1,i+j}(I)$ is an extremal Betti number of I if and only if*

$$b_{n-j',i'}(I) = 0 \quad \forall (i',j') \neq (i,j),\ i' \geq i,\ j' \geq j, \text{ and } b_{n-j,i}(I) \neq 0.$$

Moreover, in this case, $\beta_{j-1,i+j}(I) = b_{n-j,i}(I)$.

6. Associated Primes and Standard Pairs

The associated primes of a homogeneous ideal $I \subset S$ with a primary decomposition $I = Q_1 \cap Q_2 \cap \cdots \cap Q_t$ are the prime ideals $P_i := \sqrt{Q_i}$, $i = 1, \ldots, t$, where $\sqrt{Q_i}$ denotes the radical of Q_i. The set of associated primes of I, customarily denoted as $\mathrm{Ass}\,(I)$, is independent of the primary decomposition of I. The minimal elements of $\mathrm{Ass}\,(I)$ with respect to inclusion are called the *minimal primes* of I, and the non-minimal ones are called the *embedded primes* of I. We denote the set of minimal primes of I as $\mathrm{Min}\,(I)$. Recall that the irreducible (isolated) components of $V(I)$, the variety of I in $\mathbf{k}^n$, are the varieties $V(P)$ for $P \in \mathrm{Min}\,(I)$. Let $Z_i := V(P_i)$ be the variety of P_i in $\mathbf{k}^n$. The finite invariant $\deg(Z_i)$, called the *degree* of Z_i, is the cardinality of $Z_i \cap L$ for almost all linear subspaces L of dimension equal to the codimension of Z_i.

Definition 6.1. [5], [21]

(1) If P is a homogeneous prime ideal in S then the **multiplicity** of P (with respect to I), denoted as $\mathrm{mult}_I(P)$ is the length of the largest ideal of finite length in the ring S_P/IS_P.
(2) The **degree** of I, $\deg(I) := \sum_{\{\dim(Z_i)=\dim(I)\}} \mathrm{mult}_I(P_i)\deg(Z_i)$.
(3) The **geometric degree** of I, $\mathrm{geomdeg}\,(I) := \sum_{\{P_i \in \mathrm{Min}(I)\}} \mathrm{mult}_I(P_i)\deg(Z_i)$.
(4) The **arithmetic degree** of I, $\mathrm{arithdeg}\,(I) := \sum_{\{P_i \in \mathrm{Ass}(I)\}} \mathrm{mult}_I(P_i)\deg(Z_i)$.

The invariant $\mathrm{mult}_I(P) > 0$ if and only if $P \in \mathrm{Ass}\,(I)$. Our main goal in this section is to prove Theorem 6.6. We first specialize Definition 6.1 to monomial ideals. If M is a monomial ideal, then every associated prime of M is of the form $P_\sigma := \langle y_j : j \notin \sigma \rangle$ for some set $\sigma \subseteq [n]$. Hence $V(P_\sigma)$ is the $|\sigma|$-dimensional linear subspace spanned by $\{e_j : j \in \sigma\}$ and $\deg(V(P_\sigma)) = 1$. The three degrees of M from Definition 6.1 are therefore appropriate sums of multiplicities of ideals in $\mathrm{Ass}\,(M)$ with respect to M.

For a monomial ideal M the multiplicities of associated primes as well as all the degrees referred to in Definition 6.1 can be read off from the standard pairs of M (see Definition 3.2) as shown in the following lemma. The statements in this lemma are either stated or can be derived easily from the results in [21].

Lemma 6.2. *Let M be a monomial ideal. Then,*

(1) *the set of standard pairs of M is well defined,*
(2) *$*\mathbb{N}^\sigma$ is a standard pair of M if and only if $P_\sigma \in \mathrm{Ass}\,(M)$,*
(3) *$\mathbb{N}^\sigma$ is a standard pair of M if and only if $P_\sigma \in \mathrm{Min}\,(M)$,*
(4) *the dimension of M is the maximal size of a set σ such that $*\mathbb{N}^\sigma$ is a standard pair of M,*
(5) *if $P_\sigma \in \mathrm{Ass}\,(M)$, then $\mathrm{mult}_M(P_\sigma)$ is the number of standard pairs of M of the form $*\mathbb{N}^\sigma$ and*
(6) (a) *$deg(M)$ is the number of standard pairs $*\mathbb{N}^\sigma$ of M such that $|\sigma| = dim(M)$,*
  (b) *$geomdeg(M)$ is the number of standard pairs $*\mathbb{N}^\sigma$ of M such that $\mathbb{N}^\sigma$ is a standard pair of M and*
  (c) *$arithdeg(M)$ is the total number of standard pairs of M.*

Lemma 3.3 showed that $m\mathbb{N}^\sigma$ is a standard pair of $\mathrm{Gin}\,(I)$ if and only if $\sigma = [i]$ and $m \in A_i(I)$ for some $0 \le i \le n$. Combining this fact with Lemma 6.2 we obtain the following.

Corollary 6.3. *(see also* [10, Corollary 15.25]*)*

(i) *$P_{[d]}$, $d = dim(I)$, is the unique minimal prime of $\mathrm{Gin}\,(I)$ (if $I = I_\Gamma$ then $d = \dim\Gamma + 1$), and*
(ii) *all embedded primes of $\mathrm{Gin}\,(I)$ are of the form $P_{[k]}$ for some $k < d$.*

Thus the submonoids in the standard pairs of $\mathrm{Gin}\,(I)$ are initial intervals of $[n]$ while the cosets can be complicated. On the other hand, for the square free monomial ideals I_Γ and $I_{\Delta(\Gamma)}$, the cosets of the standard pairs are trivial and the submonoids determine the ideals (cf. Example 3.7).

Corollary 6.4. *If Γ is a simplicial complex then $I_\Gamma = \bigcap_{\sigma\in\max(\Gamma)} P_\sigma$ is the irredundant prime decomposition of I_Γ. In particular, I_Γ has no embedded primes and its standard pairs are $\{\mathbb{N}^\sigma : \sigma \in \max(\Gamma)\}$.*

By Corollary 3.5, $m\mathbb{N}^{[i]}$ is a standard pair of $\mathrm{Gin}\,(I_\Gamma)$ if and only if $[i-r] \cup \mathrm{supp}\,(\Phi(m))$ is a facet of $\Delta(\Gamma)$ of size i. Combining this fact with Corollary 6.4 we get the following bijection as well.

Corollary 6.5. *There is a bijection between the standard pairs of $\mathrm{Gin}\,(I_\Gamma)$ and those of $I_{\Delta(\Gamma)}$ given by: $m\mathbb{N}^{[i]}$ is a standard pair of $\mathrm{Gin}\,(I_\Gamma)$ with $\deg(m) = r$ if and only if $\mathbb{N}^{[i-r]\cup\mathrm{supp}\,(\Phi(m))}$ is a standard pair of $I_{\Delta(\Gamma)}$.*

The following theorem is a corollary of the results in Sections 3 and 4, and those stated thus far in this section.

Theorem 6.6. *The iterated Betti numbers of a homogeneous ideal I are related to the ideal $\mathrm{Gin}\,(I)$. Those of an ideal I_Γ are related to the ideals $\mathrm{Gin}\,(I_\Gamma)$, $I_{\Delta(\Gamma)}$, and the shifted complex $\Delta(\Gamma)$. The relationships are as follows.*

(1) *The multiplicity of* $P_{[i]}$ *with respect to* $\mathrm{Gin}\,(I)$ *is*

$$\mathrm{mult}_{\,\mathrm{Gin}\,(I)}(P_{[i]}) = \sum_r b_{i,r}(I).$$

If $I = I_\Gamma$ *then*

$$\mathrm{mult}_{\,\mathrm{Gin}\,(I_\Gamma)}(P_{[i]}) = \sum_r b_{i,r}(\Gamma) = |\{F \in \max(\Delta(\Gamma)) \,:\, |F| = i\}|.$$

(2) *The degree, geometric degree, and arithmetic degree of* $\mathrm{Gin}\,(I_\Gamma)$ *and* $I_{\Delta(\Gamma)}$ *have the following interpretations:*

$$(i) \qquad \deg(\mathrm{Gin}\,(I_\Gamma)) = \mathrm{geomdeg}\,(\mathrm{Gin}\,(I_\Gamma)) = \sum_r b_{d,r}(I_\Gamma)$$

$$(i') \qquad = \deg(I_{\Delta(\Gamma)}) = |\{F \in \max(\Delta(\Gamma)) \,:\, |F| = d\}| \;;$$

$$(ii) \qquad \mathrm{arithdeg}\,(\mathrm{Gin}\,(I_\Gamma)) = \sum_{i,r} b_{i,r}(I_\Gamma)$$

$$(ii') \qquad = \mathrm{arithdeg}\,(I_{\Delta(\Gamma)}) = |\max(\Delta(\Gamma))|.$$

Equations (i) and (ii) also hold for arbitrary homogeneous ideals I *in* S*.*

Proof: (1) By Lemmas 6.2 (5)

$$\begin{aligned} \mathrm{mult}_{\,\mathrm{Gin}\,(I_\Gamma)}(P_{[i]}) &= |\{\text{standard pairs of}\,\mathrm{Gin}\,(I)\,\text{of the form} * \mathbb{N}^{[i]}\}| \\ &= |A_i(I)| \qquad \text{(by Lemma 3.3)} \\ &= \sum_r b_{i,r}(I) \qquad \text{(by Theorem 4.3).} \end{aligned}$$

In particular, $P_{[i]}$ is an associated prime of $\mathrm{Gin}\,(I)$ if and only if $b_{i,r}(I) > 0$ for some r. For a simplicial complex Γ on $[n]$, by Lemma 3.4,

$$\mathrm{mult}_{\,\mathrm{Gin}\,(I_\Gamma)}(P_{[i]}) = |A_i(I_\Gamma)| = \sum_r b_{i,r}(\Gamma) = |\{F \in \max(\Delta(\Gamma)) \,:\, |F| = i\}|.$$

In particular, $P_{[i]}$ is an associated prime of $\mathrm{Gin}\,(I_\Gamma)$ if and only if $\Delta(\Gamma)$ has a facet of size i.

(2) The same lemmas along with Definition 6.1 yield these results. □

We now establish certain further facts about $\mathrm{Ass}\,(\mathrm{Gin}\,(I_\Gamma))$.

Definition 6.7. For any ideal I in a polynomial ring, its poset of associated primes $\mathrm{Ass}\,(I)$ has the **chain property** if whenever $P \in \mathrm{Ass}\,(I)$ is an embedded prime, then there exists $Q \in \mathrm{Ass}\,(I)$ such that $P \supset Q$ and $\dim(Q) = \dim(P) + 1$.

Corollary 6.8. *The poset* $\mathrm{Ass}\,(\mathrm{Gin}\,(I_\Gamma))$ *possesses the chain property if and only if* $\max(\Delta(\Gamma))$ *has the property that whenever* $\Delta(\Gamma)$ *has a facet of size* $k \le \dim(\Gamma)$ *then it also has a facet of size* $k+1$*.*

Proof: By Corollary 6.3, the poset $\mathrm{Ass}\,(\mathrm{Gin}\,(I_\Gamma))$ has the chain property if whenever $P_{[k]} \in \mathrm{Ass}\,(\mathrm{Gin}\,(I_\Gamma))$ for some $k \le \dim\Gamma$ then $P_{[k+1]} \in \mathrm{Ass}\,(\mathrm{Gin}\,(I_\Gamma))$. By Theorem 6.6 (1), this is equivalent to the condition that whenever $\Delta(\Gamma)$ has a facet of size $k \le \dim\Gamma$ then it also has a facet of size $k+1$. □

Corollary 6.9. *If Γ is a Buchsbaum complex (cf. Remark 4.9) then* $\mathrm{Ass}\,(\mathrm{Gin}\,(I_\Gamma))$ *has the chain property.*

Proof: It was shown in [19] that if Γ is a $(d-1)$-dimensional Buchsbaum complex then $b_{i,r}(\Gamma) = \binom{i-1}{r-1}\beta_{r-1}$ for $i < d$, where β_{r-1} is the reduced (topological) Betti number of Γ. Hence for $i < d$, $\mathrm{mult}_{\,\mathrm{Gin}\,(I_\Gamma)}(P_{[i]}) = \sum_r b_{i,r} = \sum_r \binom{i-1}{r-1}\beta_{r-1}$. Therefore if some $\beta_k \neq 0$ then $\mathrm{mult}_{\,\mathrm{Gin}\,(I_\Gamma)}(P_{[i]}) > 0$ for all $i \geq k$, which implies that $\mathrm{Ass}\,(\mathrm{Gin}\,(I_\Gamma))$ has the chain property. □

7. Iterated Betti numbers: exterior versus symmetric

We close the paper with several remarks and conjectures on connections between symmetric iterated Betti numbers and exterior iterated Betti numbers of a simplicial complex. The superscript e is used to denote exterior shifting.

We start with a brief description of exterior algebraic shifting extracted from [13]. Let $E = \bigwedge(\mathbf{k}[y_1, \ldots, y_n]_1) = \bigwedge S_1$ be the exterior algebra over the n-dimensional vector space S_1. A *monomial* in E is an expression of the form $m = y_{i_1} \wedge y_{i_2} \wedge \cdots \wedge y_{i_k}$, where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$; the set $\{i_1, i_2, \ldots, i_k\}$ is called the support of m, and is denoted by $\mathrm{supp}\,(m)$. The *exterior Stanley-Reisner ideal* of a simplicial complex Γ on $[n]$ is

$$J_\Gamma := \langle m \in E \,:\, m \text{ is a monomial, } \mathrm{supp}\,(m) \notin \Gamma \rangle.$$

Definition 7.1. The exterior algebraic shifting of Γ, $\Delta^e(\Gamma)$, is the simplicial complex defined by $J_{\Delta^e(\Gamma)} := \mathrm{Gin}\,(J_\Gamma)$, where $\mathrm{Gin}\,(J_\Gamma)$ is the generic initial ideal of J_Γ with respect to the reverse lexicographic order with $y_n \succ y_{n-1} \succ \cdots \succ y_1$.

The exterior iterated Betti numbers of a simplicial complex Γ were introduced by Duval and Rose [9]. They have the following combinatorial description (up to a slight change in the indexing), which we adopt as their definition.

Definition 7.2. The exterior iterated Betti numbers of a simplicial complex Γ are

$$b^e_{i,r}(\Gamma) = |\{F \in \max(\Delta^e(\Gamma)) \,:\, |F| = i,\ [i-r] \subseteq F,\ i-r+1 \notin F\}|.$$

Since $\Delta^e(\Delta(\Gamma)) = \Delta(\Gamma)$ and $\Delta(\Delta^e(\Gamma)) = \Delta^e(\Gamma)$, the above definition and Theorem 4.1 imply that

$$b_{i,r}(\Gamma) = b^e_{i,r}(\Delta(\Gamma)), \qquad b^e_{i,r}(\Gamma) = b_{i,r}(\Delta^e(\Gamma)).$$

Hence we infer the following corollary from Lemma 5.4.

Corollary 7.3. *For a simplicial complex Γ*

$$\beta_{i,i+j}(I_{\Delta^e(\Gamma^*)}) = \sum_r \binom{n-r-j}{i} b^e_{n-j,n-r-j}(\Gamma).$$

Since exterior shifting preserves extremal Betti numbers [1], the same proof as in Theorem 5.3 yields

Theorem 7.4. *$\beta_{j-1,i+j}(I_\Gamma)$ is an extremal Betti number of I_Γ if and only if*

$$b^e_{n-j',i'}(\Gamma) = 0 \quad \forall (i',j') \neq (i,j),\ i' \geq i,\ j' \geq j, \text{ and } b^e_{n-j,i}(\Gamma) \neq 0.$$

Moreover, if this is the case, then

$$\beta_{j-1,i+j}(I_\Gamma) = \beta_{j-1,i+j}(I_{\Delta^e(\Gamma)}) = \beta_{j-1,i+j}(I_{\Delta(\Gamma)}) = b^e_{n-j,i}(\Gamma) = b_{n-j,i}(\Gamma).$$

We remark that there are no known connections between general (non-extremal) graded Betti numbers of $I_{\Delta(\Gamma)}$ and $I_{\Delta^e(\Gamma)}$. However it is conjectured (see [13, Conj. 8.9]) that the following holds.

Conjecture 7.5. *For every simplicial complex* Γ, $\beta_{i,j}(I_{\Delta(\Gamma)}) \leq \beta_{i,j}(I_{\Delta^e(\Gamma)})$.

In analogy, we propose the following.

Conjecture 7.6. *For every simplicial complex* Γ, $b_{i,j}(\Gamma) \leq b^e_{i,j}(\Gamma)$ *for all* $i \leq \dim(\Gamma)$.

Note that since all the coefficients in the expression of graded Betti numbers in terms of iterated Betti numbers (see Lemma 5.4 and Corollary 7.3) are non-negative, Conjecture 7.6 if true would imply Conjecture 7.5 except for the case $j - i = n - \dim(\Gamma) - 1$ (that is, except for the last column of the Betti diagram of $I_{\Delta^{(e)}(\Gamma)}$ output by *Macaulay 2*).

Conjecture 7.5 was verified by Aramova, Herzog, and Hibi [2] in the case when Γ is the Alexander dual of a sequentially Cohen-Macaulay complex (a notion introduced by Stanley [20, Def. II.2.9]): they showed that in such a case

$$\beta_{i,j}(I_{\Delta(\Gamma)}) = \beta_{i,j}(I_{\Delta^e(\Gamma)}) = \beta_{i,j}(I_\Gamma).$$

We have the following related result.

Proposition 7.7. *Conjecture 7.6 holds for all sequentially Cohen-Macaulay complexes. More precisely, if* Γ *is sequentially Cohen-Macaulay, then*

$$b_{i,r}(\Gamma) = b^e_{i,r}(\Gamma) = h_{i,r}(\Gamma),$$

where $(h_{i,r}(\Gamma))_{0\leq r\leq i\leq \dim(\Gamma)+1}$ *is the* h*-triangle of* Γ.

The notion of f- and h-triangles was introduced by Björner and Wachs [7]. We recall the definition. For a simplicial complex Γ set

$$f_{i,j}(\Gamma) := |\{F \in \Gamma : |F| = j, \dim(\operatorname{st} F) = i-1\}|,$$

where $\operatorname{st} F$ denotes the star of F in Γ. The h-triangle of Γ, $(h_{i,j}(\Gamma))_{0\leq j\leq i\leq \dim(\Gamma)+1}$, is defined by

$$h_{i,j}(\Gamma) = \sum_{s=0}^{j} (-1)^{j-s} \binom{i-s}{j-s} f_{i,s}(\Gamma).$$

Proof of Proposition 7.7: Duval [8, Thm. 5.1] showed that if Γ is a sequentially Cohen-Macaulay simplicial complex, then $h_{i,j}(\Gamma) = h_{i,j}(\Delta^e(\Gamma))$. In his proof he relied only on the properties (P3) and (P4) of the operator Δ^e, and the fact that Γ is Cohen-Macaulay if and only if $\Delta^e(\Gamma)$ is pure. Since operator Δ (symmetric shifting) possesses all these properties as well, it follows that for a sequentially Cohen-Macaulay complex Γ,

$$h_{i,j}(\Gamma) = h_{i,j}(\Delta^e(\Gamma)) = h_{i,j}(\Delta(\Gamma)). \tag{6}$$

Another result due to Duval [8, Cor. 6.2] is that for a shifted complex K, $b^e_{i,j}(K) = h_{i,j}(K)$. Thus

$$h_{i,j}(\Delta^e(\Gamma)) = b^e_{i,j}(\Delta^e(\Gamma)) = b^e_{i,j}(\Gamma) \quad \text{and} \tag{7}$$

$$h_{i,j}(\Delta(\Gamma)) = b^e_{i,j}(\Delta(\Gamma)) = b_{i,j}(\Gamma). \tag{8}$$

Equations (6)–(8) imply the proposition. □

We close the paper with one additional conjecture for the special case of Buchsbaum complexes.

Conjecture 7.8. *If* Γ *is a Buchsbaum complex, then* $b_{i,j}(\Gamma) = b^e_{i,j}(\Gamma)$*, and hence* $\beta_{i,j}(I_{\Delta(\Gamma)}) = \beta_{i,j}(I_{\Delta^e(\Gamma)})$.

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