Simplicial spheres with $g_k = 1$

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January 15, 2026

Abstract

For $d \geq 4$, Kalai (1987) characterized all simplicial $(d-1)$-spheres with $g_2 = 0$, and for $k \geq 2$ and $d \geq 2k$, Murai and Nevo (2013) characterized all simplicial $(d-1)$-spheres with $g_k = 0$. In addition, for $d \geq 4$, Nevo and Novinsky (2011) characterized all simplicial $(d-1)$-spheres with $g_2 = 1$. Motivated by these results, we characterize, for any $k \geq 2$ and $d \geq 2k+1$, all simplicial $(d-1)$-spheres with no missing faces of dimension larger than $d - k$ that satisfy $g_k = 1$. When $d = 2k$, we obtain a characterization of simplicial $(d-1)$-spheres with $g_k = 1$ and no missing faces of dimension greater than k, under the additional assumption that there exists at least one missing face of dimension k. Finally, for $k = 3$, we are able to remove this assumption and characterize all simplicial 5-spheres with no missing faces of dimension larger than 3 that satisfy $g_3 = 1$.

1 Introduction

What is the smallest number of edges that a $(d-1)$-dimensional simplicial sphere with f_0 vertices can have? When $d \geq 3$, the answer is given by the Lower Bound Theorem (LBT) [3, 4, 9], which asserts that $f_1 \geq df_0 - \binom{d+1}{2}$. The quantity $f_1 - df_0 + \binom{d+1}{2}$ is denoted by g_2. The inequality $g_2 \geq 0$ holds not only for simplicial spheres but for all normal pseudomanifolds; see [9, 6, 29]. Furthermore, the case $g_2 = 0$ is completely characterized: when $d \geq 4$, $g_2 = 0$ holds if and only if the normal $(d-1)$-pseudomanifold in question is the boundary complex of a stacked d-polytope.

Motivated by these results, Nevo and Novinsky [21] gave, for $d \geq 4$, a complete characterization of simplicial $(d-1)$-spheres with no missing faces of dimension larger than $d - 2$ that satisfy $g_2 = 1$. Since then, several additional results characterizing simplicial manifolds (or even normal pseudomanifolds) with small values of g_2 have appeared in the literature; see, for example, [32, 5].

In complete analogy with the LBT, for any $1 \leq k \leq d/2$, McMullen and Walkup [16] conjectured a lower bound on the number of $(k-1)$-faces, f_{k-1}, in terms of the face numbers $f_{k-2}, \ldots, f_0, f_{-1}$. Using the notion of the g-numbers—certain alternating weighted sums of the f-numbers—their conjecture asserts that for a simplicial $(d-1)$-sphere Δ and any $k \leq d/2$, we have $g_k(\Delta) \geq 0$. Moreover, equality $g_k = 0$ holds if and only if Δ is $(k-1)$-stacked. This conjecture is known as the Generalized Lower Bound Conjecture (GLBC). The inequality part of the GLBC for the

boundary complexes of simplicial polytopes was proved by Stanley [26]. Much more recently, these inequalities were established for all simplicial spheres (and even for a larger class of $\mathbb{Z}/2\mathbb{Z}$-homology spheres); see [1, 24, 2, 10]. The equality part of the GLBC was proved by Murai and Nevo [17].

The goal of this paper is to understand, for a given k, spheres without large missing faces that satisfy $g_k = 1$. Throughout the paper, we work with $\mathbb{Z}/2\mathbb{Z}$-homology spheres and often refer to them simply as spheres. In particular, this class contains all simplicial spheres. We denote by $S(j, d-1)$ the collection of $\mathbb{Z}/2\mathbb{Z}$-homology $(d-1)$-spheres all of whose missing faces have dimension at most j. For instance, $S(1, d-1)$ coincides with the class of flag $(d-1)$-spheres.

Denote by $\partial\sigma^i$ the boundary complex of an i-simplex. Our main results can be summarized as follows:

1. Let $d \geq 2k$, and let $\Delta \in S(d-k, d-1)$. Assume further that if $d = 2k$, then Δ has at least one missing k-face; that is, $\Delta \notin S(d-k-1, d-1) = S(k-1, 2k-1)$. Then $g_k(\Delta) = 1$ if and only if Δ is either the join of $\partial\sigma^{d-k}$ and a $(k-1)$-sphere, or the join of $\partial\sigma^j$ and $\partial\sigma^{d-j}$ for some $k < j \leq \lfloor d/2 \rfloor$; see Theorem 4.1.

2. In the case $2k = d = 6$, we give a complete characterization of spheres in $S(3, 5)$ with $g_3 = 1$: $\Delta \in S(3, 5)$ has $g_3 = 1$ if and only if Δ is either the join of $\partial\sigma^3$ and a 2-sphere, or the join of three copies of $\partial\sigma^2$; see Theorem 5.6. In particular, all such spheres are boundaries of simplicial polytopes.

For $k = 2$, our first result recovers the theorem of Nevo and Novinsky [21].[1] Our proofs rely on the theory of (higher) stress spaces developed by Lee [13] and by Tay, White, and Whiteley [30, 31]. More specifically, we establish a new version of the cone lemma that allows us to describe the supports of certain stresses (see Section 3), which, in turn, enables us to glean some information about the structure of the complex in question. Another tool we use is McMullen's integral formula [28, Proposition 2.3].

The structure of the paper is as follows. In Section 2, we review simplicial complexes and introduce the main object of the paper, $S(j, d-1)$, along with the requisite background on Stanley–Reisner rings and stress spaces; we also derive several corollaries concerning the g-numbers. In Section 3, we establish a version of the cone lemma, and discuss one of its applications—Lemma 3.3. Sections 4 and 5 are devoted to proving the two main theorems of the paper. We conclude in Section 6 with a discussion of related open problems.

2   Preliminaries

2.1   Simplicial complexes and face numbers

An (abstract) *simplicial complex* Δ with vertex set $V = V(\Delta)$ is a non-empty collection of subsets of V that is closed under inclusion and contains all singletons; that is, $\{v\} \in \Delta$ for all $v \in V$. An example of a simplicial complex on V is the collection of all subsets of V. When $|V| = d + 1$, this complex is a *d-simplex*, and we usually denote it by σ^d or by $\overline{V}$ when the vertex set of this simplex is important.

The elements of a simplicial complex Δ are called *faces* of Δ. A face τ of Δ has *dimension i* if $|\tau| = i + 1$; in this case we say that τ is an *i-face*. We usually refer to 0-faces as *vertices*, 1-faces as *edges*, and the maximal under inclusion faces as *facets*. For brevity, we denote a vertex by v, an

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[1]The condition that for $d = 4$, $\Delta \notin S(1, 3)$ is automatically satisfied, as all spheres in $S(1, 3)$ have $g_2 \geq 2$.

edge by uv, a 2-face by uvw, instead of $\{v\}$, $\{u,v\}$, and $\{u,v,w\}$ respectively. The *dimension of* Δ is $\max\{\dim\tau : \tau \in \Delta\}$. We say that Δ is *pure* if all facets of Δ have the same dimension.

A set $\tau \subseteq V$ is a *missing face* of Δ if τ is not a face of Δ, but every proper subset of τ is a face of Δ. In analogy with faces, a *missing i-face* is a missing face of size $i+1$. The collection of the missing faces of Δ, together with its vertex set, uniquely determines Δ.

Let τ be a face of Δ. The *star* and *link* of τ are defined as

$$\mathrm{st}(\tau) = \mathrm{st}(\tau,\Delta) = \{\sigma \in \Delta \ : \ \sigma \cup \tau \in \Delta\} \quad \text{and} \quad \mathrm{lk}(\tau) = \mathrm{lk}(\tau,\Delta) = \{\sigma \in \mathrm{st}(\tau) \ : \ \sigma \cap \tau = \emptyset\}.$$

When $\tau = v$ is a vertex, we also define $\Delta \backslash v = \{\sigma \in \Delta : v \notin \sigma\}$; this subcomplex of Δ is called the *antistar* of v.

A subcomplex of Δ is called *induced* if it is of the form $\Delta[W] = \{\tau \in \Delta : \tau \subseteq W\}$ for some $W \subseteq V(\Delta)$. The subcomplex of Δ consisting of all faces of Δ of dimension $\leq k$ is called the *k-skeleton* of Δ and is denoted $\mathrm{Skel}_k(\Delta)$; the 1-skeleton of Δ is also known as the *graph* of Δ. Finally, if Δ and Γ are two simplicial complexes on disjoint vertex sets, then their *join* is

$$\Delta * \Gamma = \{\sigma \cup \tau : \sigma \in \Delta, \tau \in \Gamma\}.$$

If Γ is a 0-simplex, that is, $\Gamma = \{v, \emptyset\}$, we write $\Delta * \Gamma = \Delta * v$ and call this complex the *cone* over Δ with apex v.

Let $\mathbb{F}$ be a field. A pure $(d-1)$-dimensional simplicial complex is an $\mathbb{F}$-*homology sphere* if, for every face σ (including the empty face), the link $\mathrm{lk}(\sigma)$ has the homology of a $(d-1-|\sigma|)$-dimensional sphere (over $\mathbb{F}$). Denote by $\|\Delta\|$ the geometric realization of Δ. We say that Δ is a *simplicial $(d-1)$-sphere* if $\|\Delta\|$ is homeomorphic to a $(d-1)$-dimensional sphere. It is known that a $\mathbb{Z}/2\mathbb{Z}$-homology sphere is always an $\mathbb{R}$-homology sphere; see [10, Lemma 2.1]. Moreover, a simplicial sphere is an $\mathbb{F}$-homology sphere for any field $\mathbb{F}$.

In this paper, we work with the class of $\mathbb{Z}/2\mathbb{Z}$-homology spheres, which we refer to simply as spheres.[2] Let Δ be a $(d-1)$-sphere. Denote by $f_i = f_i(\Delta)$ the number of i-faces of Δ and by $m_i = m_i(\Delta)$ the number of missing i-faces of Δ. In particular, $f_i = 0$ if $i > d-1$. Let

$$f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1}) \quad \text{and} \quad m(\Delta) = (m_1, m_2, \ldots, m_d)$$

be the *f-vector* and the *m-vector* of Δ, respectively. The *h-vector* of Δ, $h(\Delta) = (h_0, h_1, \ldots, h_d)$, is obtained from the f-vector by the following invertible linear transformation:

$$h_j = h_j(\Delta) = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{d-j} f_{i-1}(\Delta) \quad \text{for } 0 \leq j \leq d.$$

The *g-vector* of Δ, $g(\Delta) = (g_0, g_1, \ldots, g_{\lfloor d/2 \rfloor})$, is defined by letting $g_0 = 1$ and $g_j = h_j - h_{j-1}$ for $1 \leq j \leq \lfloor d/2 \rfloor$. When d is odd, we sometimes also consider $g_{\lceil d/2 \rceil}$.

The Dehn–Sommerville relations [11] assert that the h-vector of a $(d-1)$-sphere is symmetric: $h_i = h_{d-i}$ for all $0 \leq i \leq d$. In particular, it follows that if d is odd, then $g_{\lceil d/2 \rceil} = 0$.

Following the notation in [20], we define $S(i, d-1)$ as the set of $\mathbb{Z}/2\mathbb{Z}$-homology $(d-1)$-spheres all of whose missing faces have dimension $\leq i$. Recall that the clique complex of a graph—or equivalently, a complex with no missing faces of dimension > 1—is called *flag*. Hence $S(1, d-1)$ is

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[2]The main reason for working with this class is that, unlike the class of simplicial spheres, it is closed under taking links.

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the class of flag $(d-1)$-spheres. The following element of $S(i, d-1)$ deserves a special attention. Write $d = qi + r$, where q and r are (uniquely defined) integers satisfying $1 \le r \le i$, and let $K(i, d-1) = (\partial \sigma^i)^{*q} * \partial \sigma^r$, where $(\partial \sigma^i)^{*q}$ denotes the join of q copies of $\partial \sigma^i$. It is known that the sphere $K(i, d-1)$ simultaneously minimizes all the f- and h-numbers among all spheres in $S(i, d-1)$; see [7, 20]. Furthermore, $K(1, d-1)$, the boundary complex of the d-cross-polytope, simultaneously minimizes all the g-numbers among all flag PL $(d-1)$-spheres; see [23].

To close this section, we mention that, in analogy with simplicial spheres and $\mathbb{F}$-homology spheres, one can define *simplicial balls* and $\mathbb{F}$-*homology simplicial balls*. The link of any face τ of an $\mathbb{F}$-homology ball B is either an $\mathbb{F}$-homology sphere or an $\mathbb{F}$-homology ball. We call τ an *interior face* in the former case and a *boundary face* in the latter. A *minimal interior face* is an interior face that contains no other interior faces. The collection of all boundary faces of B forms an $\mathbb{F}$-homology sphere of dimension one less than that of B. This sphere is called the *boundary complex* of B and is denoted ∂B.

In this paper, we will only use $\mathbb{R}$-homology balls, which we often refer to simply as *balls*. An $\mathbb{R}$-homology $(d-1)$-sphere Δ is called $(i-1)$-*stacked* if there exists an $\mathbb{R}$-homology d-ball B with no interior faces of dimension $\le d-i$ and with $\partial B = \Delta$. Such a B is called an $(i-1)$-*stacked triangulation* of Δ.

2.2 The Stanley–Reisner ring

Let Δ be a $(d-1)$-dimensional simplicial complex with vertex set $V = V(\Delta)$. Let $\mathbb{F}$ be a field of characteristic zero, and let $X = \{x_v : v \in V\}$ be a set of variables, one for each vertex. Denote by $\mathbb{F}[X] = \mathbb{F}[x_v : v \in V]$ the polynomial ring over $\mathbb{F}$ in these variables. The *Stanley–Reisner ideal* of Δ is the ideal of $\mathbb{F}[X]$ generated by the monomials corresponding to missing faces of Δ:

$$I_\Delta = (x_{j_1} x_{j_2} \cdots x_{j_k} : \{j_1, \dots, j_k\} \text{ is a missing face of } \Delta).$$

The *Stanley–Reisner ring* (or *face ring*) of Δ is the quotient $\mathbb{F}[\Delta] := \mathbb{F}[X]/I_\Delta$. This is a graded ring. Its Hilbert series is given by $\left(\sum_{i=0}^d h_i(\Delta) t^i \right)/(1-t)^d$; see [27, Theorem II.1.4]. A sequence Θ of d linear forms $\theta_1, \dots, \theta_d$ in $\mathbb{F}[\Delta]$ is called a *linear system of parameters* (l.s.o.p) for $\mathbb{F}[\Delta]$ if the quotient ring $\mathbb{F}[\Delta]/(\theta_1, \dots, \theta_d)$ is a finite-dimensional $\mathbb{F}$-vector space.

In what follows, assume that Δ is a $\mathbb{Z}/2\mathbb{Z}$-homology $(d-1)$-sphere. The Stanley–Reisner ring of Δ is *Cohen–Macaulay* [25]; hence any l.s.o.p. Θ for $\mathbb{F}[\Delta]$ is a *regular sequence*, that is, for each $1 \le i \le d$, θ_i is a non-zero-divisor on $\mathbb{F}[\Delta]/(\theta_1, \dots, \theta_{i-1})$. Consequently, $\dim_{\mathbb{F}} \left(\mathbb{F}[\Delta]/(\Theta) \right)_i = h_i(\Delta)$ for all $0 \le i \le d$, where R_i denotes the i-th graded component of a graded ring R. The fact that any artinian reduction of the Stanley–Reisner ring of a $(d-1)$-sphere Δ is a Poincaré duality algebra then provides an alternative proof of the Dehn–Sommerville relations: $h_i(\Delta) = h_{d-i}(\Delta)$ for all $0 \le i \le d$.

In this paper, we take $\{a_{v,j} \in \mathbb{R} : v \in V, j \in [d]\}$ to be a set **algebraically independent** over $\mathbb{Q}$, and we let $\mathbb{F} = \mathbb{Q}(a_{v,j} : v \in V, j \in [d])$. The sequence $\Theta = (\theta_1, \dots, \theta_d)$, where $\theta_j := \sum_{v \in V} a_{v,j} x_v$ for $1 \le j \le d$, then forms an l.s.o.p. for $\mathbb{F}[\Delta]$; see [27, Theorem III.2.4]. The ring $\mathbb{F}[\Delta]/(\Theta)$ is called the *generic artinian reduction* of $\mathbb{F}[\Delta]$.

Let $c = \sum_{v \in V} x_v$. The following result is the celebrated (algebraic version of the) g-theorem; see [15, 26] for the case of simplicial polytopes and [1, 24, 2, 10], or, more specifically, [10, Theorem 1.3], for the case of spheres.

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Theorem 2.1. *Let Δ be a $(d-1)$-sphere, let $\mathbb{F}[\Delta]/(\Theta)$ be the generic artinian reduction of $\mathbb{F}[\Delta]$, and let $c = \sum_{v \in V} x_v$. Then, for every $0 \leq k \leq \lfloor d/2 \rfloor$, the map $\cdot c^{d-2k} : \big(\mathbb{F}[\Delta]/(\Theta)\big)_k \to \big(\mathbb{F}[\Delta]/(\Theta)\big)_{d-k}$ is an isomorphism. In particular, the map*

$$\cdot c : \big(\mathbb{F}[\Delta]/(\Theta)\big)_k \to \big(\mathbb{F}[\Delta]/(\Theta)\big)_{k+1}$$

is injective for all $0 \leq k \leq \lceil d/2 \rceil - 1$ and surjective for all $\lfloor d/2 \rfloor \leq k \leq d-1$. Consequently, $\dim_{\mathbb{F}} \big(\mathbb{F}[\Delta]/(\Theta, c)\big)_k = g_k(\Delta)$ for all $k \leq \lceil d/2 \rceil$.

In view of this theorem, $c = \sum_{v \in V} x_v$ is called the *canonical Lefschetz element*. The inequality part of the Generalized Lower Bound Theorem (GLBT, for short) then follows immediately:

Theorem 2.2. *If Δ is a $(d-1)$-sphere, then $g_k(\Delta) \geq 0$ for all $k \leq \lfloor d/2 \rfloor$.*

Since $\mathbb{F}[\Delta]/(\Theta, c)$ is a standard graded algebra, Macaulay's theorem [27, Theorem II.2.2] implies that the g-numbers also satisfy certain nonlinear inequalities. In this paper, we will only use the following consequence: if, for some $k < \lfloor d/2 \rfloor$, $g_k = 1$, then for all j with $k < j \leq \lfloor d/2 \rfloor$, we have $g_j \leq 1$; furthermore, if for some $k < \lfloor d/2 \rfloor$, $g_k = 0$, then for all j with $k < j \leq \lfloor d/2 \rfloor$, $g_j = 0$.

2.3 More results on the g-numbers

Now we collect a few additional results on the g-numbers of spheres. We start with the following elementary but extremely useful lemma, which applies to all pure simplicial complexes. It was established for simplicial polytopes by McMullen [14, p. 183] and for all pure simplicial complexes by Swartz [28, Proposition 2.3], and is known in the literature as *McMullen's integral formula*.

Lemma 2.3. *If Δ is a pure $(d-1)$-dimensional simplicial complex, then for $0 \leq k \leq \lfloor \frac{d-1}{2} \rfloor$,*

$$\sum_{v \in V(\Delta)} g_k(\mathrm{lk}(v)) = (k+1)g_{k+1}(\Delta) + (d+1-k)g_k(\Delta).$$

The following result addresses the equality part of the GLBT; see [17, 19]. Given a simplicial complex Δ, define

$$\Delta(j) = \{\tau \subseteq V(\Delta) : \mathrm{Skel}_j(\bar{\tau}) \subseteq \Delta\}.$$

Theorem 2.4. *Let $1 \leq k \leq d/2$ and let Δ be a $(d-1)$-sphere. Then $g_k(\Delta) = 0$ if and only if Δ is $(k-1)$-stacked. Furthermore, if $g_k(\Delta) = 0$, then $\Delta(d-k) = \Delta(k-1)$, and this complex is an $\mathbb{R}$-homology d-ball that provides a $(k-1)$-stacked triangulation of Δ.*

We will also use the following characterization of k-stackedness in terms of the m-numbers, proved in [18, Corollary 1.4]:

Proposition 2.5. *Let $d \geq 4$ and let Δ be a $(d-1)$-sphere. Then for $1 \leq k \leq \lfloor \frac{d}{2} \rfloor - 1$, Δ is k-stacked if and only if $g_k(\Delta) = m_{d-k}(\Delta)$.*

Proposition 2.5 was motivated by the following result of Nagel [19, Corollary 4.6(a)]

Proposition 2.6. *Let $d \geq 4$ and let Δ be a $(d-1)$-sphere. Then for all $1 \leq k \leq \lceil \frac{d}{2} \rceil - 1$, $g_k(\Delta) \geq m_{d-k}(\Delta)$.*

Remark 2.7. Let $k \leq d/2$ and assume that Δ is a $(d-1)$-sphere with $g_k = 0$. Then, by the part of Theorem 2.4 asserting that $\Delta(k-1) = \Delta(d-k)$, Δ has no missing j-faces for any $k \leq j \leq d-k$. On the other hand, since $\Delta(d-k)$ is d-dimensional (see Theorem 2.4) and therefore strictly contains Δ, it follows that Δ must have a missing face of dimension $\geq d-k+1$. In particular, any sphere in $S(d-k, d-1)$ satisfies $g_k \geq 1$.

In this paper, we are interested in $(d-1)$-spheres with $g_k = 1$. By combining Propositions 2.5 and 2.6 with Theorem 2.4, we obtain the following result:

Corollary 2.8. *Let $k < \lfloor d/2 \rfloor$ and let Δ be a $(d-1)$-sphere with $g_k(\Delta) = 1$. Then $m_{d-k}(\Delta) \leq 1$. Furthermore, when $k \leq \lfloor d/2 \rfloor - 1$, we have $m_{d-k}(\Delta) = 1$ if and only if $g_{k+1}(\Delta) = 0$.*

2.4 The stress spaces

In this subsection, we review the basics of linear and affine stresses. For further details, we refer the reader to [12, 13] and [30, 31], and to [18, Section 2.3] for a more algebraic exposition.

Let $\mathbb{F} \subseteq \mathbb{R}$ be a field, and let Δ be a $(d-1)$-dimensional simplicial complex with vertex set $V = V(\Delta)$. A map $p : V(\Delta) \to \mathbb{F}^d \subseteq \mathbb{R}^d$ is called a *d-embedding* of Δ.

Continuing with the notation of Section 2.1, let $X = \{x_v : v \in V\}$ and let $\mathbb{F}[X]$ be the corresponding ring of polynomials. Each variable x_v acts on $\mathbb{F}[X]$ by $\frac{\partial}{\partial x_v}$; for brevity, we denote this operator by ∂_{x_v}. More generally, if $\mu = x_{i_1} \cdots x_{i_s} \in \mathbb{F}[X]$ is a monomial, define $\partial_\mu : \mathbb{F}[X] \to \mathbb{F}[X]$ by $\rho \mapsto \partial_{x_{i_1}} \cdots \partial_{x_{i_s}} \rho$. If $\ell = \sum_{v \in V} \ell_v x_v$ is a linear form in $\mathbb{F}[X]$, define

$$\partial_\ell : \mathbb{F}[X] \to \mathbb{F}[X] \quad \text{by} \quad \rho \mapsto \sum_{v \in V} \ell_v \cdot \partial_{x_v} \rho = \sum_{v \in V} \ell_v \frac{\partial \rho}{\partial x_v}.$$

For a monomial $\mu \in \mathbb{F}[X]$, the *support* of μ is $\operatorname{supp}(\mu) = \{v \in V : x_v | \mu\}$.

A d-embedding p of Δ gives rise to the sequence $\Theta(p) = (\theta_1, \ldots, \theta_d)$ of d linear forms, where $\theta_j = \sum_{v \in V} p(v)_j \, x_v$. Now, let $\mathbb{F} = \mathbb{Q}(a_{v,j} : v \in V, j \in [d])$, where $\{a_{v,j} \in \mathbb{R} : v \in V, j \in [d]\}$ is algebraically independent over $\mathbb{Q}$. In this case, the embedding $p : V \to \mathbb{F}^d$ defined by $v \mapsto p(v) = (a_{v,1}, \ldots a_{v,d})$, is called a *generic* embedding.

A homogeneous polynomial $\lambda = \lambda(X) = \sum_\mu \lambda_\mu \mu \in \mathbb{F}[X]$ of degree k is called a *linear k-stress* on (Δ, p) if it satisfies the following conditions:

- Every nonzero term $\lambda_\mu \mu$ of λ is supported on a face of Δ, that is, $\operatorname{supp}(\mu) \in \Delta$, and

- $\partial_{\theta_i} \lambda = 0$ for all $i = 1, \ldots, d$.

Recall that $c = \sum_{v \in V} x_v$. A linear k-stress λ on (Δ, p) that also satisfies $\partial_c \lambda = \sum_{v \in V} \partial_{x_v} \lambda = 0$ is called an *affine k-stress*.

It follows immediately from the definitions that the sets of linear k-stresses and affine k-stresses on Δ form vector spaces over $\mathbb{F}$, which we denote by $\mathcal{S}_k^\ell(\Delta, p)$ and $\mathcal{S}_k^a(\Delta, p)$, respectively. When we wish to emphasize the underlying field, we write $\mathcal{S}_k^\ell(\Delta, p; \mathbb{F})$ and $\mathcal{S}_k^a(\Delta, p; \mathbb{F})$. Furthermore, for all k, the space $\mathcal{S}_k^a(\Delta, p)$ is the kernel of $\partial_c : \mathcal{S}_k^\ell(\Delta, p) \to \mathcal{S}_{k-1}^\ell(\Delta, p)$.

The spaces $\mathcal{S}_k^\ell(\Delta, p)$ and $\mathcal{S}_k^a(\Delta, p)$ coincide with certain graded components of the Macaulay inverse system of $I_\Delta + (\Theta(p))$ and $I_\Delta + (\Theta(p), c)$, respectively; see [18, Section 2.3] for details. Consequently, the dimensions of $\mathcal{S}_k^\ell(\Delta, p)$ and $\mathcal{S}_k^a(\Delta, p)$ agree with the dimensions of the k-th graded components of $\mathbb{F}[\Delta]/(\Theta(p))$ and $\mathbb{F}[\Delta]/(\Theta(p), c)$, respectively (see [13] or [18]). In particular, the following (weaker) restatement of the g-theorem (Theorem 2.1) in the language of stress spaces holds:

Theorem 2.9. *Let (Δ, p) be a $(d-1)$-sphere with a generic embedding p, and let $1 \le k \le \lceil d/2 \rceil$. Then $\dim \mathcal{S}_k^\ell(\Delta, p) = \dim \mathcal{S}_{d-k}^\ell(\Delta, p) = h_k(\Delta)$, $\dim \mathcal{S}_k^a(\Delta, p) = g_k(\Delta)$, and the linear map $\partial_c : \mathcal{S}_k^\ell(\Delta, p) \to \mathcal{S}_{k-1}^\ell(\Delta, p)$ is surjective. In particular, if d is odd and $k = (d+1)/2$, this map is an isomorphism.*

Let $\lambda = \sum_\mu \lambda_\mu \mu$ be either a k-linear or a k-affine stress on (Δ, p), and let σ be a $(k-1)$-face of Δ. We write $x_\sigma := \prod_{v \in \sigma} x_v$ and $\lambda_\sigma := \lambda_{x_\sigma}$. By [13, Theorems 9,11], a k-stress λ is uniquely determined by its squarefree part $\sum_{\sigma \in \Delta, |\sigma| = k} \lambda_\sigma x_\sigma$.

We write $\mathrm{supp}(\lambda)$ to denote the subcomplex of Δ generated by all $(k-1)$-faces σ with $\lambda_\sigma \ne 0$. We also say that λ *lives on* a subcomplex $\Gamma \subseteq \Delta$ if $\mathrm{supp}(\lambda) \subseteq \Gamma$. For instance, $\partial_{x_v}\lambda$ is a $(k-1)$-stress that lives on $\mathrm{st}(v)$. Finally, we say that a face τ *participates in* λ or that τ is *in the support of* λ if $\tau \in \mathrm{supp}(\lambda)$.

3   The cone lemmas

Assume Λ is a simplicial complex of dimension $d-2$ and let Γ be the cone over Λ. It is known that, for appropriately chosen embeddings of Λ in $\mathbb{R}^{d-1}$ and of Γ in $\mathbb{R}^d$, the stress spaces of Λ and Γ are isomorphic; see, for instance, [13, Theorem 10], [22, Lemma 3.2]. Statements of this form are referred to as *cone lemmas* in the literature. The importance of the following version of the cone lemma is that it allows us to describe how the support of a stress on Λ relates to the support of its image on Γ.

Lemma 3.1. *Let Λ be a $(d-2)$-dimensional simplicial complex with $V(\Lambda) = [n]$, and let $\Gamma = 0 * \Lambda$ be the cone over Λ. Let $p' : [n] \to \mathbb{R}^{d-1}$ be an embedding of Λ such that the set of coordinates $\{p'(s)_t : s \in [n], t \in [d-1]\}$ is algebraically independent over $\mathbb{Q}$, and define $\mathbb{F}' = \mathbb{Q}(p'(s)_t : s \in [n], t \in [d-1])$. Let $a_1, \dots, a_n \in \mathbb{R}$ be algebraically independent over $\mathbb{F}'$, and let $\mathbb{F} \subseteq \mathbb{R}$ be any field that contains $\mathbb{F}'(a_1, \dots, a_n)$. Define an embedding $p : \{0, 1, \dots, n\} \to \mathbb{F}^d$ of Γ by*

$$p(0) = (0, \dots, 0, -1), \quad p(s) = ((1 + a_s)p'(s), a_s), \text{ for } s \in [n].$$

Then, for all $0 \le i \le d-1$, there exists a linear map of $\mathbb{F}'$-vector spaces $\psi_i : \mathcal{S}_i^\ell(\Lambda, p'; \mathbb{F}') \to \mathcal{S}_i^\ell(\Gamma, p; \mathbb{F})$ such that: 1) for any $\omega' \in \mathcal{S}_i^\ell(\Lambda, p'; \mathbb{F}')$, $\mathrm{supp}(\psi_i(\omega')) = \mathrm{Skel}_{i-1}\left(0 * \mathrm{supp}(\omega')\right)$, and 2) if $c' = \sum_{j=1}^n x_j$ and $c = \sum_{j=0}^n x_j$, then $\psi_{i-1}(\partial_{c'}\omega') = \partial_c(\psi_i(\omega'))$.

In other words, statement 2) implies that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{S}_i^\ell(\Lambda, p'; \mathbb{F}') & \xrightarrow{\ \psi_i\ } & \mathcal{S}_i^\ell(\Gamma, p; \mathbb{F}) \\
{\scriptstyle \partial_{c'}} \downarrow & & \downarrow {\scriptstyle \partial_c} \\
\mathcal{S}_{i-1}^\ell(\Lambda, p'; \mathbb{F}') & \xrightarrow{\ \psi_{i-1}\ } & \mathcal{S}_{i-1}^\ell(\Gamma, p; \mathbb{F})
\end{array}
$$

We refer to the map ψ_i as the *lifting* map.

Proof: The construction of ψ_i is similar to that of [22, Lemma 3.2], and so we omit some details. In our setting, the l.s.o.p. for $\mathbb{F}[\Gamma]$ corresponding to p is given by $\theta_t = \sum_{s=1}^n (1 + a_s)p'(s)_t x_s$, for $1 \le t \le d-1$, and $\theta_d = -x_0 + \sum_{s=1}^n a_s x_s$. Given $\omega' \in \mathcal{S}_i^\ell(\Lambda, p'; \mathbb{F}')$, define polynomials in

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$\mathbb{F}[x_1, \ldots, x_n]$ recursively by

$$\omega_0(x_1, \ldots, x_n) = \omega'\left(\frac{x_1}{1+a_1}, \ldots, \frac{x_n}{1+a_n}\right), \quad \text{and}$$

$$\omega_{j+1} = \frac{1}{j+1}\sum_{s=1}^n a_s\, \partial_{x_s}\omega_j, \quad \forall\, 0 \le j \le i-1. \tag{3.1}$$

In particular, $\omega_1 = \sum_{s=1}^n a_s\, \partial_{x_s}\omega_0$. We then define

$$\psi_i(\omega') := \sum_{j=0}^i x_0^j \cdot \omega_j(x_1, \ldots, x_n) \in \mathbb{F}[x_0, x_1, \ldots, x_n].$$

Computations analogous to those in [22, Lemma 3.2] show that $\psi_i(\omega')$ is in $\mathcal{S}_i^\ell(\Gamma, p; \mathbb{F})$. Hence, ψ_i is a well-defined map from $\mathcal{S}_i^\ell(\Lambda, p'; \mathbb{F}')$ to $\mathcal{S}_i^\ell(\Gamma, p; \mathbb{F})$. In fact, these computations show that if $\lambda = \sum_{j=0}^i x_0^j \cdot \lambda_j$ is any element of $\mathcal{S}_i^\ell(\Gamma, p; \mathbb{F})$, where $\lambda_j \in \mathbb{F}[x_1, \ldots, x_n]$, then for all $0 \le j \le i-1$, λ_{j+1} is determined from λ_0 via equation (3.1). This holds independently of whether λ lies in the image of ψ_i.

To see that $\psi_{i-1}(\partial_{c'}\omega') = \partial_c(\psi_i(\omega'))$, note that

$$\partial_c(\psi_i(\omega')) = \partial_{c'}\omega_0 + \sum_{j=1}^i jx_0^{j-1} \cdot \omega_j + \sum_{j=1}^i x_0^j \cdot \partial_{c'}\omega_j.$$

In particular,

$$\partial_{c'}\omega_0 + \omega_1 = \sum_{s=1}^n (1+a_s)\,\partial_{x_s}\omega_0 = \sum_{s=1}^n (\partial_{x_s}\omega')\left(\frac{x_1}{1+a_1}, \ldots, \frac{x_n}{1+a_n}\right) = (\partial_{c'}\omega')\left(\frac{x_1}{1+a_1}, \ldots, \frac{x_n}{1+a_n}\right)$$

is the constant term of $\psi_{i-1}(\partial_{c'}(\omega'))$ expanded as a polynomial in x_0. Since both $\psi_{i-1}(\partial_{c'}\omega')$ and $\partial_c(\psi_i(\omega'))$ are elements of $\mathcal{S}_{i-1}(\Gamma, p; \mathbb{F})$, and since, by the discussion in the previous paragraph, any linear stress on Γ is determined by its constant term, we conclude that these stresses are identical.

It only remains to prove that for $i \ge 1$, $\operatorname{supp}(\psi_i(\omega')) = \operatorname{Skel}_{i-1}\big(0 * \operatorname{supp}(\omega')\big)$. The definition of ω_0 ensures that every $(i-1)$ face that is in $\operatorname{supp}(\omega')$ is also in $\operatorname{supp}(\psi_i(\omega'))$. Assume now that G is an $(i-2)$ face in $\operatorname{supp}(\omega')$. To complete the proof, we have to show that $G \cup 0$ is in $\operatorname{supp}(\psi_i(\omega'))$. Write $\omega' = (\sum_{j=1}^n \ell_j x_j)x_G + \alpha$, where $\ell_j \in \mathbb{F}'$ and no term of α is divisible by x_G. (When $i = 1$, we have $G = \emptyset$, $\alpha = 0$, and $\omega' = \sum_{j=1}^n \ell_j x_j$.) The assumption that $G \in \operatorname{supp}(\omega')$ guarantees that at least one of ℓ_j is nonzero. We then obtain:

$$
\begin{aligned}
\partial_{x_{G\cup 0}}(\psi_i(\omega')) &= \partial_{x_G}\omega_1(x_1, \ldots, x_n) = \partial_{x_G}\left[\sum_{s=1}^n a_s\, \partial_{x_s}\omega_0(x_1, \ldots, x_n)\right] \\
&= \partial_{x_G}\left[\sum_{s=1}^n a_s\, \partial_{x_s}\left(\omega'\left(\frac{x_1}{1+a_1}, \ldots, \frac{x_n}{1+a_n}\right)\right)\right] \\
&= \partial_{x_G}\left[\sum_{s=1}^n a_s\, \partial_{x_s}\left(\sum_{j=1}^n \frac{\ell_j x_j}{1+a_j} \cdot \frac{x_G}{\prod_{k\in G}(1+a_k)}\right)\right] \\
&= \frac{1}{\prod_{k\in G}(1+a_k)}\left[\sum_{j\in G}\frac{2\ell_j a_j}{1+a_j} + \sum_{j\notin G}\frac{\ell_j a_j}{1+a_j}\right].
\end{aligned}
$$

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Since $\ell_j \in \mathbb{F}'$ and at least one of them is nonzero, and since $a_1, \ldots, a_n$ are algebraically independent over $\mathbb{F}'$, we conclude that $\partial_{x_{G \cup 0}}(\psi_i(\omega')) \neq 0$. Thus, $G \cup 0 \in \operatorname{supp}(\psi_i(\omega'))$. $\qquad\square$

Since the maps ψ_i, $0 \leq i \leq d-1$, of Lemma 3.1 satisfy $\psi_{i-1}(\partial_{c'}\omega') = \partial_c(\psi_i(\omega'))$, it follows that the restriction of ψ_i to $\mathcal{S}_i^a(\Lambda, p'; \mathbb{F}')$ defines a map from $\mathcal{S}_i^a(\Lambda, p'; \mathbb{F}')$ to $\mathcal{S}_i^a(\Gamma, p; \mathbb{F})$. This implies the following affine version of Lemma 3.1, which will be used in Sections 4 and 5.

Lemma 3.2. *Under the assumptions of Lemma 3.1, for all $i \leq \lfloor (d-1)/2 \rfloor$, there exists a linear map of $\mathbb{F}'$-vector spaces $\psi_i : \mathcal{S}_i^a(\Lambda, p'; \mathbb{F}') \to \mathcal{S}_i^a(\Gamma, p; \mathbb{F})$ such that for any $\omega' \in \mathcal{S}_i^a(\Lambda, p'; \mathbb{F}')$, $\operatorname{supp}(\psi_i(\omega')) = \operatorname{Skel}_{i-1}\big(0 * \operatorname{supp}(\omega')\big)$.*

Consider a $(d-1)$-sphere Δ with a generic embedding p. In this paper, we often apply Lemmas 3.1 and 3.2 to the subcomplexes $\Lambda = \operatorname{lk}(u)$ and $\Gamma = \operatorname{st}(u) = u * \Lambda$ of Δ, where $u \in V(\Delta)$ is a vertex. If $2k < d$, then by the partition of unity for affine stresses [23], $\mathcal{S}_k^a(\Delta, p) = \sum_{u \in V(\Delta)} \mathcal{S}_k^a(\operatorname{st}(u), p)$, and hence any affine k-stress on Δ can be expressed as a sum of affine k-stresses that live on vertex stars. When $2k = d$, the space $\mathcal{S}_k^a(\Delta, p)$ is much more mysterious. The following lemma, whose proof utilizes Lemma 3.1, sheds some light on this space in the case where $\Delta \in S(k-1, 2k-1)$. This lemma will be essential in Section 5.

Lemma 3.3. *Let $\Delta \in S(k-1, 2k-1)$. Let $\{a_{s,j} : s \in V(\Delta), j \in [d]\} \subset \mathbb{R}$ be a set of numbers that are algebraically independent over $\mathbb{Q}$, and let $\mathbb{F} = \mathbb{Q}(a_{s,j} : s \in V(\Delta), j \in [d])$. Define $q : V(\Delta) \to \mathbb{F}^d$ by $q(s) = (a_{s,1}, \ldots, a_{s,d})$ for all $s \in V(\Delta)$. Then for every edge $uv \in \Delta$, there exists a stress $\bar{\omega} \in \mathcal{S}_k^a(\Delta, q; \mathbb{F})$ with the following properties: $\operatorname{supp}(\bar{\omega}) \subseteq \operatorname{st}(u) \cup \operatorname{st}(v)$, and there exists a face $\sigma \in \operatorname{supp}(\bar{\omega})$ such that $u \in \sigma$ but $\sigma \notin \operatorname{st}(v)$.*

In the proof, we will consider the map ∂_c acting on the stress spaces associated with $\operatorname{lk}(u)$, $\operatorname{st}(u)$, and $\operatorname{st}(v)$. To avoid confusion, we denote these maps by ϕ_u', ϕ_u, and ϕ_v, respectively.

Proof: Consider the $d \times |V(\Delta)|$ matrix A, whose columns are labeled by the vertices of Δ and whose (j, s)-entry is $a_{s,j}$ for all $s \in V(\Delta)$, $j \in [d]$. Fix an edge uv and perform the following row operations: for each $j \in [d-1]$, divide the j-th row of A by $a_{u,j}$, and divide the d-th row of A by $-a_{u,d}$. Then, for each $1 \leq j \leq d-1$, add the resulting d-th row to the resulting j-th row to obtain a new matrix $\tilde{A}$. Thus, for $j \in [d-1]$, the (j, s)-entry of $\tilde{A}$ is $\frac{a_{s,j}}{a_{u,j}} - \frac{a_{s,d}}{a_{u,d}}$, while the (d, s)-entry of $\tilde{A}$ is $-\frac{a_{s,d}}{a_{u,d}}$.

These operations do not change the row space of the matrix A over $\mathbb{F}$. As a result, in $\mathbb{F}[x_s : s \in V(\Delta)]$, the ideal generated by the rows of A—that is, by $\theta_j = \sum_s a_{s,j} x_s$ for $j \in [d]$—coincides with the ideal generated by the rows of $\tilde{A}$. Consequently, for all i, $\mathcal{S}_i^\ell(\Delta, q; \mathbb{F})$ coincides with $\mathcal{S}_i^\ell(\Delta, p; \mathbb{F})$, where the embedding p is given by the columns of $\tilde{A}$.

Let $p' : V(\Delta)\backslash u \to \mathbb{R}^{d-1}$ be the embedding given by

$$p'(s)_j = \left(\frac{a_{s,j}}{a_{u,j}} - \frac{a_{s,d}}{a_{u,d}}\right) \bigg/ \left(1 - \frac{a_{s,d}}{a_{u,d}}\right), \text{ for } j \in [d-1],$$

and define $\mathbb{F}' = \mathbb{Q}\left(p'(s)_j : s \in V(\Delta)\backslash u, j \in [d-1]\right)$. Then $\mathbb{F} = \mathbb{F}'(a_{s,d}, a_{u,j} : s \in V(\Delta), j \in [d-1])$, and the complexes $\Lambda = \operatorname{lk}(u)$ and $\Gamma = u * \operatorname{lk}(u) = \operatorname{st}(u)$, endowed with embeddings p' and p, respectively, satisfy the conditions of Lemma 3.1. Thus, for each i, Lemma 3.1 provides the lifting map

$$\psi_i : \mathcal{S}_i^\ell(\operatorname{lk}(u), p'; \mathbb{F}') \to \mathcal{S}_i^\ell(\operatorname{st}(u), p; \mathbb{F}).$$

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Consider the map

$$\phi'_u : \mathcal{S}^\ell_k(\mathrm{lk}(u), p'; \mathbb{F}') \to \mathcal{S}^\ell_{k-1}(\mathrm{lk}(u), p'; \mathbb{F}'), \quad \lambda \mapsto \partial_c \lambda.$$

Since $\mathrm{lk}(u)$ is a $(2k-2)$-sphere, this map is an isomorphism. Moreover, since $\Delta \in S(k-1, 2k-1)$, it follows that $\mathrm{lk}(uv) \in S(k-1, 2k-3)$, and hence, by Remark 2.7,

$$0 < g_{k-1}(\mathrm{lk}(uv)) = h_{k-1}(\mathrm{lk}(uv)) - h_{k-2}(\mathrm{lk}(uv)) = h_{k-1}(\mathrm{lk}(uv)) - h_k(\mathrm{lk}(uv)).$$

Since $v * \mathrm{lk}(uv) = \mathrm{st}(v, \mathrm{lk}(u))$, we conclude that

$$
\begin{aligned}
\dim \mathcal{S}^\ell_k(\mathrm{st}(v, \mathrm{lk}(u)), p'; \mathbb{F}') &= h_k(\mathrm{st}(v, \mathrm{lk}(u))) = h_k(\mathrm{lk}(uv)) \\
&< h_{k-1}(\mathrm{lk}(uv)) = \dim \mathcal{S}^\ell_{k-1}(\mathrm{st}(v, \mathrm{lk}(u)), p'; \mathbb{F}').
\end{aligned}
$$

Hence, there exists a linear $(k-1)$-stress $\omega' \in \mathcal{S}^\ell_{k-1}(\mathrm{lk}(u), p'; \mathbb{F}')$ with the following properties: the support of ω' is a subcomplex of $v * \mathrm{lk}(uv) = \mathrm{st}(v, \mathrm{lk}(u))$, but the support of $(\phi'_u)^{-1}(\omega')$ contains faces that are not contained in $\mathrm{st}(v, \mathrm{lk}(u))$. We set $\omega = \psi_{k-1}(\omega') \in \mathcal{S}^\ell_{k-1}(\mathrm{st}(u), p; \mathbb{F}) \subseteq \mathcal{S}^\ell_{k-1}(\Delta, p; \mathbb{F})$.

Finally, we consider the maps

$$\phi_u : \mathcal{S}^\ell_k(\mathrm{st}(u), p; \mathbb{F}) \xrightarrow{\partial_c} \mathcal{S}^\ell_{k-1}(\mathrm{st}(u), p; \mathbb{F}), \quad \phi_v : \mathcal{S}^\ell_k(\mathrm{st}(v), p; \mathbb{F}) \xrightarrow{\partial_c} \mathcal{S}^\ell_{k-1}(\mathrm{st}(v), p; \mathbb{F}).$$

These maps are also isomorphisms. Observe that, by Lemma 3.1 and our choice of ω', the support of ω is a subcomplex of $\mathrm{st}(uv)$, whereas the support of $\phi_u^{-1}(\omega) = \psi_k((\phi'_u)^{-1}(\omega'))$ has some faces that are not contained in $\mathrm{st}(uv)$. Thus, we may regard ω as an element of both $\mathcal{S}^\ell_{k-1}(\mathrm{st}(u), p; \mathbb{F})$ and $\mathcal{S}^\ell_{k-1}(\mathrm{st}(v), p; \mathbb{F})$. We define $\bar{\omega} := \iota_u(\phi_u^{-1}(\omega)) - \iota_v(\phi_v^{-1}(\omega)) \in \mathcal{S}^\ell_k(\Delta, p; \mathbb{F})$, where ι_u and ι_v are the natural inclusion maps between the stress spaces on $\mathrm{st}(u)$, $\mathrm{st}(v)$, and on Δ. It is a linear k-stress with $\partial_c \bar{\omega} = \omega - \omega = 0$, and hence it is an affine k-stress. We show that $\bar{\omega}$ has the desired properties.

If there is a face $\sigma \in \mathrm{supp}(\phi_u^{-1}(\omega)) \cap \mathrm{st}(u)$ such that $u \in \sigma$ and $\sigma \notin \mathrm{st}(v)$, then $\sigma \notin \mathrm{supp}(\phi_v^{-1}(\omega))$ because $\phi_v^{-1}(\omega)$ lives on $\mathrm{st}(v)$. This implies that $\sigma \in \mathrm{supp}(\bar{\omega})$, as desired.

Otherwise, $\phi_u^{-1}(\omega)$ lives on $\mathrm{st}(uv) \cup \mathrm{lk}(u)$. Let $\sigma \in \mathrm{supp}(\phi_u^{-1}(\omega)) \cap \mathrm{lk}(u)$ be a face such that $\sigma \notin \mathrm{st}(v)$. In other words, σ is an interior face of the ball $\mathrm{lk}(u)\backslash v$, and hence σ contains a subset τ that is a minimal interior face of $\mathrm{lk}(u)\backslash v$. Then $\partial \tau \subseteq \partial(\mathrm{lk}(u)\backslash v) = \mathrm{lk}(uv)$, and hence $v \cup \tau$ is a missing face of Δ. Since $\Delta \in S(k-1, 2k-1)$, it follows that $\dim \tau \leq k-2$. However, by Lemma 3.1, the support of $\phi_u^{-1}(\omega)$ is of the form $\mathrm{Skel}_{k-1}(C * u)$, where C is the support of $(\phi'_u)^{-1}(\omega')$. Thus, $\tau * u \in \mathrm{supp}(\phi_u^{-1}(\omega))$, contradicting our assumption that $\phi_u^{-1}(\omega)$ lives on $\mathrm{st}(uv) \cup \mathrm{lk}(u)$. $\qquad \square$

Remark 3.4. In the case $k = 2$, it is known that if $\Delta \in S(1,3)$ and p is a generic embedding, then for every edge $uv \in \Delta$, the framework $(\mathrm{st}(u) \cup \mathrm{st}(v), p)$ is rigid and supports a nontrival affine 2-stress. While for $k > 2$ and $\Delta \in S(k-1, 2k-1)$ an analogous $(k-1)$-skeletal rigidity of the union of stars of adjacent vertices may fail, Lemma 3.3 still allows us to locate a nontrival affine k-stress that lives on this subcomplex.

Remark 3.5. Lemma 3.3 holds not only for spheres in $S(k-1, 2k-1)$ but also for certain other $(2k-1)$-spheres under additional conditions on the links of u and v. For example, using the same proof, one can show that given any $(2k-1)$-sphere Δ with a generic embedding, there exists a nontrivial affine k-stress supported on $\mathrm{st}(u) \cup \mathrm{st}(v)$, where uv is an edge, provided that 1) $g_{k-1}(\mathrm{lk}(uv)) > 0$, and 2) every face that lies in $\mathrm{lk}(u) \cap \mathrm{lk}(v)$ but not in $\mathrm{lk}(uv)$ has dimension at most $k-2$.

4 Proof of the first main result

We are now in a position to prove the first main result of the paper:

Theorem 4.1. *Let $d \geq 2k$, and let $\Delta \in S(d-k, d-1)$. Assume further that if $d = 2k$, then $\Delta \notin S(k-1, 2k-1)$. Then $g_k(\Delta) = 1$ if and only if one of the following holds: (1) $\Delta = \partial \sigma^{d-k} * \Gamma$, where Γ is a $(k-1)$-sphere, or (2) $\Delta = \partial \sigma^j * \partial \sigma^{d-j}$ for some j with $k < j \leq \lfloor d/2 \rfloor$.*

One direction of the theorem is immediate. Indeed, it is known (and easy to verify) that (1) $h_i(\partial \sigma^j) = 1$ for all $0 \leq i \leq j$; (2) if Γ is a $(k-1)$-sphere, then $h_k(\Gamma) = 1$; and (3) $h_i(\Lambda * \Gamma) = \sum_{j=0}^{i} h_j(\Lambda) h_{i-j}(\Gamma)$ for any simplicial complexes Λ and Γ and any i. These results imply that if Δ is one of the complexes described in the theorem, then $g_k(\Delta) = 1$.

In the remainder of this section, we focus on the other direction. Throughout, let Δ be a sphere satisfying the conditions of Theorem 4.1. To prove the theorem, we consider the following three cases separately: $d \geq 2k + 2$, $d = 2k + 1$, and $d = 2k$.

4.1 $d \geq 2k + 1$

We begin with the case $d \geq 2k + 2$. To complete the proof of Theorem 4.1 in this case, it suffices to verify the following result.

Proposition 4.2. *Let $k \leq d/2 - 1$, and let $\Delta \in S(d-k, d-1)$ satisfy $g_k(\Delta) = 1$. Suppose there exists an integer j with $k \leq j \leq (d-2)/2$ such that $g_{j+1}(\Delta) = 0$. Then either: $j = k$, in which case $\Delta = \partial \sigma^{d-k} * \Gamma$ for some $(k-1)$-sphere Γ, or $j > k$, in which case, $\Delta = \partial \sigma^{d-j} * \partial \sigma^j$. On the other hand, if no such j exists (that is, if $g_{\lfloor d/2 \rfloor}(\Delta) = 1$), then $\Delta = \partial \sigma^{\lfloor d/2 \rfloor} * \partial \sigma^{\lceil d/2 \rceil}$.*

Proof: Assume first that there exists $k \leq j \leq (d-2)/2$ such that $g_j = 1$ but $g_{j+1} = 0$. Since $g_{j+1}(\Delta) = 0$ and $j + 1 \leq d/2$, Δ is j-stacked. Furthermore, by Corollary 2.8, $m_{d-j}(\Delta) = 1$ and $m_{d-\ell}(\Delta) = 0$ for all $k \leq \ell \leq j - 1$. Our assumptions also imply that $m_{d-\ell} = 0$ for all $\ell < k$. Thus, Δ has a unique missing $(d-j)$-face, which we denote by τ, and no missing faces of dimension greater than $d - j$. Let $T = \Delta(d-j-1)$ be the j-stacked triangulation of Δ from Theorem 2.4. Then τ is the unique minimal interior face of this triangulation. Since any ball is the union of the stars of its minimal interior faces, it follows that $T = \text{st}(\tau, T)$, and therefore

$$\Delta = \partial T = \partial(\text{st}(\tau, T)) = \partial(\overline{\tau} * \text{lk}(\tau, T)) = \partial \overline{\tau} * \text{lk}(\tau, T).$$

Let $\Gamma := \text{lk}(\tau, T)$; it is a $(j-1)$-sphere, and one easily checks that $g_k(\Delta) = h_k(\Gamma)$. Thus, $h_k(\Gamma) = 1$. Observe that if $j = k$, then any $(k-1)$-sphere Γ satisfies $h_k(\Gamma) = 1$. However, if $j > k$, then the only $(j-1)$-sphere Γ with $h_k(\Gamma) = 1$ is the boundary of a j-simplex. This completes the proof of the case $g_{j+1}(\Delta) = 0$.

We now consider the case $g_{\lfloor d/2 \rfloor}(\Delta) = 1$. In this case, it suffices to show that Δ is the join of the boundary complexes of two simplices whose dimensions sum to d. The fact that dimensions of those simplices are $\lfloor d/2 \rfloor$ and $\lceil d/2 \rceil$ then follows from our assumption that $g_{\lfloor d/2 \rfloor}(\Delta) \neq 0$. By McMullen's formula, $\sum_{v \in V(\Delta)} g_k(\text{lk}(v)) = (k+1)g_{k+1}(\Delta) + (d-k+1)g_k(\Delta) = d + 2$. Hence there are exactly $d + 2$ vertices whose links have $g_k = 1$.

If the total number of vertices is $d + 2$, then Δ is the join of the boundary complexes of two simplices whose dimensions sum to d (see [8, Section 6.1]), and the result follows. Hence assume that Δ has at least $d + 3$ vertices, and let w be a vertex with $g_k(\text{lk}(w)) = 0$. Since $\text{lk}(w)$ is a $(d-2)$-sphere and $d - 1 > 2k$, it follows from Remark 2.7 that $\text{lk}(w)$ has a missing face τ of

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dimension $\geq (d-1) - k + 1 = d - k$. Then either τ or $\tau \cup w$ is a missing face of Δ. However, our assumptions that $\Delta \in S(d-k, d-1)$ and that $g_{\lfloor d/2 \rfloor} = 1$, together with Corollary 2.8, guarantee that $m_\ell = 0$ for all $\ell > \lceil d/2 \rceil$. Since $d - k > \lceil d/2 \rceil$, it follows that neither τ nor $\tau \cup w$ can be a missing face, yielding the desired contradiction. □

We now consider the second case, $d = 2k+1$. The argument proceeds in an analogous manner, again relying on McMullen's formula and the GLBT.

Proposition 4.3. If $\Delta \in S(k+1, 2k)$ satisfies $g_k(\Delta) = 1$, then Δ is the join of $\partial\sigma^{k+1}$ and a $(k-1)$-sphere.

Proof: Let Δ be a sphere in $S(k+1, 2k)$ with $g_k(\Delta) = 1$. By McMullen's formula,

$$\sum_{v \in V(\Delta)} g_k(\mathrm{lk}(v)) = (k+1)g_{k+1}(\Delta) + (k+2)g_k(\Delta) = k+2.$$

Hence there are exactly $k+2$ vertices whose links satisfy $g_k = 1$; for every other vertex w, $\mathrm{lk}(w)$ is a $(2k-1)$-sphere with $g_k = 0$. It follows from Remark 2.7 that for such w, $\mathrm{lk}(w)$ has a missing face τ of dimension $\geq (d-1) - k + 1 = k+1$. Then either τ or $\tau \cup w$ would be a missing face of Δ. However, since $\Delta \in S(k+1, 2k)$, the latter case is impossible. Therefore, τ is a missing $(k+1)$-face of Δ. Now, for every vertex $u \in \tau$, the set $\tau \backslash u$ is a missing k-face of $\mathrm{lk}(u)$. By Remark 2.7, this implies $g_k(\mathrm{lk}(u)) = 1$. Since there are $k+2$ vertices in τ, it follows that the vertices of Δ whose links satisfy $g_k = 1$ are precisely the vertices of τ. Moreover, Δ has no other missing $(k+1)$-faces; otherwise there would be more than $k+2$ vertices whose links satisfy $g_k = 1$.

We conclude from the above discussion that for every vertex w not in τ, the link of w is a $(k-1)$-stacked $(2k-1)$-sphere and τ is its unique missing $(k+1)$-face. As in the proof of Proposition 4.2, this implies that $\mathrm{lk}(w) = \partial\overline{\tau} * S_w$ for some $(k-2)$-sphere S_w. In other words, $\mathrm{st}(w) = \partial\overline{\tau} * w * \mathrm{lk}(w)$. Now, every facet of Δ contains a vertex that is not in τ. Thus,

$$\Delta = \cup_{w \notin \tau} \mathrm{st}(w) = \partial\overline{\tau} * [\cup_{w \notin \tau} w * S_w].$$

To summarize, Δ is the join of two complexes one of which is $\partial\overline{\tau} = \partial\sigma^{k+1}$. Since Δ is a $2k$-sphere, the other complex must be a $(k-1)$-sphere. The result follows. □

4.2 $d = 2k$

As our final case, we consider a sphere Δ with $g_k(\Delta) = 1$ that lies in $S(k, 2k-1)$ but not in $S(k-1, 2k-1)$. Let the vertex set be $V = V(\Delta)$, and let $\tau = \{v'_1, \ldots, v'_{k+1}\}$ be a missing k-face of Δ. Our goal is to show that Δ is the join of $\partial\overline{\tau}$ with a $(k-1)$-sphere; see Proposition 4.6 below. The proof relies on several lemmas. To start, choose a generic embedding $q : V(\Delta) \to \mathbb{R}^{2k}$ that satisfies the conditions of Lemma 3.3, and let ω be the unique (up to scalar multiplication) non-zero affine k-stress on $(\Delta, q; \mathbb{F})$.

Note that for each $1 \leq i \leq k+1$, $\tau \backslash v'_i$ is a missing $(k-1)$-face of the $(2k-2)$-sphere $\mathrm{lk}(v'_i)$. Consider the complex $\Lambda = \mathrm{lk}(v'_i) \cup \{\tau \backslash v'_i\}$, i.e., $\mathrm{lk}(v'_i)$ together with the new face $\tau \backslash v'_i$. We endow Λ (equivalently, $\mathrm{lk}(v'_i)$) with the embedding p' defined as in the proof of Lemma 3.3, with v'_i playing the role of u. Following the notation of Section 3, we observe that since $\dim(\mathbb{F}'[\Lambda]/(\Theta'))_k = h_k(\mathrm{lk}(v'_i)) + 1 = h_{k-1}(\mathrm{lk}(v'_i)) + 1$, while $\dim(\mathbb{F}'[\Lambda]/(\Theta'))_{k-1} = h_{k-1}(\mathrm{lk}(v'_i))$, the stress space $\mathcal{S}_k^a(\Lambda, p'; \mathbb{F}')$ is 1-dimensional. Lemma 3.2 then implies that $\mathcal{S}_k^a(\Lambda * v'_i, p; \mathbb{F})$ is also 1-dimensional. Since $\mathrm{st}(v'_i) \cup \{\tau \backslash v'_i\} \subseteq \Delta$ and $\Lambda * v'_i$ have the same $(k-1)$-skeleton, and since ω is the unique (up to scalar multiplication) non-zero affine k-stress on $(\Delta, q; \mathbb{F})$, we obtain the following result.

Lemma 4.4. *For every $1 \leq i \leq k+1$, the stress ω lives on $\mathrm{st}(v_i') \cup \{\tau \backslash v_i'\}$. Furthermore, for each $v_i' \in \tau$, the support of ω is of the form $\mathrm{Skel}_{k-1}(v_i' * C_i)$ for some subcomplex C_i of $\mathrm{lk}(v_i') \cup \{\tau \backslash v_i'\}$.*

Lemma 4.4 leads to the following structural result about Δ.

Lemma 4.5. *If a $(k-1)$-face $G \in \Delta[V \backslash \tau]$ participates in ω, then $\partial \overline{\tau} * \overline{G}$ is a subcomplex of Δ.*

Proof: Pick any $v_i' \in \tau$. By Lemma 4.4, the stress ω lives on $\mathrm{st}(v_i') \cup \{\tau \backslash v_i'\}$. Hence if $G \in \mathrm{supp}(\omega)$ and $\tau \cap G = \emptyset$, then $G \in \mathrm{lk}(v_i')$. Moreover, the second part of Lemma 4.4 implies that for every $v \in G$, the $(k-1)$-face $(G \backslash v) \cup v_i'$ also lies in the support of ω. Now pick any $v_j' \in \tau$ with $j \neq i$. Since ω also lives on $\mathrm{st}(v_j') \cup \{\tau \backslash v_j'\}$, it follows that $(G \backslash v) \cup v_i'$ must be in the star of v_j'. This implies that replacing any two vertices of G with any two vertices of τ yields a $(k-1)$-face of Δ. By Lemma 4.4, this face also participates in ω. Iterating this argument, we conclude that $\mathrm{Skel}_{k-1}(\partial \overline{\tau} * \overline{G})$ is a subcomplex of Δ.

To complete the proof, observe that any two missing k-faces K, K' of Δ must be disjoint. (Indeed, if $v \in K \cap K'$, then $\mathcal{S}_k^a\big(\mathrm{st}(v) \cup \{K \backslash v, K' \backslash v\}\big)$ has dimension 2, contradicting the assumption that $g_k(\Delta) = 1$.) Since $|\tau \cup G| = 2k+1 < 2k+2$, it follows that τ is the unique missing k-face of $\Delta[\tau \cup G]$. Moreover, because $\mathrm{Skel}_{k-1}(\partial \overline{\tau} * \overline{G})$ is a subcomplex of Δ, we conclude that $\mathrm{Skel}_k(\partial \overline{\tau} * \overline{G})$ is also a subcomplex of Δ. Finally, since Δ has no missing faces of dimension greater than k, it follows that $\partial \overline{\tau} * \overline{G}$ is a subcomplex of Δ. $\qquad\square$

We are ready to complete the proof of Theorem 4.1 by verifying the following result.

Proposition 4.6. *Let $\Delta \in S(k, 2k-1) \backslash S(k-1, 2k-1)$ be a sphere with $g_k = 1$, and let τ be a missing k-face of Δ. Then Δ is the join of $\partial \overline{\tau}$ and a $(k-1)$-sphere.*

Proof: We continue with the notation introduced above; in particular, we write $\tau = \{v_1', \ldots, v_{k+1}'\}$. We also let $V \backslash \tau = \{v_1, \ldots, v_m\}$. Then for each $i = 1, \ldots, k+1$, the link of $\tau \backslash v_i'$ is a $(k-1)$-sphere, which we denote by S_i. Our goal is to show that $S_1 = \cdots = S_k$.

By Lemma 4.5, if a face $G \in \Delta[V \backslash \tau]$ participates in ω, then G must belong to the intersection of $S_1, \ldots, S_{k+1}$. Consequently, if not all $(k-1)$-spheres $S_1, \ldots, S_{k+1}$ are identical, then $(\mathrm{supp}(\omega))[V \backslash \tau]$ is a proper (possibly empty) subcomplex of S_1. It follows that either no $(k-1)$-face of $\Delta[V \backslash \tau]$ participates in ω or there exists a $(k-2)$-face $H \in \Delta[V \backslash \tau]$ that is contained in a unique $(k-1)$-face $G \in (\mathrm{supp}(\omega))[V \backslash \tau]$.

We claim that neither of these two cases can occur. We begin with the latter case. By relabeling the vertices if necessary, we may assume that $H = \{v_1, \ldots, v_{k-1}\}$ and $G = \{v_1, \ldots, v_k\}$. Then the stress ω can be written in the form

$$\omega = x_H \left[\sum_{i=1}^{k-1} \alpha_i x_{v_i} + \beta x_{v_k} + \sum_{j=1}^{k+1} \gamma_j x_{v_j'} \right] + \sum_{x_H \nmid \mu} \delta_\mu \mu.$$

Consequently, $\partial_{x_H} \omega = \sum_{i=1}^{k-1} 2\alpha_i x_{v_i} + \beta x_{v_k} + \sum_{j=1}^{k+1} \gamma_j x_{v_j'}$ is a non-trivial affine 1-stress on Δ. However, the corresponding affine dependence involves only $2k+1$ points in $\mathbb{R}^{2k}$. Since the embedding is generic, these points cannot be affinely dependent, and therefore no such affine 1-stress can exist.

The former case can be handled in a similar manner. For each $(k-2)$-face H' in $\mathrm{supp}(\omega)$, consider the size of $H' \cap \{v_1, \ldots, v_m\}$, and let H be a face for which this size is maximized. Say,

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$H = \{v_1, \ldots, v_\ell, v'_1, \ldots, v'_{k-1-\ell}\}$ for some $0 \leq \ell \leq k - 1$. Then ω can be written as

$$\omega = x_{H'} \left[\sum_{i=1}^{\ell} \alpha_i x_{v_i} + \sum_{j=1}^{k+1} \gamma_j x_{v'_j} \right] + \sum_{x_{H'} \nmid \mu} \delta_\mu \mu.$$

Taking the partial derivative $\partial_{x_{H'}}$, we obtain an affine dependence among $\ell + (k + 1) \leq 2k$ generic points in $\mathbb{R}^{2k}$. This is again impossible.

Hence all the spheres $S_1, \ldots, S_{k+1}$ are the same sphere S, and $\partial \overline{\tau} * S$ is a subcomplex of Δ. Since $\partial \overline{\tau} * S$ is a $(2k - 1)$-sphere, we must have $\Delta = \partial \overline{\tau} * S$. $\qquad \square$

5 Proof of the second main result

The goal of this section is to characterize all 5-spheres in $S(2, 5)$ with $g_3 = 1$; see Theorem 5.6. Throughout this section, let $\Delta \in S(2, 5)$ be a sphere with $g_3(\Delta) = 1$, equipped with an embedding q that satisfies the conditions of Lemma 3.3. Following the notation of Lemma 3.3, let $\omega \in \mathcal{S}_3^a(\Delta, q; \mathbb{F})$ be the unique (up to a scalar) nontrivial affine 3-stress on Δ. By Lemma 3.3, for each edge $ab \in \Delta$, the support of ω is contained in $\mathrm{st}(a) \cup \mathrm{st}(b)$. Below, we exploit this observation to provide several properties of the support of ω, which, in turn, enable us to prove results about the structure of Δ.

Lemma 5.1. *Every vertex of Δ is in $\mathrm{supp}(\omega)$.*

Proof: By Lemma 3.3, for every edge $ab \in \Delta$, the support of ω is contained in $\mathrm{st}(a) \cup \mathrm{st}(b)$; furthermore, there exist faces $F_a, F_b \in \mathrm{supp}(\omega)$ such that $a \in F_a$, $b \in F_b$, but $F_a, F_b \notin \mathrm{st}(ab)$. In particular, a and b are in the support of ω. As every vertex is in some edge of Δ, this implies that every vertex of Δ participates in ω. $\qquad \square$

Lemma 5.2. *Every missing 2-face of Δ has at least two edges that lie in $\mathrm{supp}(\omega)$.*

Proof: Consider a missing 2-face aby. It suffices to show that for every vertex of aby, at least one of the edges of aby containing that vertex lies in $\mathrm{supp}(\omega)$. Without loss of generality, consider vertex y and view ω as an affine 3-stress living on $\mathrm{st}(a) \cup \mathrm{st}(b)$. Following the notation of the proof of Lemma 3.3, we can write (a multiple of) ω as $\omega = \iota_a(\phi_a^{-1}(\alpha)) - \iota_b(\phi_b^{-1}(\alpha))$ for some linear 2-stress α that lives on $\mathrm{st}(ab)$. By Lemma 5.1, we have $y \in \mathrm{supp}(\omega)$, and hence y must lie in the support of either $\phi_a^{-1}(\alpha)$ or $\phi_b^{-1}(\alpha)$. Assume without loss of generality that y lies in the support of $\phi_a^{-1}(\alpha) = \psi_a((\phi'_a)^{-1}(\alpha'))$. Here, following the notation of the proof of Lemma 3.3, ψ_a is the lifting map from $(\mathrm{lk}(a), p')$ to $(\mathrm{st}(a), p)$, $\phi'_a : \mathcal{S}_3^\ell(\mathrm{lk}(a), p') \to \mathcal{S}_2^\ell(\mathrm{lk}(a), p')$ is an isomorphism, and α' is a linear 2-stress on $(\mathrm{lk}(a), p')$. It then follows from Lemma 3.2 that the edge ay also lies in the support of $\phi_a^{-1}(\alpha)$. Since $ay \notin \mathrm{st}(b)$, it does not lie in the support of $\phi_b^{-1}(\alpha)$. Consequently, $ay \in \mathrm{supp}(\omega)$, as desired. $\qquad \square$

We now use the properties established above to show that the graph of Δ must be a complete graph. We begin by proving the following weaker result.

Lemma 5.3. *The graph of Δ is a complete multipartite graph with at least six parts.*

Proof: Assume that ab is not an edge. We claim that the set of neighbors of a coincides with that of b. Indeed, if y is a neighbor of a but not of b, then $b \notin \operatorname{st}(y) \cup \operatorname{st}(a)$. Since $ay \in \Delta$, we have $\operatorname{supp}(\omega) \subseteq \operatorname{st}(a) \cup \operatorname{st}(y)$, and hence $b \notin \operatorname{supp}(\omega)$. This contradicts Lemma 5.1.

Let I be a maximal independent set of Δ (with respect to inclusion). The above claim implies that each vertex in I is connected to all vertices of Δ that are not in I. Since this holds for all maximal independent sets, we conclude that the maximal independent sets partition the vertex set of Δ, and that the graph of Δ is the complete multipartite graph with these parts. Moreover, since $\dim \Delta = 5$, there are at least six parts. $\qquad\square$

To show that Δ has a complete graph, we need a few additional definitions. An edge uv of Δ is called *contractible* if uv is not contained in any missing face. If uv is contractible, then one can contract uv and obtain a new simplicial complex Δ' by replacing the vertices u and v with a new vertex u'. Every face τ that contains u, v, or both is then replaced by $(\tau \backslash \{u, v\}) \cup \{u'\}$, while all other faces remain unchanged. The resulting complex Δ' is a (homology) 5-sphere; see [21, Proposition 2.3].

In what follows, we write $(u_1, u_2, \ldots, u_j)$ to denote the *j-cycle* with edges $u_1 u_2, \ldots, u_{j-1}u_j, u_j u_1$. We also write $H_i(\Delta)$ to denote the i-th homology group of Δ with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Lemma 5.4. *Assume that $\Delta \in S(2,5)$ satisfies $g_3(\Delta) = 1$. If Δ has a contractible edge, then Δ is the join of three 3-cycles.*

Proof: Let uv be a contractible edge of Δ, and let Δ' be the 5-sphere obtained from Δ by contracting uv to a new vertex u'. Then $g_3(\Delta') = g_3(\Delta) - g_2(\operatorname{lk}(uv, \Delta))$. Since $\Delta \in S(2,5)$, $\operatorname{lk}(uv) \in S(2,3)$. Hence $g_2(\operatorname{lk}(uv)) = 1$ and $g_3(\Delta') = 0$. By Theorem 2.4, Δ' is therefore 2-stacked. Moreover, by [21, Theorem 1.3], the conditions $g_2(\operatorname{lk}(uv)) = 1$ and $\operatorname{lk}(uv) \in S(2,3)$ imply that $\operatorname{lk}(uv) = \partial\overline{\tau} * C$, where $\overline{\tau}$ is a 2-simplex and C is a cycle. Note that $\tau \notin \Delta'$, for otherwise τ would be part of a missing face of dimension greater than 2 in Δ. Hence $\partial\tau$ is an induced cycle of Δ'. Therefore, Δ' is 2-stacked but not stacked, and as such it must have at least one missing 4-face.

Let $\{a, b, y, z, u'\}$ be a missing 4-face of Δ', and write $\sigma = \{a, b, y, z\}$. Then $\Delta[\sigma \cup uv]$ contracts to $\Delta'[\sigma \cup u'] = \partial\overline{\sigma \cup u'}$; in particular, $\sigma \in \Delta$. We claim that $\Delta[\sigma \cup uv]$ is the join of two 3-cycles. First, observe that the subcomplex of Δ induced by $V(\Delta)\backslash(\sigma \cup uv)$ and the subcomplex of Δ' induced by $V(\Delta')\backslash(\sigma \cup u')$ are identical. Thus, by Alexander duality, $H_3(\Delta[\sigma \cup uv]) = H_3(\Delta'[\sigma \cup u']) \neq 0$. Since Δ has no missing 4-faces and $\Delta[\sigma \cup uv]$ has only six vertices, it follows that $\Delta[\sigma \cup uv]$ must contain the join of two 3-cycles as a subcomplex. Finally, since Δ has no missing faces of dimension greater than 2, we conclude that $\Delta[\sigma \cup uv]$ is precisely the join of two 3-cycles. Since uv is not contained in any missing 2-face, we may assume without loss of generality that $\Delta[\sigma \cup uv] = (a, b, u) * (y, z, v)$. Then $\operatorname{lk}(uv, \Delta)$ contains the 4-cycle (a, y, b, z). Thus, by the discussion in the previous paragraph, $\operatorname{lk}(uv, \Delta)$ must be $\partial\overline{\tau} * (a, y, b, z)$. In particular, since every missing face of dimension ≥ 4 in Δ' contains u', we see that $\sigma \cup u'$ is the only missing 4-face of Δ'.

Similarly, if Δ' has a missing 5-face, then it is of the form $\sigma' \cup u'$. Consider the complex $\Delta[\sigma' \cup uv]$ that contracts to $\Delta'[\sigma' \cup u'] = \partial\overline{\sigma' \cup u'}$. The same argument as above shows that $\Delta[\sigma \cup uv]$ is a complex on 7 vertices with $H_4 \neq 0$ and no missing faces of dimension larger than 2. Since no such 7-vertex complex exists, it follows that Δ' has no missing 5-faces.

Therefore, the 2-stacked 6-ball $\Delta'(2)$ is the join of the simplex $\overline{\sigma \cup u'}$ and a cycle. Since $\partial\overline{\tau} \subseteq \operatorname{lk}(uv) \subseteq \Delta$, this cycle must be $\partial\overline{\tau}$. It follows that $\Delta = \partial\overline{\tau} * \Delta[\sigma \cup uv] = \partial\overline{\tau} * (a, b, u) * (y, z, v)$, as desired. $\qquad\square$

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Proposition 5.5. *The graph of Δ is a complete graph. Moreover, every edge in the support of ω is contained in at most one missing 2-face.*

Proof: Recall from Lemma 5.3 that the graph of Δ, $G(\Delta)$, is a complete multipartite graph. Thus, to show that $G(\Delta)$ is a complete graph, it suffices to show that each part of $G(\Delta)$ consists of a single vertex.

First, we prove that at most one part of $G(\Delta)$ can have size greater than 1. Assume, for contradiction, that there are at least two such parts. Then there exist four vertices a, b, y, z such that the subcomplex induced by these vertices is the 4-cycle $C = (a, b, y, z)$. Since yz is not in $\mathrm{st}(a) \cup \mathrm{st}(b)$, it follows that $yz \notin \mathrm{supp}(\omega)$. By a similar argument applied to other edges of C, we conclude that none of the edges of C are in $\mathrm{supp}(\omega)$. On the other hand, since $z \in \mathrm{supp}(\omega)$ and $z \notin \mathrm{st}(b)$, the argument in the proof of Lemma 5.2 implies that az lies in $\mathrm{supp}(\omega)$, a contradiction.

It remains to consider the case where exactly one part of $G(\Delta)$, say V_1, has size larger than 1. Let a and y be vertices of V_1, and let b and z be vertices from other parts of $G(\Delta)$. The subgraph of Δ induced by these four vertices is the union of two 3-cycles, (a, b, z) and (b, y, z). Consider $\mathrm{st}(a) \cup \mathrm{st}(b) \supseteq \mathrm{supp}(\omega)$. Since $y \in \mathrm{st}(b)$, $y \notin \mathrm{st}(a)$, and $y \in \mathrm{supp}(\omega)$, the argument from Lemma 5.2 shows that $by \in \mathrm{supp}(\omega)$. By symmetry, all edges ab, by, yz, and za lie in $\mathrm{supp}(\omega)$. Now, since $\mathrm{supp}(\omega)$ is also a subcomplex of $\mathrm{st}(b) \cup \mathrm{st}(y)$ and since $az \notin \mathrm{st}(y)$, it follows that az is in $\mathrm{st}(b)$, so $abz \in \Delta$. In other words, the edge ab, together with any vertex in $V(\Delta) \setminus (V_1 \cup b)$, forms a 2-face of Δ. Hence, ab is not contained in any missing 2-face and is therefore a contractible edge. Lemma 5.4 then implies that Δ is the join of three 3-cycles, and consequently, $G(\Delta)$ is a complete graph. This concludes the proof of the first part of the theorem.

Finally, suppose an edge uv is contained in two missing faces uvy and uvz. Since yz is an edge, we have $\mathrm{supp}(\omega) \subseteq \mathrm{st}(y) \cup \mathrm{st}(z)$. However, uv is contained in neither $\mathrm{st}(y)$ nor $\mathrm{st}(z)$, and hence $uv \notin \mathrm{supp}(\omega)$. This completes the proof. $\square$

We are now ready to prove the second main result of the paper:

Theorem 5.6. *Let $\Delta \in S(2, 5)$ be a sphere with $g_3(\Delta) = 1$. Then Δ is the join of three 3-cycles.*

Proof: Let $n = f_0(\Delta)$. By Proposition 5.5, $f_1(\Delta) = \binom{n}{2}$. The assumption that $g_3(\Delta) = 1$ then implies that $f_2(\Delta) = 5\binom{n}{2} - 15n + 36$.

Assume that Δ is not the join of three 3-cycles. By Lemma 5.4, every edge of Δ is contained in at least one missing 2-face. Hence

$$3\left(5\binom{n}{2} - 15n + 36\right) = 3f_2 = \sum_{e \in \Delta} f_0(\mathrm{lk}(e)) \leq \binom{n}{2}(n - 3).$$

Simplifying this inequality yields $n^3 - 19n^2 + 108n - 216 \geq 0$, and therefore $n \geq 11$.

On the other hand, by Lemma 5.2, every missing 2-face has at least two edges in $\mathrm{supp}(\omega)$. By Proposition 5.5, each of these two edges is not contained in any other missing 2-face. Thus, $2m_2(\Delta) \leq f_1(\Delta) = \binom{n}{2}$. Since $m_2 = \binom{n}{3} - f_2 = \binom{n}{3} - 5\binom{n}{2} + 15n - 36$, we obtain

$$\binom{n}{3} - 5\binom{n}{2} + 15n - 36 \leq \frac{1}{2}\binom{n}{2}.$$

Simplifying this inequality yields $h(n) := 2n^3 - 39n^2 + 217n - 432 \leq 0$. For $n \geq 9$, the function $h(n)$ is strictly increasing, and since $h(12) = 12 > 0$, it follows that $n \leq 11$. We conclude that Δ has exactly 11 vertices.

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Since Δ has a complete graph, a direct computation yields $f_1(\Delta) = 55$, $f_2(\Delta) = 146$, and $m_2(\Delta) = \binom{n}{3} - f_2 = 19$. Also, by Lemma 5.4, each edge of Δ is contained in at least one missing 2-face. Since $19 \cdot 3 = 55 + 2$, we obtain that, with an exception of at most two edges, each edge of Δ, is contained in a unique missing 2-face. It follows that there exists a vertex v such that each edge incident to v lies in a unique missing 2-face. Then $\mathrm{lk}(v)$ is a 4-sphere whose graph contains exactly five missing edges $e_1, \ldots, e_5$ and they are pairwise disjoint. Hence $\mathrm{lk}(v)$ is a subcomplex of the octahedral 4-sphere. Since $\mathrm{lk}(v)$ is itself a 4-sphere, it must be precisely the octahedral 4-sphere. In particular, every triple of vertices of $\mathrm{lk}(v)$ that does not contain one of the edges e_i is a 2-face. We conclude that every missing 2-face of Δ contains one of the edges e_i. But then the total number of missing 2-faces is at most $5 + 2 = 7 < 19$, which yields a contradiction. □

6 Open problems

We close the paper with a few open problems. First, it is natural to ask whether one can remove the assumptions on the dimensions of missing faces and provide a characterization of *all* $(d-1)$-spheres with $g_k = 1$. To this end, and in view of the GLBT, we propose the following conjecture.

Conjecture 6.1. *Let $k \geq 2$, $d \geq 2k$, and let Δ be a $(d-1)$-sphere with $g_k(\Delta) = 1$. Then there exists a d-dimensional cell complex $\mathcal{C}$ with the following properties:*[3]

1. *all faces of $\mathcal{C}$ of dimension $\leq d - k$ are faces of Δ; furthermore, all faces of $\mathcal{C}$ except one d-face are simplices;*

2. *the one exceptional d-face is a homology d-ball whose boundary is a sphere in $S(d - k, d - 1)$ with $g_k = 1$;*

3. *every two faces of $\mathcal{C}$ intersect in a common (possibly empty) face;*

4. *the geometric realization of $\mathcal{C}$ is a homology d-ball, and the boundary complex of $\mathcal{C}$ is Δ.*

Furthermore, if Δ is the boundary complex of a simplicial d-polytope, then the the exceptional d-cell is also the boundary complex of a simplicial d-polytope.

It is not hard to see that if such a cell complex exists, then $g_k(\Delta) = 1$. The converse direction appears to be much harder: the case $k = 2$ follows from [21], but all other cases remain open.

While in Theorem 5.6, we characterized all spheres in $S(2,5)$ with $g_3 = 1$, for $k > 3$, the question of which spheres in $S(k - 1, 2k - 1)$ can have $g_k = 1$ remains open. Recall the definition of $K(i, d - 1) \in S(i, d - 1)$: for fixed integers d and i, write $d = qi + r$ with $1 \leq r \leq i$, and define $K(i, d - 1) := (\partial \sigma^i)^{*q} * \partial \sigma^r$. In particular, the unique sphere in $S(2,5)$ with $g_3 = 1$ is $K(2,5)$.

Using the fact that $h_j(\partial \sigma^i) = 1$ for all $0 \leq j \leq i$, together with the identity $h_j(\Delta * \Gamma) = \sum_{0 \leq \ell \leq j} h_\ell(\Delta) h_{j-\ell}(\Gamma)$ for all j, one can easily prove the following result.

Lemma 6.2. *Let $2i < d \leq 3i$. Then*

$$g_j(K(i, d - 1)) = \begin{cases} j + 1 & \text{if } 0 \leq j \leq r \\ r + 1 & \text{if } r + 1 \leq j \leq i \\ d + 1 - 2j & \text{if } i + 1 \leq j \leq d/2 \end{cases}.$$

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[3]The complex $\mathcal{C}$ is more general than a CW complex because one of the cells of $\mathcal{C}$ may not be homeomorphic to a ball.

In particular, for $k \geq 3$, we have $g_k(K(i, 2k-1)) = 1$ whenever $\frac{2k}{3} \leq i < k$.

In light of Theorem 5.6, it is natural to pose the following question.

Question 6.3. *Let $k \geq 4$. Let $\Delta \in S(k-1, 2k-1)$ be a sphere with $g_k(\Delta) = 1$. Must Δ be one of the spheres $K(i, 2k-1)$, where $\frac{2k}{3} \leq i < k$?*

Finally, we note that among all spheres in $S(i, d-1)$, the sphere $K(i, d-1)$ simultaneously minimizes all the f- and h-numbers; see [7, 20]. This observation leads to another natural question.

Question 6.4. *In $S(i, d-1)$, does $K(i, d-1)$ simultaneously minimize all the g-numbers? If so, is it the unique minimizer?*

In the class of PL flag $(d-1)$-spheres, $K(1, d-1)$—namely the octahedral $(d-1)$-sphere—is indeed the unique minimizer of all g-numbers [23]. All other cases remain open.

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