Face ring multiplicity via CM-connectivity sequences

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Abstract

The multiplicity conjecture of Herzog, Huneke, and Srinivasan is verified for the face rings of the following classes of simplicial complexes: matroid complexes, complexes of dimension one and two, and Gorenstein complexes of dimension at most four. The lower bound part of this conjecture is also established for the face rings of all doubly Cohen-Macaulay complexes whose 1-skeleton's connectivity does not exceed the codimension plus one as well as for all $(d-1)$-dimensional d-Cohen-Macaulay complexes. The main ingredient of the proofs is a new interpretation of the minimal shifts in the resolution of the face ring $\mathbf{k}[\Delta]$ via the Cohen-Macaulay connectivity of the skeletons of Δ.

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1 Introduction

In this paper we prove the multiplicity conjecture of Herzog, Huneke, and Srinivasan for the face rings of several classes of simplicial complexes. Throughout the paper we work with the polynomial ring $S = \mathbf{k}[x_1, \ldots, x_n]$ over an arbitrary field $\mathbf{k}$. If $I \subset S$ is a homogeneous ideal, then the ($\mathbb{Z}$-graded) **Betti numbers** of I, $\beta_{i,j} = \beta_{i,j}(I)$, are the invariants that appear in the minimal free resolution of S/I as an S-module:

$$0 \to \bigoplus_j S(-j)^{\beta_{l,j}(I)} \to \ldots \to \bigoplus_j S(-j)^{\beta_{2,j}(I)} \to \bigoplus_j S(-j)^{\beta_{1,j}(I)} \to S \to S/I \to 0$$

Here $S(-j)$ denotes S with grading shifted by j and l denotes the length of the resolution. In particular, $l \geq \mathrm{codim}\,(I)$.

Our main objects of study are the **maximal and minimal shifts** in the resolution of S/I defined by $M_i = M_i(I) = \max\{j : \beta_{i,j} \neq 0\}$ and $m_i = m_i(I) = \min\{j : \beta_{i,j} \neq 0\}$ for $i = 1, \ldots, l$, respectively. The following conjecture due to Herzog, Huneke, and Srinivasan [12] is known as the multiplicity conjecture.

Conjecture 1.1 *Let $I \subset S$ be a homogeneous ideal of codimension c. Then the multiplicity of S/I, $e(S/I)$, satisfies the following upper bound:*

$$e(S/I) \leq (\prod_{i=1}^{c} M_i)/c!.$$

Moreover, if S/I is Cohen-Macaulay, then also

$$e(S/I) \geq (\prod_{i=1}^{c} m_i)/c!.$$

This conjecture was motivated by the result due to Huneke and Miller [16] that if S/I is Cohen-Macaulay and $M_i = m_i$ for all i (in which case S/I is said to have a **pure resolution**), then $e(S/I) = (\prod_{i=1}^{c} m_i)/c!$. Starting with the paper of Herzog and Srinivasan [12] a tremendous amount of effort was put in establishing Conjecture 1.1 for various classes of rings S/I (see [6, 9, 10, 11, 12, 13, 19, 21, 25, 26] and the survey article [7]). In particular, the conjecture was proved in the following cases: S/I has a **quasi-pure resolution** (that is, $m_i(I) \geq M_{i-1}(I)$ for all i) [12], I is a stable or squarefree strongly stable ideal [12], I is a codimension 2 ideal [9, 12, 25], and I is a codimension 3 Gorenstein ideal [19].

We investigate Conjecture 1.1 for squarefree monomial ideals or, equivalently, face ideals of simplicial complexes. If Δ is a simplicial complex on the vertex set $[n] = \{1, 2, \ldots, n\}$, then its **face ideal** (or the **Stanley-Reisner ideal**), I_Δ, is the ideal generated by the squarefree monomials corresponding to non-faces of Δ, that is,

$$I_\Delta = \langle x_{i_1} \cdots x_{i_k} \,:\, \{i_1 < \cdots < i_k\} \notin \Delta \rangle,$$

and the **face ring** (or the **Stanley-Reisner ring**) of Δ is $\mathbf{k}[\Delta] := S/I_\Delta$ [28].

Various combinatorial and topological invariants of Δ are encoded in the algebraic invariants of I_Δ and vice versa [2, 28]. The Krull dimension of $\mathbf{k}[\Delta]$, $\dim \mathbf{k}[\Delta]$, and the topological dimension of Δ, $\dim \Delta$, are related by $\dim \mathbf{k}[\Delta] = \dim \Delta + 1$ and so

$$\operatorname{codim}(I_\Delta) = n - \dim \Delta - 1.$$

The Hilbert series of $\mathbf{k}[\Delta]$ is determined by knowing the number of faces in each dimension. Specifically, let f_i be the number of i-dimensional faces. By convention, the empty set is the unique face of dimension minus one. Then,

$$\sum_{i=0}^{\infty} \dim_{\mathbf{k}} \mathbf{k}[\Delta]_i \lambda^i = \frac{h_0 + h_1\lambda + \cdots + h_d\lambda^d}{(1-\lambda)^d},$$

where, $\mathbf{k}[\Delta]_i$ is the i-th graded component of $\mathbf{k}[\Delta]$, $d = \dim \Delta + 1 = \dim \mathbf{k}[\Delta]$, and

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}. \tag{1}$$

The multiplicity $e(\mathbf{k}[\Delta])$ equals the number of top-dimensional faces of Δ which in turn is $h_0 + \cdots + h_d$. The minimal and maximal shifts have the following interpretation in terms of the reduced homology:

$$M_i(I_\Delta) = \max\{|W| : W \subseteq [n] \text{ and } \tilde{H}_{|W|-i-1}(\Delta_W; \mathbf{k}) \neq 0\}, \tag{2}$$

$$m_i(I_\Delta) = \min\{|W| : W \subseteq [n] \text{ and } \tilde{H}_{|W|-i-1}(\Delta_W; \mathbf{k}) \neq 0\}. \tag{3}$$

(Here Δ_W denotes the **induced subcomplex** of Δ whose vertex set is W, and $\tilde{H}(\Delta_W; \mathbf{k})$ stands for the reduced simplicial homology of Δ_W with coefficients in $\mathbf{k}$. The above expressions for M_i and m_i follow easily from Hochster's formula on the Betti numbers $\beta_{i,j}(I_\Delta)$ [28, Theorem II.4.8].) Thus, for face ideals, Conjecture 1.1 can be considered as a purely combinatorial-topological statement. As we will see below, the upper bound part of the conjecture is closely related to the celebrated Upper Bound Theorem for polytopes and Gorenstein* complexes [27].

In this paper we prove Conjecture 1.1 for the face rings of the following classes of simplicial complexes.

- Matroid complexes. A simplicial complex is called a **matroid complex** if it is **pure** (that is, all its maximal under inclusion faces have the same dimension) and all its induced subcomplexes are pure.
- One- and two-dimensional complexes.
- Three- and four-dimensional Gorenstein complexes.

The first result about matroid complexes has a flavor similar to that of squarefree strongly stable ideals, while the last two results complement the fact that the multiplicity conjecture holds for codimension 2 ideals and for codimension 3 Gorenstein ideals.

Recall that a simplicial complex is called **Gorenstein** over $\mathbf{k}$ if its face ring is Gorenstein. Similarly, a simplicial complex Δ is said to be **Cohen-Macaulay over k** (CM, for short) if its face ring $\mathbf{k}[\Delta]$ is Cohen-Macaulay. A simplicial complex is **q-Cohen-Macaulay** (q-CM, for short) if for every set $U \subset [n]$, $0 \leq |U| \leq q-1$, the induced subcomplex $\Delta_{[n]-U}$ (that is, the complex obtained from Δ by removing all vertices in U) is a CM complex of the same dimension as Δ. Sometimes 2-CM complexes are called **doubly CM complexes**.

By Reisner's criterion [24], Δ is CM if and only if

$$\tilde{H}_i(\operatorname{lk} F; \mathbf{k}) = 0 \qquad \text{for all } F \in \Delta \text{ and } i < \dim \Delta - |F|.$$

Here, $\operatorname{lk} F = \{G \in \Delta \ : \ F \cap G = \emptyset,\ F \cup G \in \Delta\}$ is the **link of face F in Δ** (e.g., $\operatorname{lk} \emptyset = \Delta$). Thus, a 1-dimensional complex (that is, a graph) is CM if and only if it is connected. Moreover, a 1-dimensional complex is q-CM if it is q-connected in the usual graph-theoretic sense. A 0-dimensional complex on n vertices is n-CM.

Theorem 3.6 verifies the lower-bound part of Conjecture 1.1 for the face rings of the following classes of simplicial complexes

- 2-CM $(d-1)$-dimensional simplicial complexes on n vertices whose 1-skeleton is at most $(n-d+1)$-connected.
- d-CM $(d-1)$-dimensional complexes.

The restriction that the 1-skeleton is at most $(n-d+1)$-connected is a rather mild one: it means that there exists a subset $W \subset [n]$ of size $d-1$ such that Δ_W is disconnected. This condition is satisfied, for instance, by the order complex of an arbitrary graded poset of rank d with $\geq d^2$ elements as well as numerous other classes of complexes (see the last paragraph of Section 3).

The key notion in the proof of the lower bound part of Conjecture 1.1 is that of the **CM-connectivity sequence of a CM complex**. Recall that the i-skeleton of a simplicial complex Δ, $\operatorname{Skel}_i(\Delta)$, is the collection of all faces of Δ of dimension $\leq i$.

Definition 1.2 *For a CM simplicial complex Δ define $q(\Delta) := \max\{q \ : \ \Delta \text{ is } q\text{-CM}\}$, and $q_i = q_i(\Delta) := q(\operatorname{Skel}_i(\Delta))$ for $0 \leq i \leq \dim \Delta$. The sequence $(q_0, \ldots, q_{\dim(\Delta)})$ is called the* **CM-connectivity sequence** *of Δ.*

A recent result of Fløystad [5, Corollary 2.5] implies that the CM-connectivity sequence is strictly decreasing. (See Proposition 3.2 and Remark 3.3 for a new and very short proof of this result.) Thus for a $(d-1)$-dimensional CM complex Δ on n vertices,

$$1 \leq q_{d-1} < q_{d-2} < \cdots < q_1 < q_0 = n.$$

On the other hand, it follows easily from the minimality of the resolution (see [3, Proposition 1.1]) that for any $(d-1)$-dimensional complex Δ on n vertices

$$2 \leq m_1(I_\Delta) < m_2(I_\Delta) \cdots < m_{n-d}(I_\Delta) \leq n.$$

This suggests that there may be a connection between the CM-connectivity sequence and the sequence of the minimal shifts. Establishing such a connection is one of the two main ingredients of our proofs.

Theorem 1.3 *Let Δ be a CM complex on n vertices, and let $(q_0, \ldots, q_{d-1})$ be the CM-connectivity sequence of Δ, where $d-1$ is the dimension of Δ. Then*

$$[n] - \{m_1(I_\Delta), \ldots, m_{n-d}(I_\Delta)\} = \{n - q_0 + 1, n - q_1 + 1, \ldots, n - q_{d-1} + 1\},$$

and hence

$$\frac{\prod_{i=1}^{n-d} m_i(I_\Delta)}{(n-d)!} = \frac{n!}{(n-d)!} \cdot \frac{\prod_{i=1}^{n-d} m_i(I_\Delta)}{n!} = \frac{n(n-1)\cdots(n-(d-1))}{(n-q_0+1)\cdots(n-q_{d-1}+1)}.$$

In view of this theorem, we refer to $n - q_i + 1$ as the ***i*-th skip** in the m-sequence.

The second ingredient is the following purely combinatorial fact.

Theorem 1.4 *Let Δ be a CM $(d-1)$-dimensional complex, and let $(q_0, \ldots, q_{d-1})$ be its CM-connectivity sequence. Then*

$$e(\mathbf{k}[\Delta]) = f_{d-1}(\Delta) \geq \frac{q_0 q_1 \cdots q_{d-1}}{d!}.$$

Combining the last two theorems we obtain that the lower bound part of Conjecture 1.1 holds for all CM complexes whose CM-connectivity sequence satisfies

$$\frac{n(n-1)\cdots(n-(d-1))}{(n-q_0+1)\cdots(n-q_{d-1}+1)} \leq \frac{q_0 q_1 \cdots q_{d-1}}{d!}. \tag{4}$$

We then work out which sequences satisfy this inequality. (Not all decreasing sequences $\{q_i : 0 \leq i \leq d-1\}$ satisfy (4) as a sequence of the form $q_i = n - i$ if $0 \leq i \leq d-2$ and $1 \leq q_{d-1} \leq d-1$ shows.)

When $\mathbf{k}[\Delta]$ has a pure resolution, the combination of Huneke and Miller's formula for rings with pure resolutions and Theorems 1.3, and 1.4 puts a strong restriction on the CM-connectivity sequence of Δ. Indeed, $(q_0, \ldots, q_{d-1})$ must satisfy

$$\frac{q_0 \cdots q_{d-1}}{d!} \leq \frac{n(n-1)\cdots(n-d+1)}{(n-q_0+1)\cdots(n-q_{d-1}+1)}.$$

For instance, when $d = 2$, it is immediate that the only possible values for q_1 are $1, 2$ or $n-1$. Each of these values does occur as trees, circuits and complete graphs all have pure resolutions [4].

It is worth mentioning that there is a more recent conjecture [13, 20] asserting that if for a ring S/I, the multiplicity of S/I equals the lower bound or the upper bound of Conjecture 1.1, then S/I is Cohen-Macaulay and has a pure resolution. For the case of Cohen-Macaulay rings with a quasi-pure resolution this conjecture was established in [13, Theorem 3.4]. Here we verify this conjecture in all the cases where we are able to prove the original multiplicity conjecture (Conjecture 1.1) with the exception of the upper bound for 2-dimensional complexes.

The structure of the paper is as follows. Section 2 is devoted to the class of matroid complexes. In Section 3 we prove Theorems 1.3 and 1.4 and apply them to 2-CM complexes with $q_1 \leq n-d+1$ and d-CM complexes. In Section 4 we treat one- and two-dimensional complexes. Finally in Section 5 we discuss three- and four-dimensional Gorenstein complexes.

2 Matroid complexes

In this section we verify Conjecture 1.1 (together with the treatment of equality) for matroid complexes, namely we prove the following theorem. Here and in the rest of the paper we abuse notation and write $m_i(\Delta)$ and $M_i(\Delta)$ instead of $m(I_\Delta)$ and $M_i(I_\Delta)$, respectively.

Theorem 2.1 *Let Δ be a $(d-1)$-dimensional matroid complex on n vertices. Then*

$$(\prod_{i=1}^{n-d} m_i(\Delta))/(n-d)! \leq f_{d-1}(\Delta) \leq (\prod_{i=1}^{n-d} M_i(\Delta))/(n-d)!.$$

Moreover, if one of the bounds is achieved, then $\mathbf{k}[\Delta]$ has a pure resolution.

We start by reviewing necessary background on matroid complexes. A complex Δ on the vertex set $[n]$ is a matroid complex if for every subset $\emptyset \subseteq W \subseteq [n]$, the induced subcomplex Δ_W is pure. Equivalently (see [22, Proposition 2.2.1]), a matroid complex is a complex that consists of the independent sets of a matroid.

The following lemma summarizes several well known properties of matroid complexes.

Lemma 2.2 *Let Δ be a matroid complex. Then*

1. *Every induced subcomplex of Δ is a matroid complex.*
2. *Δ is CM. Moreover, Δ is the join of a simplex and a 2-CM complex.*
3. *Δ is either a cone or has a non-vanishing top homology.*

Proof: Part 1 is obvious from the definition of a matroid complex. For Part 2 see Proposition III.3.1 and page 94 of [28]. Part 3 is a consequence of Part 2 and the fact that a 2-CM complex has a nonvanishing top homology (see top of page 95 in [28]). $\square$

Another fact we need for the proof of Theorem 2.1 is

Lemma 2.3 *Let Δ be a $(d-1)$-dimensional complex on the vertex set $[n]$. Then*

1. *$M_{n-d}(\Delta) = n$ unless $\tilde{H}_{d-1}(\Delta; \mathbf{k}) = 0$, in which case $M_{n-d}(\Delta) < n$.*
2. *$M_i(\Delta) = \max\{M_i(\Delta_{[n]-x}) : x \in [n]\}$ and $m_i(\Delta) = \min\{m_i(\Delta_{[n]-x}) : x \in [n]\}$ for all $1 \le i < n-d$.*
3. *If Δ is 2-CM, then $m_{n-d}(\Delta) = M_{n-d}(\Delta) = n$.*

Proof: Parts 1 and 2 are immediate from equations (2) and (3). Part 3 follows from [28, Proposition III.3.2(e)]. $\square$

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1: The proof is by induction on n. The assertion clearly holds if $n = 1$ or $n = 2$. If $n > 2$, then by Lemma 2.2 either Δ is a cone with apex x or $\tilde{H}_{d-1}(\Delta; \mathbf{k}) \neq 0$. In the former case, $\Delta_{[n]-x}$ is a $(d-2)$-dimensional matroid complex on $n-1$ vertices. Since in this case $f_{d-1}(\Delta) = f_{d-2}(\Delta_{[n]-x})$, and

$$M_i(\Delta) = M_i(\Delta_{[n]-x}) \text{ and } m_i(\Delta) = m_i(\Delta_{[n]-x}) \ \forall\ 1 \le i \le n-d,$$

all parts of the theorem follow from the induction hypothesis on $\Delta_{[n]-x}$. In the latter case, each of $\Delta_{[n]-x}$ is a $(d-1)$-dimensional matroid complex on $n-1$ vertices. Thus we have

$$\begin{aligned} f_{d-1}(\Delta) &= \frac{1}{n-d} \sum_{x\in[n]} f_{d-1}(\Delta_{[n]-x}) \le \frac{1}{n-d} \cdot \sum_{x\in[n]} \frac{\prod_{i=1}^{n-1-d} M_i(\Delta_{[n]-x})}{(n-1-d)!} \\ &\le \frac{1}{(n-d)!} \cdot n \cdot \prod_{i=1}^{n-1-d} \max_{x\in[n]} M_i(\Delta_{[n]-x}) = \frac{\prod_{i=1}^{n-d} M_i(\Delta)}{(n-d)!}. \end{aligned}$$

In the above equation, the first step is implied by the fact that every $(d-1)$-dimensional face has d vertices, and hence is contained in exactly $n-d$ of the complexes $\Delta_{[n]-x}$. The second step is by induction hypothesis on $\Delta_{[n]-x}$, and the last step is an application of Parts 1 and 2 of Lemma 2.3. Moreover, if equality $f_{d-1}(\Delta) = (\prod_{i=1}^{n-d} M_i(\Delta))/(n-d)!$ holds, then all the inequalities in the above equation are equalities. Hence

$$f_{d-1}(\Delta_{[n]-x}) = \frac{\prod_{i=1}^{n-d-1} M_i(\Delta_{[n]-x})}{(n-d-1)!} \text{ and } M_i(\Delta_{[n]-x}) = M_i(\Delta) \quad \forall x \in [n], 1 \le i \le n-d-1.$$

The induction hypothesis on $\Delta_{[n]-x}$ then yields $M_i(\Delta) = M_i(\Delta_{[n]-x}) = m_i(\Delta_{[n]-x})$ for all $x \in [n]$, and so, by Part 2 of Lemma 2.3, $M_i(\Delta) = m_i(\Delta)$ for $i \le n-d-1$. In addition, $M_{n-d}(\Delta) = m_{n-d}(\Delta)$ as Δ is 2-CM. Thus, $\mathbf{k}[\Delta]$ has a pure resolution.

The proof of the lower bound (together with the treatment of equality) is completely analogous and is omitted. $\square$

We close this section with several remarks.

Remark 2.4 Translating the circuit axiom for matroids into commutative algebra leads to the following algebraic characterization:

- A proper squarefree monomial ideal I is the face ideal of a matroid complex if and only if for every pair of monomials $\mu_1, \mu_2 \in I$ and for every i such that x_i divides both μ_1 and μ_2, the monomial $\mathrm{lcm}\,(\mu_1, \mu_2)/x_i$ is in I as well.

We refer to such an ideal as a **matroid ideal**. In the commutative algebra literature the term "matroid ideal" is sometimes used for the ideal whose minimal generators correspond to the bases of a matroid. Our notion of a matroid ideal is different as in our case the ideal is generated by the circuits of the matroid rather than by its bases.

The above definition of a matroid ideal is reminiscent of that of a squarefree strongly stable ideal (that is, the face ideal of a shifted complex). In fact, matroid complexes and shifted complexes share certain properties. For instance, an induced subcomplex of a shifted complex is shifted (cf, Lemma 2.2(1)), and if Δ has a vanishing top homology and is a shifted complex on $[n]$ with respect to the ordering $1 \succ 2 \succ \cdots \succ n$, then Δ is a subcomplex of the cone over $\Delta_{[n-1]}$ with apex n (cf, Lemma 2.2(3)). Thus, the same reasoning as above provides a new simple proof of the upper-bound part of the multiplicity conjecture for squarefree strongly stable ideals, the result originally proved in [12].

Remark 2.5 The same argument as in the proof of Theorem 2.1 shows that if the multiplicity upper bound conjecture holds for all complexes of dimension $< d-1$ and for all $(d-1)$-dimensional complexes with vanishing top homology, then it holds for all $(d-1)$-dimensional complexes.

Remark 2.6 In view of Theorem 2.1 it is natural to ask for which matroid complexes Δ, does $\mathbf{k}[\Delta]$ have a pure resolution. Using either the methods of Theorem 3.11 of [23], or Lemmas 2.2, 2.3, (2), (3), and the basic matroid properties of duality, it is not hard to show that $k[\Delta]$ has a pure resolution if and only if Δ is the matroid dual of a perfect matroid design. Perfect matroid designs are matroids such that for every r the cardinality of a flat of rank r is some fixed number $f(r)$. Thus, face rings of duals of affine geometries and projective geometries provide a large collection of examples of squarefree monomial ideals with a pure resolution.

3 Lower bounds and the connectivity sequence

As noted in the introduction, the key to the lower bounds are Theorems 1.3 and 1.4. Before proving these, we recall an alternative characterization of Cohen-Macaulay complexes due to Hochster.

Theorem 3.1 *[15] Let Δ be $(d-1)$-dimensional. Then Δ is CM if and only if for every subset W of vertices*

$$\tilde{H}_p(\Delta_W; \mathbf{k}) = 0, \text{ for all } p + (n - |W|) < d - 1.$$

Thus, $\mathrm{Skel}_j(\Delta)$ *is q-CM if and only if*

$$\tilde{H}_p(\Delta_W;\mathbf{k})=0,\ \textit{for all } p+(n-q+1-|W|)\le j-1.$$

We also need the following fact.

Proposition 3.2 *If* Δ' *is a* j*-dimensional CM complex, then* $\mathrm{Skel}_{j-1}(\Delta')$ *is 2-CM. Hence if* Δ' *is q-CM complex, then its codimension one skeleton is* $(q+1)$*-CM.*

Proof: By a result of Hibi [14], the codimension one skeleton of a CM complex is level. In addition, $\mathrm{Skel}_{j-1}(\Delta')$ has a nonvanishing top homology, as every j-face of Δ' is attached to a $(j-1)$-cycle of $\mathrm{Skel}_{j-1}(\Delta')$. Thus the assertion on 2-CM follows from the last paragraph of [28, p. 94]. For the second part apply the first one to Δ' with $q-1$ points removed. □

Remark 3.3 *A corollary of the above is the previously mentioned result of Fløystad [5], that for Cohen-Macaulay* Δ *with* $\dim\Delta=d-1$, $q_{d-1}<q_{d-2}<\cdots<q_1<q_0$.

We are now ready to prove Theorems 1.3 and 1.4. For convenience, we repeat their statements.

Theorem 1.3. *Let* Δ *be a CM complex on* n *vertices, and let* $(q_0,\ldots,q_{d-1})$ *be the CM-connectivity sequence of* Δ*, where* $d-1$ *is the dimension of* Δ*. Then*

$$[n]-\{m_1(\Delta),\ldots,m_{n-d}(\Delta)\}=\{n-q_0+1,n-q_1+1,\ldots,n-q_{d-1}+1\},$$

and hence

$$\frac{\prod_{i=1}^{n-d}m_i(\Delta)}{(n-d)!}=\frac{n!}{(n-d)!}\cdot\frac{\prod_{i=1}^{n-d}m_i(\Delta)}{n!}=\frac{n(n-1)\cdots(n-(d-1))}{(n-q_0+1)\cdots(n-q_{d-1}+1)}. \tag{5}$$

Proof: The m-sequence is a strictly increasing sequence of length $n-d$ of integers contained in $[1,n]$. Hence, there are d numbers skipped which we denote by $s_0<s_1<\cdots<s_{d-1}$. We must prove that $q_j=n-s_j+1$. We argue by induction on j. For $j=0$ this follows immediately from the fact that $m_1\ge 2$ and $q_0=n$.

Let $m_i'=m_i-i-1$. So, m_i' is the dimension in which $\tilde{H}_{|W_i|-i-1}(\Delta_{W_i};\mathbf{k})$ is nonzero, where W_i is a subset of vertices of cardinality m_i. Since the m-sequence is strictly increasing, the m'-sequence is nondecreasing. Define t_j to be the largest i such that $m_i'<j$. With this definition, $s_j=t_j+j+1$. (See Example 3.4 below on how these invariants relate.)

Note that since $m_i'\ge j$ for all $i\ge t_j+1$, there can be no subsets W of the vertex set with $|W|\ge(t_j+1)+j=s_j$ and $\tilde{H}_{j-1}(\Delta_W;\mathbf{k})\neq 0$ (for all $j=0,1,\ldots,d-1$). Thus for a fixed j, there is no subset W with $|W|\ge s_j$ and $\tilde{H}_{j-1}(\Delta_W;\mathbf{k})\ne 0$, no subset W with $|W|\ge s_j-1(\ge s_{j-1})$ and $\tilde{H}_{j-2}(\Delta_W;\mathbf{k})\neq 0$, etc. Theorem 3.1 then implies that $\mathrm{Skel}_j(\Delta)$ is $(n-s_j+1)$-CM, that is, $q_j\ge n-s_j+1$.

Now if $j-1$ appears as some m_i', then $j-1=m_{t_j}'$, $m_{t_j}=s_j-1$, and $\tilde{H}_{j-1}(\Delta_{W_{t_j}})\neq 0$. Hence we also have $q_j\le n-|W_{t_j}|=n-s_j+1$, and so $q_j=n-s_j+1$. What happens if $j-1$ does not appear in the m'-sequence? In this case $s_j=s_{j-1}+1$, and we infer from Remark 3.3 and the induction hypothesis that $q_j\le q_{j-1}-1=(n-s_{j-1}+1)-1=n-s_j+1$ which again yields $q_j=n-s_j+1$. □

Example 3.4 *In this example $n = 19$ and $d = 9$.*

i	1	2	3	4	5	6	7	8	9	10
m_i	2	3	4	6	7	11	13	16	17	18
m'_i	0	0	0	1	1	4	5	7	7	7

j	t_j	s_j	q_j
0	0	1	19
1	3	5	15
2	5	8	12
3	5	9	11
4	5	10	10
5	6	12	8
6	7	14	6
7	7	15	5
8	10	19	1

Theorem 1.4. *Let Δ be a CM $(d-1)$-dimensional complex, and let $(q_0, \ldots, q_{d-1})$ be its CM-connectivity sequence. Then*

$$e(\mathbf{k}[\Delta]) = f_{d-1}(\Delta) \geq \frac{q_0 q_1 \cdots q_{d-1}}{d!}.$$

Proof: The proof is by induction on d, with the initial case $d = 0$ being self-evident. Since links of q-CM-complexes are q-CM [1] and the i-skeleton of the link of a vertex v is the link of v in the $(i+1)$-skeleton of Δ, $q_i(\mathrm{lk}\, v) \geq q_{i+1}(\Delta)$. Therefore the induction hypothesis implies that

$$f_{d-2}(\mathrm{lk}\, v) \geq \frac{q_1 \cdots q_{d-1}}{(d-1)!}.$$

Summing up over all $n = q_0$ vertices finishes the proof. □

Another theorem we will make a frequent use of is the following result. Its first part is [12, Theorem 1.2] and its second part is [13, Theorem 3.4]

Theorem 3.5 *If $\mathbf{k}[\Delta]$ is Cohen-Macaulay and has a quasi-pure resolution, then $\mathbf{k}[\Delta]$ satisfies the multiplicity conjecture. Furthermore, if the multiplicity of $\mathbf{k}[\Delta]$ equals the lower bound or the upper bound of the multiplicity conjecture, then $\mathbf{k}[\Delta]$ has a pure resolution.*

With Theorems 1.3, 1.4, and 3.5 in hand we are ready to discuss 2-CM and d-CM complexes. We remark that although the M-sequence need not be strictly increasing in general, it does strictly increase if Δ is CM. (This follows from the well-known fact that the dual of the resolution of a CM-algebra up to a shift resolves the canonical module of the algebra, and hence the M-sequence of the algebra is determined by the m-sequence of its canonical module; see e.g. [3, Proposition 1.1].) Thus in the CM case we can also talk about skips in the M-sequence.

Theorem 3.6 *Let Δ be a $(d-1)$-dimensional complex. If Δ is 2-CM with $q_1 \le n-d+1$ or Δ is d-CM, then $\mathbf{k}[\Delta]$ satisfies the multiplicity lower bound conjecture. Furthermore, if the bound is achieved, then $\mathbf{k}[\Delta]$ has a pure resolution.*

Proof: By virtue of Theorems 1.3 and 1.4, to prove the multiplicity lower bound conjecture it suffices to verify that $\frac{n(n-1)\cdots(n-(d-1))}{(n-q_0+1)\cdots(n-q_{d-1}+1)} \le \frac{q_0 q_1 \cdots q_{d-1}}{d!}$, or equivalently (via $q_0 = n$) that

$$d! \prod_{i=1}^{d-1} (n-i) \le \prod_{i=1}^{d-1} q_i (n - q_i + 1). \tag{6}$$

Assume first that Δ is 2-CM with $q_1 \le n-d+1$. Then $n-d+1 \ge q_1 > q_2 > \cdots > q_{d-1} \ge 2$ which implies that

$$q_{d-i} \in [i+1, n-d+1] \subseteq [i+1, n-i] \quad \forall\ 1 \le i \le d-1.$$

Since $f(q) := q(n-q+1)$ is a concave function of q symmetric about $q = (n+1)/2$, it follows that $q_{d-i}(n - q_{d-i} + 1) \ge (i+1)(n-i)$ for all $1 \le i \le d-1$. Multiplying these inequalities over all $1 \le i \le d-1$ yields (6).

When can the bound $f_{d-1} = (\prod_{i=1}^{n-d} m_i)/(n-d)!$ be achieved? Since for all $i < d-1$, $q_{d-i} \in [i+1, n-d+1] \subset [i+1, n-i)$, analysis of the above inequalities and the proof of Theorem 1.4 reveals that this can happen only if $q_1 \in \{d, n-d+1\}$, $q_{d-i} = i+1$ for $i < d-1$, and $f_{d-3}(\operatorname{lk} E) = (q_2 \cdots q_{d-1})/(d-2)! = d-1$ for every edge E. As $\operatorname{lk} E$ is a $(d-3)$-dimensional 2-CM complex, the latter condition implies that for every edge E, $\operatorname{lk} E$ is the boundary of a $(d-2)$-simplex, which in turn implies that Δ itself is the boundary of a d-simplex as long as $d > 3$, and hence that $\mathbf{k}[\Delta]$ has a pure resolution. This completes the treatment of equality in the $d > 3$ case. If $d = 2$ then the resolution is always quasi-pure and the result follows from Theorem 3.5. What if $d = 3$? Then either (i) $q_2 = 2$, $q_1 = 3$, and $f_1(\operatorname{lk} v) = 2 \cdot 3/2! = 3$ for every vertex v or (ii) $q_2 = 2$ and $q_1 = n-2$. In the former case, the link of each vertex must be the boundary of a 2-simplex, and hence Δ itself must be the boundary of a 3-simplex. In the latter case the 1^{st} skip in the m-sequence is $s_1 = 3$. Since the 2^{nd} (and the last) skip in the M-sequence is at least 3, the resolution is quasi-pure, and we are done by Theorem 3.5.

We now turn to the case of d-CM Δ. In this case $d \le q_{d-1} < \cdots < q_2 < q_1 \le n-1$. Thus $q_i \in [d, n-i] \subseteq [i+1, n-i]$, and the same computation as in the 2-CM case implies (6). Moreover, if the multiplicity lower bound is achieved, then $q_{d-1} \in \{d, n-d+1\}$ and $q_i = n-i$ for all $i < d-1$. The latter implies (via Theorem 1.3) that all integers from 1 to $d-1$ are skipped from the m-sequence, and hence that the resolution is quasi-pure. $\square$

The hypothesis $q_1 \le n-d+1$ is very mild as it only requires that there be some $(d-1)$-subset of vertices which is disconnected. For instance, (reduced) order complexes of all of the following posets satisfy this condition: face posets of 2-CM cell complexes with the intersection property (that is, intersection of any two faces is a face; this class includes face posets of all polytopes and face posets of all 2-CM simplicial complexes), geometric lattices, supersolvable lattices with nonzero Möbius function on every interval, rank selected subposets of any of these. We remark that the multiplicity upper bound conjecture for the order complexes of face posets of all simplicial complexes was very recently verified by Kubitzke and Welker [18].

4 One- and two-dimensional complexes

The goal of this section is to establish the multiplicity conjecture for one- and two-dimensional complexes. We start with the multiplicity upper bound conjecture. This requires the following strengthening of Theorem 3.5.

Definition 4.1 *The face ring* $\mathbf{k}[\Delta]$ *of a* $(d-1)$*-dimensional complex* Δ *is* **almost Cohen-Macaulay** *if the length of its minimal free resolution is at most* $n-d+1$. *(Equivalently,* $\mathbf{k}[\Delta]$ *is almost CM if the codimension one skeleton of* Δ *is CM.)*

Theorem 4.2 *[12, Theorem 1.5] If* $\mathbf{k}[\Delta]$ *is almost Cohen-Macaulay and has a quasi-pure resolution, then* $\mathbf{k}[\Delta]$ *satisfies the multiplicity upper bound conjecture.*

Theorem 4.3 *If* Δ *is one or two-dimensional, then* $\mathbf{k}[\Delta]$ *satisfies the multiplicity upper bound conjecture. Furthermore, if* Δ *is one-dimensional and*

$$f_1 = \frac{1}{(n-2)!}\prod_{i=1}^{n-2} M_i,$$

then $\mathbf{k}[\Delta]$ *is Cohen-Macaulay and has a pure resolution.*

Proof: First we consider $\dim\Delta = 1$. Using (2) and (3) we see that in this case $m_i, M_i \in \{i+1, i+2\}$, and so $\mathbf{k}[\Delta]$ has a quasi-pure resolution. Also, since $M_i \leq n$, it follows that $i \leq n-1$. Thus $\mathbf{k}[\Delta]$ is almost Cohen-Macaulay, and hence satisfies the multiplicity upper bound conjecture. If Δ is not connected and has t components, let Δ' be any connected complex obtained by adding $t-1$ edges to Δ. Then Δ' is connected, has the same 1-cycles as Δ and has more edges than Δ. Since the M_i only depend on the cardinality of the cycles, this reasoning shows that if equality occurs, then Δ is connected and hence Cohen-Macaulay. In this case, Theorem 3.5 implies that $\mathbf{k}[\Delta]$ has a pure resolution.

Now assume that Δ is 2-dimensional, connected, and $\tilde{H}_2(\Delta;\mathbf{k}) = 0$. Again, by (2) and (3), Δ is almost Cohen-Macaulay and has a quasi-pure resolution, and hence satisfies the multiplicity upper bound conjecture. When Δ is not connected we can add edges as above to get a connected complex with the same number of triangles and identical M_i. What if $\tilde{H}_2 \neq 0$? Remark 2.5 completes the proof in this case. $\square$

For one-dimensional complexes, i.e. graphs, the M_i encode the size of the smallest circuit. If the graph is acyclic, then the multiplicity conjecture gives the best possible bound, $f_1 \leq n-1$. When the graph contains a triangle Conjecture 1.1 says that $f_1 \leq n(n-1)/2$ which, in view of the complete graph, is best possible. However, for all other possible smallest circuit sizes the asymptotic upper bound for f_1 is known to be much less than that given by the multiplicity upper bound conjecture. Determination of the optimal upper bounds with this data is an area of ongoing research. Triangle free graphs have at most $n^2/4$ edges (this is Mantel's theorem [29, p. 30] — a special case of Turán's theorem). The maximum number of edges in a graph without triangles or squares is asymptotically bounded above by $n\sqrt{n-1}/2$ [8].

We now turn to the multiplicity lower bound conjecture. Recall that for a CM complex Δ, $h_i(\Delta) \geq 0$ for all i [27], [28, Cor. II.3.2].

Theorem 4.4 *If Δ is a 1 or 2-dimensional CM complex, then it satisfies the multiplicity lower bound conjecture. Furthermore, if*

$$f_{d-1} = \frac{1}{(n-d)!}\prod_{i=1}^{n-d} m_i, \tag{7}$$

then $\mathbf{k}[\Delta]$ has a pure resolution.

Proof: If Δ is one-dimensional, then there are only two skips in the m and M-sequence and 1 is the 0^{th} skip in both. Therefore, the minimal resolution is quasi-pure and Theorem 3.5 applies. So, we assume that $\dim \Delta = 2$. There are several cases to consider.

1. $\tilde{H}_2(\Delta; \mathbf{k}) = 0$. Under these conditions there are only two skips in the M-sequence (as $M_{n-3} < n$), and so $\mathbf{k}[\Delta]$ has a quasi-pure resolution.

2. $q_1 = n-1$. This is equivalent to the 1-skeleton being the complete graph on n vertices. Now the 1^{st} skip of the m-sequence is 2. Once again $\mathbf{k}[\Delta]$ has a quasi-pure resolution.

3. $q_2 \geq 2$. If $q_1 = n-1$, then the previous case holds. Otherwise, $q_1 \leq n-2$ and Theorem 3.6 applies.

4. $q_2 = 1$ and q_1 is 2 or 3. When $q_1 = 2$ Theorem 1.3 says that we must show that $f_2 \geq n-2$. However, for any two-dimensional CM complex, $h_0 = 1$, $h_1 = n-3$, and $h_2, h_3 \geq 0$. Hence $f_2 = h_0 + h_1 + h_2 + h_3 \geq n-2$. If $q_1 = 3$, then we must show that $f_2 \geq n-1$. If $f_2 < n-1$, then the h-vector must be $(1, n-3, 0, 0)$. This implies that $\dim_{\mathbf{k}} \tilde{H}_2(\Delta; \mathbf{k}) = h_3 = 0$ and that case 1 applies.

5. $q_2 = 1$ and $q_1 \geq 4$. By Theorem 1.4 applied to the 1-skeleton, $f_1 \geq nq_1/2$. Using (1), $h_2 \geq nq_1/2 - 2(n-3) - 3$ and hence,

   $$f_2 = 1 + (n-3) + h_2 + h_3 \geq \frac{n(q_1-2)+2}{2}.$$

   Comparing this to the multiplicity lower bound conjecture via (5),

   $$\frac{n(q_1-2)+2}{2} - \frac{(n-1)(n-2)}{n-q_1+1} = \frac{[n-q_1-1][(q_1-4)n+2]}{2(n-q_1+1)} \geq 0.$$

The last inequality holds since $q_1 \geq 4$ and any Cohen-Macaulay complex Δ with $\dim \Delta \geq 1$ has at least q_1+1 vertices. If (7) holds, then $n = q_1+1$ and the 1-skeleton is the complete graph. In addition, we must have $0 = h_3 (= \dim \tilde{H}_3(\Delta; \mathbf{k}))$ since this is implicit in the above estimate. Thus the skips for both the m and the M-sequences are $1, 2$ and n. $\square$

5 Gorenstein complexes

In this section we discuss the multiplicity conjecture for Gorenstein complexes. Since every such complex is the join of a simplex and a Gorenstein* complex, it suffices to treat the case of Gorenstein* complexes only. As in the previous section we start with the multiplicity upper bound conjecture. This will require the following facts.

Lemma 5.1 *Let Δ be a $(d-1)$-dimensional Gorenstein* complex on n vertices, and let $(q_0, \dots, q_{d-1})$ be its CM-connectivity sequence. If $d \geq 3$, then*

1. *$q_{d-1} = 2$ and $q_{d-2} \leq 5$.*
2. *The sequence $M_1, \dots M_{n-d}$ is strictly increasing and satisfies $[n] - \{M_1, \dots, M_{n-d}\} = \{q_{d-1}-1, q_{d-2}-1, \dots, q_1-1, q_0-1\}$. Hence*

$$\frac{\prod_{i=1}^{n-d} M_i}{(n-d)!} = \frac{n!}{(n-d)!} \cdot \frac{\prod_{i=1}^{n-d} M_i}{n!} = n \cdot \frac{(n-2)\cdots(n-(d-1))}{(q_1-1)\cdots(q_{d-2}-1)}.$$

Proof: If Δ is a 2-dimensional Gorenstein* complex on n vertices, then $f_1(\Delta) = 3n-6$. (This is a consequence of the Euler relation $f_0 - f_1 + f_2 = 2$ and the fact that every 2-face has exactly three edges, while every edge is contained in exactly two 2-faces.) Thus Δ has a vertex of degree ≤ 5, and hence the 1-skeleton of Δ is at most 5-connected. To see that $q_{d-2} \leq 5$ for a general $d > 3$, note that the link of a $(d-4)$-dimensional face F of Δ is a 2-dimensional Gorenstein* complex and that $q_{d-2}(\Delta) \leq q_1(\mathrm{lk}\, F)$ [1].

Since Δ is Gorenstein*, the minimal free resolution of I_Δ is self-dual. Thus

$$M_i + m_{n-d-i} = n \quad \text{for all } 0 \leq i \leq n-d,$$

where we set $m_0 = M_0 = 0$. This fact together with Theorem 1.3 implies Part 2. Finally, since $2 \leq M_1 < \dots < M_{n-d}$, we have $1 \in [n] - \{M_1, \dots M_{n-d}\}$, and so $q_{d-1} - 1 = 1$. $\square$

In view of the lemma, we refer to $q_{d-i-1} - 1$ $(0 \leq i \leq d-1)$ as the ***i*-th skip** in the M-sequence. Note that for a Gorenstein* complex, $q_0 - 1 = n - q_{d-1} + 1$, and so the $(d-1)$-th skip of the M-sequence coincides with that of the m-sequence. Note also that $q_{d-2} > 3$ implies that 2 is not skipped from the M-sequence, and hence that $M_1 = 2$.

Theorem 5.2 *If Δ is a three- or four-dimensional Gorenstein* complex, then $\mathbf{k}[\Delta]$ satisfies the multiplicity upper bound conjecture. Furthermore, if the bound is achieved then $\mathbf{k}[\Delta]$ has a pure resolution.*

Proof: We can assume without loss of generality that $\mathbf{k}[\Delta]$ does not have a quasi-pure resolution (since otherwise the result follows from Theorem 3.5).

First consider the case $d-1 = \dim\Delta = 4$. Since the resolution is not quasi-pure, either the first skip in the m-sequence occurs after the second skip in the M-sequence or the second skip in m occurs after the third skip in M, that is, either $n-q_1+1 > q_2-1$ or $n-q_2+1 > q_1-1$. Both of these expressions are equivalent to $(q_1-1)+(q_2-1) \leq n-1$ which together with $q_1 \geq q_2+1$ yields

$$(q_1-1)(q_2-1) \leq \frac{n}{2}\left(\frac{n}{2}-1\right).$$

Substituting this inequality along with $q_3 \leq 5$ in Part 2 of Lemma 5.1, we obtain

$$\frac{\prod_{i=1}^{n-5} M_i}{(n-5)!} = \frac{n(n-2)(n-3)(n-4)}{(q_1-1)(q_2-1)(q_3-1)} \geq (n-3)(n-4) = 2\binom{n-3}{2} \geq f_4,$$

where the last inequality is the Upper Bound Theorem for Gorenstein* complexes [27]. Furthermore, equality $(\prod_{i=1}^{n-5} M_i)/(n-5)! = f_4$ is impossible under above assumptions as it would imply $q_3-1=4$, and hence $M_1 = 2$, on one hand, and $f_4 = 2\binom{n-3}{2}$, and hence 2-neighborliness of the Gorenstein* complex, on the other.

Now assume that Δ is 3-dimensional. Then the assumption that $\mathbf{k}[\Delta]$ does not have a quasi-pure resolution implies that $q_1-1 < n-q_1+1$, and so $2(q_1-1) \leq n-1$. We treat separately two cases: $q_2 = 3$ and $q_2 \in \{4,5\}$. If $q_2 = 3$, then we have

$$\frac{\prod_{i=1}^{n-4} M_i}{(n-4)!} = \frac{n(n-2)(n-3)}{(q_1-1)(q_2-1)} \geq \frac{n(n-2)(n-3)}{n-1} > (n-2)(n-3) > \frac{n(n-3)}{2} \geq f_3,$$

where the first step is Lemma 5.1 and the last step is again the Upper Bound Theorem. The $q_2 \in \{4,5\}$ case requires a little bit more work. First we note that in this case $n-1 \geq 2(q_1-1) \geq 2q_2 \geq 8$, and so $n \geq 9$. Also, since $q_2 > 3$, we have $M_1 = 2$. Thus Δ is a 3-dimensional *flag* complex. In particular, the 1-skeleton of Δ is a 5-clique-free graph. Turán's theorem [29, Theorem 4.1] then implies that

$$f_1(\Delta) \leq \frac{n(n-\lfloor n/4 \rfloor)}{2} \leq \frac{n(n-2)}{2}.$$

It remains to note that for a 3-dimensional Gorenstein* complex $f_3 = f_1 - n$. (This is a consequence of the Euler relation $f_0 - f_1 + f_2 - f_3 = 0$ and the fact that every 3-face has exactly four 2-faces, while every 2-face is contained in exactly two 3-faces). Hence

$$f_3 = f_1 - n \leq \frac{n(n-4)}{2} < \frac{n(n-2)(n-3)}{2(n-1)} \leq \frac{n(n-2)(n-3)}{(q_1-1)(q_2-1)} = \frac{\prod_{i=1}^{n-4} M_i}{(n-4)!}.$$

$\square$

Generalizing case $d-1=3$, $q_2=3$ of the above proof leads to the following observation.

Proposition 5.3 *The multiplicity upper bound conjecture holds for the following classes of Gorenstein* complexes.*

1. $(d-1)$*-dimensional complexes with* $M_1 \geq \lfloor d/2 \rfloor + 1$*;*

2. $(d-1)$*-dimensional complexes on* n *vertices, where* $n - d = 4$ *and* $d \leq 22$*.*

Proof: To prove the first assertion note that $M_i \geq m_i = n - M_{n-d-i}$, and so $M_i + M_{n-d-i} \geq n$ for $i \geq 1$. A routine computation using these inequalities together with $\lfloor d/2 \rfloor + 1 \leq M_1 < M_2 < \cdots < M_{n-d} = n$ shows that the upper bound of the multiplicity conjecture is at least as large as the bound provided by the Upper Bound Theorem for Gorenstein* complexes [27] implying the result.

Similarly, for the second assertion one verifies that either $\mathbf{k}[\Delta]$ has a quasi-pure resolution or the upper bound of the multiplicity conjecture is at least as large as the bound of the Upper Bound Theorem. We omit the details. $\square$

Finally we discuss the lower bounds in the three- and four-dimensional cases.

Theorem 5.4 *If* Δ *is a* 3 *or* 4*-dimensional Gorenstein* complex, then it satisfies the multiplicity lower bound conjecture. Furthermore, if the lower bound is achieved then* $\mathbf{k}[\Delta]$ *has a pure resolution.*

Proof: Δ is Gorenstein*, hence 2-CM, and so as long as $q_1 \leq n - \dim \Delta$ Theorem 3.6 applies. Thus we assume for the rest of the proof that $q_1 \geq n - \dim \Delta + 1$.

By Theorem 3.5, we can also assume that $\mathbf{k}[\Delta]$ does not have a quasi-pure resolution which, as was observed in the proof of Theorem 5.2, is equivalent to

$$\begin{aligned} 2(q_1 - 1) &\leq n - 1 \qquad \text{if} \quad \dim \Delta = 3, \text{ and} \\ (q_1 - 1) + (q_2 - 1) &\leq n - 1 \qquad \text{if} \quad \dim \Delta = 4. \end{aligned} \tag{8}$$

This finishes the proof of the 3-dimensional case as the inequalities $q_1 \geq n - 2$ and $2(q_1 - 1) \leq n - 1$ imply $2(n-3) \leq 2(q_1 - 1) \leq n - 1$, and hence $n \leq 5$, on one hand, and $n - 1 \geq 2(q_1 - 1) \geq 2(q_3 + 1) = 6$ on the other.

Assume now $\dim \Delta = 4$. If $q_1 \geq n - 2$, then, by (8), $q_2 \leq 3$ which contradicts the fact that $q_2 \geq q_4 + 2 = 4$. Thus $q_1 = n - 3$, and $q_2 = 4$, $q_3 = 3$, $q_4 = 2$, and via Theorem 1.3 to complete the proof we must show that $f_4 > n(n-4)/4$.

By Theorem 1.4 applied to the 1-skeleton, $f_1 \geq nq_1/2 = n(n-3)/2$. Hence, by (1), $h_2 = f_1 - 4f_0 + 10 \geq (n^2 - 11n + 20)/2$. The Dehn-Sommerville relations [17] then yield

$$f_4 = 2(h_0 + h_1 + h_2) \geq 2[1 + (n-5) + (n^2 - 11n + 20)/2] = n^2 - 9n + 12.$$

This completes the proof if $n \geq 9$, since in this case $n^2 - 9n + 12 > n(n-4)/4$.

What if $n \leq 8$? Since $n - 3 = q_1 \geq q_2 + 1 = 5$, n must equal 8. Then $h_0 = 1$, $h_1 = 8 - 5 = 3$, and $h_2 > 0$. Thus $f_4 = 2(h_0 + h_1 + h_2) \geq 2[1 + 3 + 1] = 10 > 8(8-4)/4$, as desired. $\square$

References

[1] K. Baclawski, Cohen-Macaulay connectivity and geometric lattices, *European J. Combinatorics* **3** (1982), 293-305.

[2] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, **39**, Cambridge University Press, Cambridge, 1993.

[3] W. Bruns and T. Hibi, Stanley-Reisner rings with pure resolutions, *Comm. Algebra* **23** (1995), 1201–1217.

[4] W. Bruns and T. Hibi, Cohen-Macaulay partially ordered sets with pure resolutions, *European J. Combinatorics*, **19** (1998), 779–785.

[5] G, Fløystad, Cohen-Macaulay cell complexes, *Proceedings of the Conference on Algebraic and Topological Combinatorics*, Anogia 2005, to appear.

[6] C. A, Francisco, New approaches to bounding the multiplicity of an ideal, *J. Algebra* **299** (2006), 309-328.

[7] C. A. Francisco and H. Srinivasan, Multiplicity conjectures, *Lect. Notes Pure Appl. Math.*, to appear.

[8] K. Garnick, Y. H. H. Kwong and F. Lazebnik, Extremal graphs without three cycles or four cycles, *J. Graph Theory* **17** (1993), 633–645.

[9] L.H. Gold, A degree bound for codimension two lattice ideals, *J. Pure Appl. Algebra* **182** (2003), 201–207.

[10] L. Gold, H. Schenck, and H. Srinivasan, Betti numbers and degree bounds for some linked zero schemes, *J. Pure Appl. Algebra*, **210** (2007), 481-491.

[11] E. Guardo and A. Van Tuyl, Powers of complete intersections: graded Betti numbers and applications, *Illinois J. Math.* **49** (2005), 265–279.

[12] J. Herzog and H. Srinivasan, Bounds for multiplicities, *Trans. Amer. Math. Soc.* **350** (1998), 2879–2902.

[13] J. Herzog and X. Zheng, Notes on the multiplicity conjecture, *Collect. Math.* **57** (2006), no. 2, 211–226.

[14] T. Hibi, Level rings and algebras with straightening laws, *J. Algebra*, **117** (1988), 343–362.

[15] M. Hochster, Cohen-Macaulay rings, combinatorics, and simplicial complexes, *Ring Theory II* (Proc. Second Oklahoma Conf.), Dekker, NY, 1977, 171–223.

[16] C. Huneke and M. Miller, A note on the multiplicity of Cohen-Macaulay algebras with pure resolutions, *Canad. J. Math.* **37** (1985), 1149–1162.

[17] V. Klee, A combinatorial analogue of Poincare's duality theorem, *Canad. J. Math.* **16** (1964), 517–531.

[18] M. Kubitzke and V. Welker, The multiplicity conjecture for barycentric subdivisions, preprint, 2006, math.AC/0606274.

[19] J. Migliore, U. Nagel, and T. Römer, The multiplicity conjecture in low codimensions, *Math. Res. Lett.* **12** (2005), 731–747.

[20] J. Migliore, U. Nagel, and T. Römer, Extensions of the Multiplicity Conjecture, *Trans. Amer. Math. Soc.*, to appear.

[21] R. Miro-Roig, A note on the multiplicity of determinantal ideals, *J. Algebra* **299** (2006), no. 2, 714–724.

[22] G. Nicoletti and N. White, Axiom systems, in *Theory of matroids* (ed. by N. White), 29–44, Encyclopedia Math. Appl., vol. 26, Cambridge Univ. Press, Cambridge, 1986.

[23] I. Novik, A. Postnikov, and B. Sturmfels, Syzygies of oriented matroids, *Duke Mathematical Journal*, **111** (2002), 287-317.

[24] G. Reisner, Cohen-Macaulay quotients of polynomial rings, *Advances in Math.* **21** (1976), 30–49.

[25] T. Römer, Note on bounds for multiplicities, *J. Pure Appl. Algebra* **195** (2005), 113–123.

[26] H. Srinivasan, A note on the multiplicities of Gorenstein algebras, *J. Algebra* **208** (1998), 425–443.

[27] R. Stanley, The Upper Bound Conjecture and Cohen-Macaulay rings, *Studies in Appl. Math.* **54** (1975), 135–142.

[28] R. Stanley, *Combinatorics and Commutative Algebra*, Second Edition, Birkhäuser, Boston, 1996.

[29] J. H. van Lint and R. M. Wilson, *A Course in Combinatorics* Cambridge University Press, Cambridge, 1992.