ON FACE NUMBERS OF MANIFOLDS WITH SYMMETRY

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Abstract. Necessary conditions on the face numbers of Cohen-Macaulay simplicial complexes admitting a proper action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ of a prime order are given. This result is extended further to necessary conditions on the face numbers and the Betti numbers of Buchsbaum simplicial complexes with a proper $\mathbb{Z}/p\mathbb{Z}$-action. Adin's upper bounds on the face numbers of Cohen-Macaulay complexes with symmetry are shown to hold for all $(d-1)$-dimensional Buchsbaum complexes with symmetry on $n \geq 3d-2$ vertices. A generalization of Kühnel's conjecture on the Euler characteristic of $2k$-dimensional manifolds and Sparla's analog of this conjecture for centrally-symmetric $2k$-manifolds are verified for all $2k$-manifolds on $n \geq 6k+3$ vertices. Connections to the Upper Bound Theorem are discussed and its new version for centrally symmetric manifolds is established.

1. Introduction

One of the central problems in geometric combinatorics is to characterize or at least to obtain significant necessary conditions on the face numbers of different classes of simplicial complexes. Among the earliest results in this quest was the Kruskal–Katona theorem [21], [17] that characterized the face numbers of all simplicial complexes. Since then many powerful tools and techniques have been developed, among them are the theory of Stanley-Reisner rings, one of whose first applications was the characterization (due to Stanley [34]) of the face numbers of all Cohen-Macaulay complexes, and the theory of algebraic shifting (introduced by Kalai and closely related to the notion of generic initial ideals) that culminated in the characterization of the face numbers of all simplicial complexes with prescribed Betti numbers (due to Björner–Kalai [7]).

In this paper we strengthen Stanley's characterization of the face numbers of Cohen-Macaulay simplicial complexes (e.g., simplicial spheres) to certain necessary conditions on the face numbers of Cohen-Macaulay complexes with a proper $\mathbb{Z}/p\mathbb{Z}$-action (for a prime p). We then generalize those conditions further for the class of Buchsbaum complexes (e.g., simplicial manifolds) with a proper $\mathbb{Z}/p\mathbb{Z}$-action. The latter result is similar in spirit to the necessity portion of the theorem of Björner – Kalai on face numbers and Betti numbers of simplicial complexes.

Our proofs are based on a variant of algebraic shifting, namely the study of the combinatorics of the initial ideal of (the isomorphic image of) the Stanley-Reisner ideal. The main novelty of the approach is that instead of the generic initial ideal (used in the classical algebraic shifting) we explore the behavior of a special (only partially generic) initial ideal appropriate for complexes with symmetry. The idea for such a technique was motivated by Stanley's approach of exploiting special systems of parameters when the simplicial complex at hand has additional structure (see for example [35, 36, 37, 2, 3, 4]). What makes this non-generic shifting work is roughly the fact (due to Schenzel) that in the case of Buchsbaum complexes all linear systems of parameters "behave alike" (see Theorems 2.6 and 2.7).

The search for bounds on the face numbers and Betti numbers of Buchsbaum complexes was motivated by the desire to extend the Upper Bound Theorem (abbreviated, UBT) for polytopes and spheres to the class of simplicial manifolds having the same Euler characteristic as a sphere. In its original form the Upper Bound Conjecture, proposed by Motzkin in 1957, asserted that among all d-dimensional polytopes on n-vertices the cyclic polytope maximizes the number of i-dimensional faces for every i. Over the last 40 years this conjecture has been treated extensively by many mathematicians, and a large number of generalizations have been proved (see [20, 25, 33, 30, 6, 28, 12]). Related to the UBT is a conjecture by Kühnel [22] for the maximum value of the Euler characteristic of simplicial even-dimensional manifolds with n vertices. In the second part of the paper we provide several applications of our main theorem to this topic: we prove a single generalization of the Kühnel conjecture and its companion conjecture for centrally symmetric manifolds (due to Sparla) for all manifolds with a sufficiently large number of vertices, and establish upper bounds on the face numbers of manifolds with symmetry (leading in particular to a new version of the UBT for centrally symmetric manifolds).

We start by describing the basic concepts and the main results, deferring most of the definitions until the next section.

Let $S(k_1, \ldots, k_n)$ (where $1 \leq k_i \leq \infty$) denote the (po)set of all monomials $x_1^{a_1} \cdots x_n^{a_n}$ with $0 \leq a_i \leq k_i$ for $i = 1, 2, \ldots, n$ (ordered by divisibility). In particular, $S(\infty, \ldots, \infty) =: S(\infty^n)$ is the set of all monomials in n variables, and $S(1, \ldots, 1) =: S(1^n)$ is the set of all squarefree monomials. A *multicomplex* is a collection of monomials $\Gamma \subseteq S(\infty^n)$ such that it is closed under divisibility (i.e., $\mu' | \mu \in \Gamma \Longrightarrow \mu' \in \Gamma$) and $x_i \in \Gamma$ for all $1 \leq i \leq n$. The F-vector of a multicomplex Γ is the vector $F(\Gamma) = (F_0, F_1, \ldots)$, where $F_i = F_i(\Gamma) := |\{\mu \in \Gamma : \deg \mu = i\}|$. (Thus $F_1 = n$ and $F_0 = 1$, since $1 = x_1^0 \cdots x_n^0$ is the only monomial of degree 0.)

A multicomplex Γ is a *simplicial complex* if $\Gamma \subseteq S(1^n)$. The elements of Γ are called *faces*, and the maximal ones (under divisibility) are called *facets*. We say that $\mathcal{F} \in \Gamma$ is an *i-dimensional face* (or an *i-face*) if $\deg \mathcal{F} = i + 1$. (0-faces are usually referred to as *vertices*.) We also define the *dimension of* Γ, $\dim \Gamma$, as the maximal dimension of its faces. The *f-vector* of a $(d-1)$-dimensional simplicial complex Γ is the vector $f(\Gamma) = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$, where f_i denotes the number of i-faces of Γ. Thus for a simplicial complex Γ, $f(\Gamma)$ differs from $F(\Gamma)$ only by a shift in the indexing. The h-vector of a $(d-1)$-dimensional simplicial complex Γ is the vector $h(\Gamma) = (h_0, h_1, \ldots, h_d)$ such that

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1} (x-1)^{d-i}.$$

Equivalently,

$$(1) \quad f_{i-1} = \sum_{j=0}^{i} \binom{d-j}{d-i} h_j \quad \text{and} \quad h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}, \quad i = 0, 1, \ldots, d.$$

Unlike the f-numbers f_{i-1}, the h-numbers h_i may, in general, be negative. An important class of simplicial complexes with non-negative h-numbers is the class of *Cohen–Macaulay* complexes over a field $\mathbf{k}$ (in short: $\mathbf{k}$-CM or CM). Moreover, a celebrated theorem due to Stanley [34], [38, Thm. II.3.3] asserts that a sequence

$h = (h_0, h_1, \ldots, h_d) \in \mathbb{Z}^{d+1}$ is the h-vector of a $(d-1)$-dimensional CM complex on n vertices if and only if h is the F-vector of a multicomplex $\mathcal{M} \subseteq S(\infty^{n-d})$ on $n-d$ variables. Our first result is the following generalization of Stanley's theorem. (Write $S(a^r, b^s)$ for the set $S(k_1, \ldots, k_{r+s})$ with $k_1 = \ldots = k_r = a$ and $k_{r+1} = \ldots = k_{r+s} = b$.)

Theorem 1.1. *Let Γ be a $(d-1)$-dimensional CM complex on $n = l + pm$ vertices endowed with a proper action of $G = \mathbb{Z}/p\mathbb{Z}$, where l is the number of G-invariant vertices and m is the number of G-orbits of non-invariant vertices. Then $h(\Gamma) = (h_0, h_1, \ldots, h_d)$ is the F-vector of a multicomplex $\mathcal{M} \subseteq S\left((p-1)^m, \infty^{n-d-m}\right)$.*

The significance of Theorem 1.1 is that it imposes strong restrictions on possible face numbers of a CM complex with a group action. The following (tight) upper bounds on the h-numbers (originally due to Adin [2]) are immediate consequences of Theorem 1.1.

Corollary 1.2. *Let Γ be a $(d-1)$-dimensional CM complex on n vertices admitting a proper action of $G = \mathbb{Z}/p\mathbb{Z}$. Then*

$$h_j(\Gamma) \leq F_j\left(S\left((p-1)^m, \infty^{n-d-m}\right)\right), \quad 0 \leq j \leq d$$

where m denotes the number of G-orbits of non-invariant vertices.

A characterization of the F-vectors of multicomplexes $\mathcal{M} \subseteq S(\infty^n)$ is due to Macaulay [24], and those of simplicial complexes $\Gamma \subseteq S(1^n)$ is due to Kruskal [21] and Katona [17]. Clements and Lindström [10], [13, Section 8] generalized both of these results by characterizing all possible F-vectors of multicomplexes $\mathcal{M} \subseteq S(k_1, \ldots, k_n)$. Their result asserts the existence and provides an explicit description of the function $\partial_{S,j}$ (for every $S = S(k_1, \ldots, k_n)$ and $j \geq 1$) such that a non-negative integer vector $F = (F_0, F_1, \ldots)$ is the F-vector of some multicomplex $\mathcal{M} \subseteq S$ if and only if $F_0 = 1$, $F_1 = n$, and $\partial_{S,j}(F_{j+1}) \leq F_j$ for all $j \geq 1$.

Our next result is the following generalization of Theorem 1.1 for a ($\mathbf{k}$-)Buchsbaum complex Γ, that is, a complex whose local homology (with coefficients in $\mathbf{k}$) vanishes below the top dimension; see Definition 2.5. Denote by $\widetilde{H}_i(\Gamma, \mathbf{k})$ the i-th reduced simplicial homology of Γ with coefficients in $\mathbf{k}$, and by $\beta_i(\Gamma) = \dim_{\mathbf{k}} \widetilde{H}_i(\Gamma; \mathbf{k})$ the reduced Betti numbers of Γ. Define

$$h'_j(\Gamma) := h_j(\Gamma) + \binom{d}{j} \sum_{i=0}^{j-1} (-1)^{j-i-1} \beta_{i-1}(\Gamma) \text{ for } j = 0, 1, \ldots, d. \tag{2}$$

Theorem 1.3. *Let Γ be a $(d-1)$-dimensional $\mathbf{k}$-Buchsbaum complex on n vertices endowed with a proper action of $G = \mathbb{Z}/p\mathbb{Z}$. Let m be the number of G-orbits of non-invariant vertices, and let $S = S\left((p-1)^m, \infty^{n-d-m}\right)$. Then $h'_0(\Gamma) = 1$, $h'_1(\Gamma) = n - d$, and*

$$\partial_{S,j}(h'_{j+1}(\Gamma)) \leq h'_j(\Gamma) - \binom{d-1}{j} \beta_{j-1}(\Gamma) \quad \textit{for all } 1 \leq j \leq d-1,$$

where $\partial_{S,j}$ is the Clements–Lindström function.

We remark that the main reason for defining the modified h-vector, h', (Eq. (2)) is the Schenzel theorem (Thm. 2.6 below), and that $h'(\Gamma) = h(\Gamma)$ if Γ is $\mathbf{k}$-CM complex (since in this case $\beta_i(\Gamma) = 0$ for $i < d-1$). A special case of Theorem 1.3 for the trivial group-action (with additional restriction that $\mathbf{k}$ is a field of characteristic zero) was verified in [28], where it was used as a key step in the proof of the UBT for several classes of simplicial manifolds and the proof of the Kühnel conjecture for $2k$-dimensional manifolds with at least $4k+3$ vertices. Here we prove the general result as well as simplify the original construction.

While the reduced Euler characteristic of all odd-dimensional manifolds equals -1 (as follows from Poincaré duality), the Euler characteristic of even-dimensional manifolds may vary. Kühnel's conjecture [22] asserts that for every $2k$-dimensional simplicial manifold Γ with n vertices, its reduced Euler characteristic, $\widetilde{\chi}(\Gamma) = \sum_{i=-1}^{2k}(-1)^i f_i = \sum_{i=0}^{2k}(-1)^i\beta_i$, satisfies the inequality

$$(-1)^k\binom{2k+1}{k}(\widetilde{\chi}(\Gamma)-1) \leq \binom{n-k-2}{k+1}.$$

An analogous conjecture for the case of centrally symmetric manifolds (that is, simplicial manifolds admitting a free action of $\mathbb{Z}/2\mathbb{Z}$) of dimension $2k$ on $n=2m$ vertices was proposed by Sparla [31]. It asserts that for such a complex Γ

$$(-1)^k\binom{2k+1}{k}(\widetilde{\chi}(\Gamma)-1) \leq 4^{k+1}\binom{\frac{1}{2}(m-1)}{k+1}.$$

While Kühnel's bound $\binom{n-k-2}{k+1} = F_{k+1}\left(S(\infty^{n-2k-1})\right) - F_k\left(S(\infty^{n-2k-1})\right)$, Sparla's bound equals $F_{k+1}\left(S(1^m,\infty^{m-2k-1})\right) - F_k\left(S(1^m,\infty^{m-2k-1})\right)$ (see [32]). We use Theorem 1.3 to prove the following generalization of both of these conjectures for $2k$-dimensional manifolds with $n\geq 6k+3$ vertices.

Theorem 1.4. *Let Γ be a $2k$-dimensional simplicial manifold on n vertices endowed with a proper action of $G=\mathbb{Z}/p\mathbb{Z}$. Let m be the number of G-orbits of non-invariant vertices, and let $S=S\left((p-1)^m,\infty^{n-2k-1-m}\right)$. If $n\geq 6k+3$, then*

$$(-1)^k\binom{2k+1}{k}(\widetilde{\chi}(\Gamma)-1) \leq F_{k+1}(S)-F_k(S).$$

Although at present we are able to establish Theorem 1.4 only for manifolds with sufficiently many vertices, our belief is that the result holds in general (see Conjecture 6.2 below).

As an additional application of Theorem 1.3, we prove the following generalization of Adin's upper bounds (Cor. 1.2) for Buchsbaum complexes (that leads to a new version of the UBT for centrally symmetric manifolds — Theorem 5.7).

Theorem 1.5. *Let Γ be a $(d-1)$-dimensional Buchsbaum complex on $n\geq 3d-2$ vertices admitting a proper action of $G=\mathbb{Z}/p\mathbb{Z}$. Then*

$$h_j(\Gamma)\leq F_j\left(S\left((p-1)^m,\infty^{n-d-m}\right)\right), \quad 0\leq j\leq d$$

where m denotes the number of G-orbits of non-invariant vertices.

The structure of the paper is as follows. Section 2 contains the necessary background on Buchsbaum complexes and complexes with a group action. In Section 3, after defining and studying certain monomial sets that are at the root of all our proofs, we verify Theorems 1.1 and 1.3 (in a more general setting of subcomplexes of a certain complex $\Lambda(p,m,l)$ — see Eq. (4)). The remaining sections are devoted to applications of Theorem 1.3: Section 4 contains proofs of Theorems 1.4 and 1.5, and Section 5 deals with the Upper Bound Theorem and, in particular, with its new version for centrally symmetric manifolds. The last section discusses several further conjectures and open problems.

2. Preliminaries on group actions and Buchsbaum complexes

In this section we set up notation and review certain facts and definitions related to simplicial complexes. Throughout the paper let $\Gamma \subseteq S(1^n)$ be a $(d-1)$-dimensional simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$, and let $G = \mathbb{Z}/p\mathbb{Z}$ be the cyclic group of a prime order.

2.1. **Group actions.** A bijection $\sigma : V \to V$ defines a natural map $\sigma : S(1^n) \to S(1^n)$. We call this map a *(simplicial) automorphism* of Γ if for every face $\mathcal{F} \in \Gamma$, $\sigma(\mathcal{F}) \in \Gamma$ as well. Let $\mathrm{Aut}(\Gamma)$ denotes the group of all automorphisms of Γ. An *action* of group G on Γ is a homomorphism $\pi : G \to \mathrm{Aut}(\Gamma)$. An action π of G is *proper* if

$$\pi(h)(\mathcal{F}) = \mathcal{F} \text{ for some } h \in G,\ \mathcal{F} = x_{i_1} \ldots x_{i_k} \in \Gamma \implies \pi(h)(x_{i_j}) = x_{i_j}\ \forall 1 \leq j \leq k.$$

An action π of G is *free* if

$$\pi(h)(\mathcal{F}) = \mathcal{F} \text{ for some } \mathcal{F} \in \Gamma,\ \mathcal{F} \neq 1 \Longrightarrow h \text{ is the unit element of } G.$$

Example 2.1.
1. Let $\Gamma = \Delta^{p-1}$ be a $(p-1)$-dimensional simplex with all its faces. Define the action of G on Γ by letting the generator of G cyclically permute the p vertices of the complex. This is a non-free (and also non-proper) G-action. On the other hand, the same action but restricted to the boundary complex $\partial\Delta^{p-1}$ is a free G-action.
2. Recall that if Γ_1 and Γ_2 are simplicial complexes on two disjoint vertex sets V_1 and V_2, then their join $\Gamma_1 * \Gamma_2 := \{\mu_1 \cdot \mu_2 : \mu_1 \in \Gamma_1, \mu_2 \in \Gamma_2\}$ is a simplicial complex on $V_1 \cup V_2$. A pair of proper G-actions $\pi_i : G \to \mathrm{Aut}(\Gamma_i)$ $(i = 1, 2)$ defines a proper action $\pi : G \to \mathrm{Aut}(\Gamma_1 * \Gamma_2)$ via $\pi(h)(\mu_1 \cdot \mu_2) = \pi_1(h)(\mu_1) \cdot \pi_2(h)(\mu_2)$.

Assume that Γ is endowed with a G-action π. For a vertex v of Γ, define the *G-orbit* of v as $\mathrm{Orb}\,(v) := \{\pi(h)(v) \ : \ h \in G\}$. We say that v is *G-invariant* if $\mathrm{Orb}\,(v) = \{v\}$. (Hence the action of G is trivial iff all the vertices of Γ are G-invariant.) Since G is a group of prime order p, it follows that each vertex v of Γ is either G-invariant or $|\mathrm{Orb}\,(v)| = p$ (we call such an orbit a *free G-orbit*). Thus $n = l + pm$, where l is the number of G-invariant vertices and m is the number of free G-orbits. For notational convenience assume from now on that the last l vertices x_{pm+i}, $1 \leq i \leq l$, are G-invariant and that $\mathrm{Orb}\,(x_i) = \{x_{i+jm} \ : \ 0 \leq j \leq p-1\}$ for $1 \leq i \leq m$. Note that if the G-action on Γ is proper then no free G-orbit forms a face of Γ:

$$x_i x_{i+m} \cdots x_{i+(p-1)m} \notin \Gamma \quad \text{for all } 1 \leq i \leq m. \tag{3}$$

Hence every face of Γ misses at least one vertex from each of m free G-orbits. Thus $d \leq n - m$, and so the poset $S((p-1)^m, \infty^{n-d-m})$ appearing in all the statements of the Introduction is indeed well defined.

Definition and Example 2.2. Let $p \geq 2$, $m, l \geq 0$ be arbitrary integers (with p prime or composite). Let $\partial\Delta_i^{p-1}$ $(i = 1, \ldots, m)$ be the boundary complex of the $(p-1)$-dimensional simplex on the vertex set $\{x_{i+jm} : 0 \leq j \leq p-1\}$, and let Δ^{l-1} be the simplex (with all its faces) on the vertex set $\{x_{pm+j} : 1 \leq j \leq l\}$. Define

$$\Lambda(p, m, l) := \partial\Delta_1^{p-1} * \ldots * \partial\Delta_m^{p-1} * \Delta^{l-1}. \tag{4}$$

Then $\Lambda(p, m, l)$ is the maximal subcomplex of $S(1^n)$ satisfying Eq. (3), where $n = l + pm$. If p is a prime, let $G = \mathbb{Z}/p\mathbb{Z}$ act freely on $\partial\Delta_i^{p-1}$ $(1 \leq i \leq m)$, and trivially on Δ^{l-1}. This defines a proper G-action on $\Lambda(p,m,l)$. Moreover $\Lambda(p,m,l)$ (its $(d-1)$-dimensional skeleton, $\mathrm{Skel}_{d-1}\Lambda(p,m,l)$, respectively) is the maximal subcomplex of $S(1^n)$ among all the complexes ($(d-1)$-dimensional complexes, respectively) that are endowed with a proper G-action and have m free G-orbits and l G-invariant vertices. (Note that in the special case of $p=2$ and $l=0$, the complex $\Lambda(2,m,0)$ is the boundary of the m-dimensional cross polytope C_m^{Δ}.)

Remark 2.3. The theorems stated in the Introduction remain valid and are proved below for all subcomplexes Γ of $\Lambda(p,m,l)$, where $p \geq 2$, $m \geq 0$, $l \geq 0$ are arbitrary integers. We suspect that the techniques in this paper are also applicable to CM and Buchsbaum subcomplexes of $\partial\Delta^{p_1-1} * \cdots * \partial\Delta^{p_m-1} * \Delta^{l-1}$, where $2 \leq p_1 \leq \cdots \leq p_m$. In particular we conjecture that if Γ is a $(d-1)$-dimensional CM subcomplex of $\partial\Delta^{p_1-1} * \cdots * \partial\Delta^{p_m-1} * \Delta^{l-1}$, then $h(\Gamma)$ is the F-vector of a multicomplex $\mathcal{M} \subseteq S(p_1-1,\ldots,p_m-1,\infty^{n-d-m})$, where $n = l + \sum p_i$.

2.2. **Cohen-Macaulay and Buchsbaum rings.** Let $\mathbf{k}$ be a field. Denote by $\mathbf{k}[\mathbf{x}] = \mathbf{k}[x_1,\ldots,x_n]$ the polynomial ring over $\mathbf{k}$ in n variables with the grading $\deg x_i$=1 for all $1 \leq i \leq n$.

Definition 2.4. The *Stanley-Reisner ideal* of Γ, $I_\Gamma \subseteq \mathbf{k}[\mathbf{x}]$, is the ideal generated by the squarefree monomials $x_{i_1}\cdots x_{i_s} \in S(1^n) \setminus \Gamma$. The ring $\mathbf{k}[\mathbf{x}]/I_\Gamma$ is called the *Stanley-Reisner ring* (or the *face ring*) of Γ.

Example 2.2 continued: The Stanley-Reisner ideal of $\Lambda(p,m,l)$ is minimally generated by m monomials $x_i x_{i+m} \cdots x_{i+(p-1)m}$, $1 \leq i \leq m$.

Historically, a simplicial complex Γ is called a *Cohen-Macaulay complex* (a *Buchsbaum complex* respectively) if its Stanley-Reisner ring is Cohen-Macaulay (Buchsbaum, respectively). In this paper we adopt the following topological characterizations of Cohen-Macaulay and Buchsbaum simplicial complexes due to Munkres [27] and Schenzel [30] respectively as their definitions.

Definition 2.5. A $(d-1)$-dimensional simplicial complex Γ is a *Buchsbaum complex* over $\mathbf{k}$ (in short: $\mathbf{k}$-Buchsbaum or Buchsbaum) if its geometric realization $X = |\Gamma|$ satisfies

$$H_i(X, X-\{p\};\mathbf{k}) = 0 \quad \text{for all } p \in X, i < d-1,$$

where $H_i(X, X-\{p\};\mathbf{k})$ denotes the i-th relative singular homology with coefficients in $\mathbf{k}$. If, in addition, the i-th reduced singular homology $\widetilde{H}_i(X;\mathbf{k}) = 0$ for all $i < d-1$, then Γ is called a *Cohen-Macaulay complex* over $\mathbf{k}$ (in short: $\mathbf{k}$-CM or CM).

Thus all simplicial spheres (i.e., simplicial complexes whose geometric realization is homeomorphic to a sphere) are $\mathbf{k}$-CM for any $\mathbf{k}$, and all simplicial manifolds (i.e., simplicial complexes whose geometric realization is a compact topological manifold without boundary) are Buchsbaum. Another example of a CM complex is the complex $\Lambda(p,m,l)$. It is also worth noting that every Buchsbaum complex is *pure* (that is, all its facets have the same dimension as Γ), and that if a complex is $\mathbf{k}$-CM (or $\mathbf{k}$-Buchsbaum) then it is $\mathbf{K}$-CM ($\mathbf{K}$-Buchsbaum, respectively) for each field $\mathbf{K}$ of the same characteristic as $\mathbf{k}$.

Every $g \in GL_n(\mathbf{k})$ defines a graded automorphism of $\mathbf{k}[\mathbf{x}]$ via $g(x_j) = \sum_{i=1}^n g_{ij}x_i$, and in particular gI_Γ is a *homogeneous ideal* of $\mathbf{k}[\mathbf{x}]$. Thus $\mathbf{k}[\mathbf{x}]/gI_\Gamma$ is a *graded* algebra over $\mathbf{k}$ isomorphic to the Stanley-Reisner ring of Γ. Hence (by [38, Thm. II.1.3]) the *Krull dimension* of $\mathbf{k}[\mathbf{x}]/gI_\Gamma$ (i.e., the maximum number of algebraically independent elements over $\mathbf{k}$ in $\mathbf{k}[\mathbf{x}]/gI_\Gamma$) is d. In fact, by the result due to Kind and Kleinschmidt [18], [38, Lemma III.2.4], $x_{n-d+1}, \ldots, x_n$ considered as elements of $\mathbf{k}[\mathbf{x}]/gI_\Gamma$ are algebraically independent over $\mathbf{k}$ iff g possesses the following property (referred to as the Kind-Kleinschmidt condition):

- for every facet $x_{i_1}\cdots x_{i_k} \in \Gamma$, the submatrix of g^{-1} defined by the intersection of its last d columns and the rows numbered $i_1, \ldots, i_k$ has rank k.

From now on assume that g satisfies this condition. (If $\mathbf{k}$ is infinite, then at least one such matrix exists.)

Consider homogeneous ideals $J(\Gamma, g)^s = J^s := gI_\Gamma + \langle x_{n-s+1}, \ldots, x_n\rangle$ in $\mathbf{k}[\mathbf{x}]$ and graded $\mathbf{k}[\mathbf{x}]$-modules $M(\Gamma, g)^s = M^s := \mathbf{k}[\mathbf{x}]/J^s$, where $0 \leq s \leq d$ and g satisfies the Kind-Kleinschmidt condition. Stanley ([33], [38, Thm. I.5.10]) showed that if Γ is a $\mathbf{k}$-CM complex, then

$$\dim_{\mathbf{k}}(M(\Gamma, g)^d)_j = h_j(\Gamma) \quad (0 \leq j \leq d) \quad \text{for every such } g, \tag{5}$$

where $(-)_j$ stands for the j-th homogeneous component. A generalization of this result for Buchsbaum complexes is due to Schenzel [30]:

Theorem 2.6. *Let Γ be a $(d-1)$-dimensional $\mathbf{k}$-Buchsbaum complex and let β_i, $-1 \leq i \leq d-1$, be its reduced Betti numbers over $\mathbf{k}$. Then*

$$\dim_{\mathbf{k}}(M(\Gamma, g)^d)_j = h_j(\Gamma) + \binom{d}{j}\sum_{i=0}^{j-1}(-1)^{j-i-1}\beta_{i-1}(\Gamma) = h'_j(\Gamma) \text{ for } j = 0, 1, \ldots, d,$$

where g is any matrix satisfying the Kind-Kleinschmidt condition.

We close this section by two additional properties of Buchsbaum complexes (rings). The first one is [39, Prop. I.1.10] and the second one is [39, Cor. II.4.16].

Theorem 2.7. *Let Γ be a $(d-1)$-dimensional $\mathbf{k}$-Buchsbaum complex and let β_i, $-1 \leq i \leq d-1$ be its reduced Betti numbers over $\mathbf{k}$. Then*

(a) $0 :_{M^s} x_{n-s}^2 = 0 :_{M^s} x_{n-s} = \bigcap_{i=s}^n 0 :_{M^s} x_{n-i}$ *for all* $0 \leq s \leq d-1$, *where for* $y \in \mathbf{k}[\mathbf{x}]$, $0 :_{M^s} y := \{u \in M^s \ : \ yu = 0\}$.

(b) $\dim_{\mathbf{k}}(0 :_{M^s} x_{n-s})_j = \binom{s}{j}\beta_{j-1}$ *for all* $0 \leq j$ *and* $0 \leq s \leq d-1$.

3. Monomial bases and shadows

In this section after defining and studying certain monomial sets that are at the root of all our proofs, we verify Theorems 1.1 and 1.3. We fix the reverse lexicographic order $\succ$ on $S(\infty^n)$ – the set of all monomials in $x_1, \ldots x_n$ – with $x_1 \succ x_2 \succ \ldots \succ x_n$ extending the partial ordering by degree. We write $\mathrm{in}(I)$ for the reverse lexicographic *initial ideal* of a homogeneous ideal $I \in \mathbf{k}[\mathbf{x}]$ (see [11, Section 15.2]).

Recall that for a $(d-1)$-dimensional simplicial complex Γ and a matrix $g \in GL_n(\mathbf{k})$ satisfying the Kind-Kleinschmidt condition,

$$J(\Gamma, g)^s = J^s := gI_\Gamma + \langle x_{n-s+1}, \ldots, x_n\rangle \text{ and } M(\Gamma, g)^s = M^s := \mathbf{k}[\mathbf{x}]/J^s.$$

Definition 3.1. For $0 \leq s \leq d$ and $j \geq 0$ define

$$B^s(\Gamma, g) = B^s := S(\infty^n) \setminus \mathrm{in}(J^s) \ \text{ and } \ B^s_j := \{\mu \in B^s \ : \ \deg(\mu) = j\},$$

that is, B^s is the set of all monomials in variables $x_1, \dots x_n$ that do not belong to $\text{in}(J^s)$. Equivalently,

$$B^s(\Gamma, g) = B^s := \{\mu \in S(\infty^n) \ : \ \mu \notin \text{span}_{\mathbf{k}}(\{\nu : \mu \succ \nu\} \cup J^s)\} \tag{6}$$

For a generic matrix g, the definitions of B^s and B^s_j were given in [16]. The following lemma summarizes some elementary properties of these sets.

Lemma 3.2. *Let Γ be any simplicial complex. Then*

(a) *B^s is an order ideal — if $\mu \in B^s$ and μ' divides μ, then $\mu' \in B^s$.*
(b) *B^s (B^s_j) is a $\mathbf{k}$-basis of M^s ($(M^s)_j$ respectively).*
(c) *$B^s = S(\infty^{n-s}) \cap B^0$, where $S(\infty^{n-s})$ is the set of all monomials in variables $x_1, \dots, x_{n-s}$, and hence $B^s = \{\mu \in B^{s-1} \ : \ x_{n-s+1} \not| \mu\}$ (for $1 \le s \le d$).*

Proof: (a) follows form the fact that $\text{in}(J^s)$ is an ideal. (b) is a consequence of [11, Thm. 15.3]. (It also follows from Eq. (6).) (c) is implied by the following property of reverse lexicographic order [11, Thm. 15.12a]:

$$\text{in}(J^s) = \text{in}(J^0 + \langle x_{n-s+1}, \dots, x_n \rangle) = \text{in}(J^0) + \langle x_{n-s+1}, \dots, x_n \rangle.$$

□

We are now ready to verify Theorem 1.1. In fact we prove the following stronger fact.

Theorem 3.3. *Let Γ be a $(d-1)$-dimensional CM complex on $n = l + mp$ vertices, where $p \ge 2$, $m, l \ge 0$ are arbitrary integers. If Γ is a subcomplex of $\Lambda(p, m, l)$, then the h-vector of Γ is the F-vector of a multicomplex $\mathcal{M} \subseteq S((p-1)^m, \infty^{n-d-m})$.*

Proof: We construct a matrix g satisfying the Kind-Kleinschmidt condition and such that $B^d(\Gamma, g) \subseteq S((p-1)^m, \infty^{n-d-m})$. By Lemma 3.2(ab) and Eq. (5), $\mathcal{M} = B^d(\Gamma, g)$ then provides the required multicomplex. For the proof, we replace field $\mathbf{k}$ by a larger field $\mathbf{K}$ defined below and perform all computations inside $\mathbf{K}[\mathbf{x}]$ rather than $\mathbf{k}[\mathbf{x}]$. In particular, we regard J^s (M^s) as an ideal of $\mathbf{K}[\mathbf{x}]$ (module over $\mathbf{K}[\mathbf{x}]$, respectively).

We first consider the case of $l = 0$. Let $\mathbf{K} = \mathbf{k}\,(y_{ij} : 1 \le i, j \le (p-1)m)$ be the field of rational functions in $(p-1)^2 m^2$ variables y_{ij}, let I_m denote the $m \times m$ identity matrix, let $E = [I_m | I_m | \cdots | I_m]$ be the $m \times (p-1)m$ matrix consisting of $(p-1)$ blocks of I_m, and let $Y = (y_{ij}) \in GL_{(p-1)m}(\mathbf{K})$. Define

$$g^{-1} = \begin{bmatrix} I_m & -EY \\ O & Y \end{bmatrix} \in GL_{pm}(\mathbf{K}), \qquad \text{so that} \qquad g = \begin{bmatrix} I_m & E \\ O & Y^{-1} \end{bmatrix}.$$

(Here O stands for the zero-matrix.) In particular $g_{k,i+jm} = 0$ for all $1 \le k < i \le m$.

Since $\prod_{j=0}^{p-1} x_{i+jm} \in I_{\Lambda(p,m,l)} \subseteq I_\Gamma$ for $1 \le i \le m$, it follows that

$$J^0 = gI_\Gamma \ni \prod_{j=0}^{p-1} g(x_{i+jm}) = \prod_{j=0}^{p-1} \left(x_i + \sum_{k>i} g_{k,i+jm} x_k \right) = x_i^p + \sum \{\alpha_\mu \mu \ : \ \mu \prec x_i^p\}.$$

Thus $x_i^p \in \text{in}(J^0)$, $1 \le i \le m$, implying that $x_i^p \notin B^0(\Gamma, g)$ for $1 \le i \le m$. Lemma 3.2(ac) then yields that $B^d(\Gamma, g) \subseteq S((p-1)^m, \infty^{n-d-m})$.

To check the Kind-Kleinschmidt condition, note that the i-th row of EY equals the sum of the rows of Y indexed by $\{k : k \equiv i \mod m\}$ (for all $1 \le i \le m$), and that for every face $\mathcal{F} \in \Gamma$, $\prod_{j=0}^{p-1} x_{i+jm} = \prod_{k \equiv i \mod m} x_k \not| \mathcal{F}$ (for all $1 \le i \le m$).

Hence for every facet $\mathcal{F} = x_{j_1} \cdots x_{j_d} \in \Gamma$, the determinant of submatrix $C_{\mathcal{F}}$ of g^{-1} defined by the intersection of the last d columns and rows numbered $j_1, \ldots, j_d$ can be written as a non-empty sum $\det C_{\mathcal{F}} = \sum \pm \det C_\gamma$ for certain distinct submatrices C_γ of Y. Since the entries of Y are algebraically independent over $\mathbf{k}$, we infer that $\det C_{\mathcal{F}} \neq 0$, and the Kind-Kleinschmidt condition follows.

For any $l > 0$, we replace $\mathbf{k}$ by the field $\mathbf{K} = \mathbf{k}(y_{ij}, w_{ij}, z_{ij})$ of rational functions in $(p-1)^2m^2 + l^2 + pml$ variables. Let $W = (w_{ij})$ and $Z = (z_{ij})$ be $l \times l$ and $pm \times l$ matrices respectively, and let Y and E be as above. Define

$$g^{-1} = \begin{bmatrix} \begin{bmatrix} I_m & -EY \\ O & Y \end{bmatrix} & Z \\ O & W \end{bmatrix}, \quad \text{so that} \quad g = \begin{bmatrix} \begin{bmatrix} I_m & E \\ O & Y^{-1} \end{bmatrix} & * \\ O & W^{-1} \end{bmatrix}$$

The same reasoning as above shows that $B^d(\Gamma, g) \subseteq S((p-1)^m, \infty^{n-d-m})$ and that g satisfies the Kind-Kleinschmidt condition, completing the proof. $\square$

To generalize Theorem 3.3 to the case of Buchsbaum complexes we introduce certain additional sets.

Definition 3.4. For $0 \leq s \leq d-1$ and $j \geq 0$ define

$$A^s(\Gamma, g) = A^s := \{\mu \in B^s \ : \ x_{n-s}\mu \notin B^s\} \ \text{ and } \ A^s_j := \{\mu \in A^s \ : \ \deg(\mu) = j\}.$$

Several remarks are in order. First, if $\mu \in B^s$, then either $x_{n-s}|\mu$, in which case $\mu/x_{n-s} \in B^s \setminus A^s$ (as follows from Lemma 3.2(a)), or $\mu \in B^{s+1}$ (Lemma 3.2(c)). Therefore

$$B^s_{j+1} = B^{s+1}_{j+1} \cup x_{n-s} \cdot (B^s_j \setminus A^s_j), \qquad \text{and so}$$

$$|B^s_{j+1}| = |B^{s+1}_{j+1}| + |B^s_j| - |A^s_j|. \tag{7}$$

Second, $\mu \in A^s_j$ iff $x_{n-s}\mu \notin B^s_{j+1}$, which (by Eq. (6)) happens iff there is a (unique) expansion in M^s:

$$\begin{aligned} x_{n-s}\mu &= \sum \{a_\nu \nu \ : \ \nu \prec x_{n-s}\mu, \ \nu \in B^s_{j+1}, \ a_\nu \in \mathbf{k}\} \\ &= x_{n-s} \sum \{a_\nu \nu' \ : \ \nu' \prec \mu, \ \nu' = \nu/x_{n-s} \in B^s_j, \ a_\nu \in \mathbf{k}\} \end{aligned}$$

(since for a monomial ν in variables $x_1, \ldots, x_{n-s}$ and the reverse lexicographic order $\succ$, if $\nu \prec x_{n-s}\mu$, then $x_{n-s}|\nu$). This defines a map

$$\phi : A^s \to 0 :_{M^s} x_{n-s}, \quad \mu \mapsto \mu - \sum \{a_\nu \nu' \ : \ \mu \succ \nu' \in B^s\}.$$

The set $\{\phi(\mu) \ : \ \mu \in A^s\}$ is a linearly independent subset of $0 :_{M^s} x_{n-s} \subset M^s$ since its initial terms are. The following lemma shows that in fact $\{\phi(\mu) \ : \ \mu \in A^s\}$ is a $\mathbf{k}$-basis of $0 :_{M^s} x_{n-s}$.

Lemma 3.5. *For any simplicial complex Γ, $|A^s_j| = \dim_{\mathbf{k}}(0 :_{M^s} x_{n-s})_j$.*

Proof: Since the sequence of $\mathbf{k}$-vector spaces

$$0 \longrightarrow (0 :_{M^s} x_{n-s})_j \longrightarrow (M^s)_j \xrightarrow{\cdot x_{n-s}} (M^s)_{j+1} \longrightarrow (M^{s+1})_{j+1} \longrightarrow 0$$

is exact,

$$\dim_{\mathbf{k}}(0 :_{M^s} x_{n-s})_j - \dim_{\mathbf{k}}(M^s)_j + \dim_{\mathbf{k}}(M^s)_{j+1} - \dim_{\mathbf{k}}(M^{s+1})_{j+1} = 0.$$

Lemma 3.2(b) together with Eq. (7) completes the proof. $\square$

The next lemma describes combinatorics of A^s, B^s for Buchsbaum complexes.

Lemma 3.6. *Let Γ be a $(d-1)$-dimensional Buchsbaum complex. Then*

(a) $|B^d_j| = h'_j(\Gamma)$ *and* $|A^s_j| = \binom{s}{j}\beta_{j-1}(\Gamma)$ *for all* $0 \le s \le d-1$ *and* $j \ge 0$.

(b) *If* $\mu \in A^s$, *then* $x_{n-s} \not| \mu$, *and so* A^s *is a subset of* B^{s+1} *rather than* B^s.

(c) *For every* $\mu \in A^s$ *and every* $i \le n-s$, $x_i\mu \notin B^s$.

Proof: (a) follows from Lemma 3.2(b) and Theorem 2.6, and Lemma 3.5 and Theorem 2.7(b) respectively.

(b): If $x_{n-s}|\mu \in A^s$, then all summands of

$$\phi(\mu) = \mu - \sum\{a_\nu \nu' : \mu \succ \nu'\} \in 0 :_{M^s} x_{n-s}$$

are divisible by x_{n-s}. Hence

$$\mu/x_{n-s} - \sum\{a_\nu \nu'/x_{n-s} \ : \ \nu' \prec \mu\} \in 0 :_{M^s} x^2_{n-s} \overset{\text{Thm. 2.7}}{=} 0 :_{M^s} x_{n-s},$$

and so $\mu/x_{n-s} \in A^s$, contradicting the fact that $\mu \in B^s$.

(c): For $\mu \in A^s$ and $i \le n-s$,

$$\phi(\mu) = \mu - \sum\{a_\nu \nu' \ : \mu \succ \nu' \in B^s\} \in 0 :_{M^s} x_{n-s} \overset{\text{Thm. 2.7}}{\subseteq} 0 :_{M^s} x_i,$$

yielding that $x_i\mu = \sum\{a_\nu x_i \nu' \ : \mu \succ \nu' \in B^s\}$ as elements of M^s, and hence that $x_i\mu \notin B^s$. □

The proof of Theorem 1.3 requires one additional definition.

Definition 3.7. The *shadow* of a set $C \subseteq S(\infty^{n-s})_{j+1}$ of monomials of degree $j+1$ is $\partial_j C := \cup_{\mu\in C}\partial_j\{\mu\}$, where $\partial_j\{\mu\}$ — the *shadow* of a monomial μ — is

$$\partial_j\{\mu\} = \{\mu' \in S(\infty^{n-s})_j \ : \ \mu'|\mu\}.$$

Since $B^s \subseteq S(\infty^{n-s})$ (by Lemma 3.2(c)), Lemma 3.6(bc) implies the following.

Theorem 3.8. *Let Γ be a $(d-1)$-dimensional Buchsbaum complex and let g be a matrix satisfying the Kind-Kleinschmidt condition. Then*

$$A^{d-1}_j(\Gamma, g) \subseteq B^d_j(\Gamma, g) \text{ and } \partial_j B^d_{j+1}(\Gamma, g) \subseteq B^d_j(\Gamma, g) \setminus A^{d-1}_j(\Gamma, g)\ \forall 0 \le j \le d-1.$$

Recall that if $\Gamma \subseteq \Lambda(p, m, l)$ and g is the matrix constructed in the proof of Theorem 3.3, then $B^d(\Gamma, g) \subseteq S((p-1)^m, \infty^{n-d-m}) := S$. Since by Clements-Lindström theorem [10], $\partial_{S,j}(c) = \min\{|\partial_j(C)| \ : \ C \subseteq S_{j+1}, \ |C| = c\}$ (where $j \ge 1$, $c \le F_{j+1}(S)$, and $\partial_{S,j}$ is the Clements-Lindström function), we infer from Theorem 3.8 and Lemma 3.6(a) the following stronger version of Theorem 1.3.

Theorem 3.9. *Let Γ be a $(d-1)$-dimensional Buchsbaum complex on $n = l + mp$ vertices, where $p \ge 2$, $m, l \ge 0$ are arbitrary integers. If Γ is a subcomplex of $\Lambda(p, m, l)$, then $h'_0(\Gamma) = 1$, $h'_1(\Gamma) = n - d$, and*

$$\partial_{S,j}(h'_{j+1}(\Gamma)) \le h'_j(\Gamma) - \binom{d-1}{j}\beta_{j-1}(\Gamma) \quad \textit{for all } 1 \le j \le d-1,$$

where $S = S((p-1)^m, \infty^{n-d-m})$, $\partial_{S,j}$ is the Clements–Lindström function, and

$$h'_j(\Gamma) := h_j(\Gamma) + \binom{d}{j}\sum_{i=0}^{j-1}(-1)^{j-i-1}\beta_{i-1}(\Gamma) \textit{ for } j = 0, 1, \ldots, d.$$

We note that a special case of Theorem 3.9 for $m = 0$ (with additional restriction that $\mathbf{k}$ is a field of characteristic zero) was verified in [28]. The proof of the general result given here is simpler than the previous argument: this simplification comes from reliance on Theorem 2.7 rather than on the later results due to Eisenbud and Stanley involving the notion of peeling (see appendix to [28]).

Remark. Theorem 3.8 together with a suitably chosen matrix g leads to a generalization of Stanley's result on the h-vectors of balanced Cohen-Macaulay complexes [35] to the case of balanced Buchsbaum complexes. However, this result is beyond the scope of this paper.

4. The h-numbers and Euler characteristic

In this section we use Theorem 1.3 (and 3.9) and combinatorial computations similar to those used in [28] to obtain upper bounds on the h-numbers of Buchsbaum subcomplexes of $\Lambda(p, m, l)$ (e.g., complexes with symmetry) and on the Euler characteristic (and moreover on the (weighted) sum of the Betti numbers) of simplicial manifolds that are subcomplexes of $\Lambda(p, m, l)$. Instead of working directly with the Clements-Lindström function (whose description is rather complicated) we utilize the *normalized matching property* of the poset $S((p-1)^m, \infty^{n-d-m})$. We start by reviewing necessary background on finite posets. (Since we work with monomials of degree $\leq d$ only, we replace poset $S((p-1)^m, \infty^{n-d-m})$ with $S((p-1)^m, d^{n-d-m})$.)

Let $(P, <)$ be a finite ranked poset with the rank function rk, (e.g., $P = S(k_1, \dots k_r)$, where $1 \leq k_i < \infty$, ranked by degree) and let $P_j = \{\rho \in P : \mathrm{rk}\,\rho = j\}$. For $C \subseteq P_{j+1}$ we define $\partial_j C := \cup_{\rho \in C} \partial_j\{\rho\}$, where for an element $\rho \in P_{j+1}$, $\partial_j\{\rho\} := \{\rho' \in P_j : \rho' < \rho\}$.

Definition 4.1. We say that a ranked poset P possesses the *normalized matching property* if for every $j < \mathrm{rk}\,P$ and every $\emptyset \neq C \subseteq P_{j+1}$

$$\frac{|\partial_j C|}{|C|} \geq \frac{|P_j|}{|P_{j+1}|}.$$

We also say that P is *log-concave* if $|P_j|^2 \geq |P_{j-1}| \cdot |P_{j+1}|$ for all j.

It is well-known [13, Thm. 4.10] that if posets P and Q are such that they both

(i) have the normalized matching property, and
(ii) are log-concave

then their Cartesian product $P \times Q$ satisfies (i) and (ii) as well. Since $S(k_1, \dots, k_r) \cong S(k_1) \times \cdots \times S(k_r)$ and since $S(k_i)$ clearly satisfies (i) and (ii) we obtain:

Lemma 4.2. *The poset $S = S(k_1, \dots, k_r)$ has the normalized matching property and is log-concave. Thus for every $j < \sum k_i$ and $c \leq F_{j+1}(S)$, $c \leq \frac{F_{j+1}(S)}{F_j(S)} \partial_{S,j}(c)$.*

The following lemma provides a simple lower bound on $F_{j+1}(S)/F_j(S)$.

Lemma 4.3. *Let $S = S(k_1, \dots, k_{r-1}, k_r)$ and let $\widetilde{S} = S(k_1, \dots, k_{r-1}, k_r - 1)$. Then $F_{j+1}(S)/F_j(S) \geq F_{j+1}(\widetilde{S})/F_j(\widetilde{S})$ for all $j < \sum k_i - 1$. In particular*

$$\frac{F_{j+1}(S)}{F_j(S)} \geq \frac{F_{j+1}(S(1^r))}{F_j(S(1^r))} = \frac{\binom{r}{j+1}}{\binom{r}{j}} \quad \text{for } j < r$$

Proof: Consider $T = S(k_1, \dots, k_{r-1})$. Every element μ in S $(\widetilde{S})$ can be written uniquely as $\mu = \mu' x_r^s$, where $\mu' \in T$. Hence

$$F_i(\widetilde{S}) = \sum_{s=0}^{k_r-1} F_{i-s}(T) \quad \text{and} \quad F_i(S) = F_{i-k_r}(T) + F_i(\widetilde{S}) \quad \text{for } i \geq k_r.$$

Also it follows from the log-concavity of T that

$$\frac{F_{i-k_r+1}(T)}{F_{i-k_r}(T)} \geq \frac{F_{i-k_r+2}(T)}{F_{i-k_r+1}(T)} \geq \cdots \geq \frac{F_i(T)}{F_{i-1}(T)} \quad \text{for } i \geq k_r,$$

and we infer that

$$\frac{F_{j+1}(S)}{F_j(S)} = \frac{F_{j+1-k_r}(T) + F_{j+1}(\widetilde{S})}{F_{j-k_r}(T) + F_j(\widetilde{S})} \geq \frac{F_{j+1}(\widetilde{S})}{F_j(\widetilde{S})} \quad \text{for } j \geq k_r$$

(since $\frac{a+b}{a'+b'} \geq \frac{b}{b'}$ for all positive a, a', b, b' satisfying $\frac{a}{a'} \geq \frac{b}{b'}$).

For $j < k_r$, $S_j = \widetilde{S}_j$ and $S_{j+1} \supseteq \widetilde{S}_{j+1}$. Hence the lemma clearly holds in this case as well. ☐

We now verify (a stronger version of) Theorem 1.5. The proof uses the above lemmas and Theorem 3.9, but otherwise is very similar to that of [28, Lemma 6.5].

Theorem 4.4. *Let Γ be a $(d-1)$-dimensional Buchsbaum complex on $n = l + mp$ vertices. If Γ is a subcomplex of $\Lambda(p, m, l)$ and $n \geq 3d - 2$, then*

$$h_j(\Gamma) \leq F_j\left(S\left((p-1)^m, d^{n-d-m}\right)\right) = h_j(\operatorname{Skel}_{d-1}\Lambda(p, m, l)) \quad \textit{for all } 0 \leq j \leq d.$$

Proof: Set $S = S\left((p-1)^m, d^{n-d-m}\right)$. Repeated applications of Lemma 4.2 and Theorem. 3.9 imply that for $j \leq d - 1 < n - d$

$$\begin{aligned} h'_{j+1} &\leq \frac{F_{j+1}(S)}{F_j(S)} \partial_{S,j}(h'_{j+1}) \leq \frac{F_{j+1}(S)}{F_j(S)} \left(h'_j - \binom{d-1}{j}\beta_{j-1}\right) \\ &\leq \frac{F_{j+1}(S)}{F_j(S)} \left(\frac{F_j(S)}{F_{j-1}(S)} \left(h'_{j-1} - \binom{d-1}{j-1}\beta_{j-2}\right) - \binom{d-1}{j}\beta_{j-1}\right) \\ &\leq \dots \leq \frac{F_{j+1}(S)}{F_0(S)} h'_0 - \sum_{i=0}^{j} \frac{F_{j+1}(S)}{F_i(S)} \binom{d-1}{i}\beta_{i-1} \qquad (8) \\ &\leq F_{j+1}(S) - \sum_{i=0}^{j} \frac{\binom{n-d}{j+1}}{\binom{n-d}{i}} \binom{d-1}{i}\beta_{i-1}, \end{aligned}$$

where in the last step we used Lemma 4.3. Since as follows from Eq. (2)

$$h_{j+1}(\Gamma) = h'_{j+1}(\Gamma) - \binom{d}{j+1}\beta_{j-1} + \sum_{i=0}^{j-1} (-1)^{j-i+1} \binom{d}{j+1}\beta_{i-1},$$

and since all $\beta_{i-1} \geq 0$, to prove that $h_{j+1}(\Gamma) \leq F_{j+1}(S)$ it suffices to check that

$$\frac{\binom{n-d}{j+1}}{\binom{n-d}{i}} \binom{d-1}{i} \geq \binom{d}{j+1} \quad \text{for } 0 \leq i \leq j-1. \qquad (9)$$

And indeed a straightforward computation shows that

$$\frac{\binom{n-d}{j+1}}{\binom{n-d}{i}} \binom{d-1}{i} = \frac{n-d-i}{d} \cdot \frac{\binom{n-i-j-2}{d-i-1}}{\binom{n-i-j-2}{d-j-1}} \cdot \binom{d}{j+1}.$$

Since $n \geq 3d-2$ and $i \leq j-1 \leq d-2$, we infer that

$$n-d-i \geq d \quad \text{and} \quad \binom{n-i-j-2}{d-i-1} \geq \binom{n-i-j-2}{d-j-1},$$

(for the latter inequality we observe that $d-i-1 > d-j-1$ and $n-i-j-2 \geq (d-i-1)+(d-j-1)$), and Eq. (9) follows. Finally, the fact that $F_j(S) = h_j(\mathrm{Skel}_{d-1}\Lambda(p,m,l))$ for all $0 \leq j \leq d$ is due to Adin [2, (6.21)–(6.24)]. □

Remarks.

1. The same proof as above implies that if $n \geq 2d+r-2$ (for some $r \leq d$) then $h_j(\Gamma) \leq F_j\left(S\left((p-1)^m, d^{n-d-m}\right)\right)$ for all $j \leq r$.

2. Since $\Lambda(2,m,0) = \partial C_m^\Delta$ — the boundary of the m-dimensional cross polytope — in the case of the centrally symmetric complexes (i.e, complexes with a free action of $\mathbb{Z}/2\mathbb{Z}$) Theorem 4.4 and the previous remark can be restated as follows.

Corollary 4.5. *Let Γ be a $(d-1)$-dimensional centrally symmetric Buchsbaum complex on $n = 2m$ vertices. Then $h_j(\Gamma) \leq h_j(\mathrm{Skel}_{d-1}C_m^\Delta)$ for all $0 \leq j \leq \min\{r,d\}$ provided that $n \geq 2d+r-2$.*

In the rest of the section we obtain upper bounds on the (reduced) Euler characteristic $\widetilde{\chi}(\Gamma) = \sum_{i=-1}^{2k}(-1)^i f_i(\Gamma) = \sum_{i=0}^{2k}(-1)^i\beta_i(\Gamma)$ of a manifold $\Gamma \subseteq \Lambda(p,m,l)$ (e.g., a manifold with symmetry), and moreover on the (weighted) sum of its Betti numbers. The calculations are very similar to those used for the (partial) proof of Kühnel's conjecture [28, Section 5]. (The only difference is that we are working in the poset $S((p-1)^m, \infty^{n-d-m})$ rather than in $S(\infty^{n-d})$.) We start by reviewing several facts and definitions.

Definition 4.6. A simplicial $(d-1)$-dimensional complex Γ (or more precisely, its geometric realization $X = |\Gamma|$) is called *semi-Eulerian* if for all $p \in X$, $\chi(X, X-p) = (-1)^{\dim X}$, where $\chi(X, X-p) = \sum_i (-1)^i \dim_{\mathbf{k}} H_i(X, X-p; \mathbf{k})$ is the relative Euler characteristic. A complex that is both ($\mathbf{k}$-)Buchsbaum and semi-Eulerian is called a *($\mathbf{k}$-)homology manifold*. A connected $\mathbf{k}$-homology manifold Γ is *orientable* (over $\mathbf{k}$) if $H_{d-1}(\Gamma;\mathbf{k}) = \mathbf{k}$. An arbitrary $\mathbf{k}$-homology manifold is orientable if all its connected components are.

Note that $X = |\Gamma|$ is a $(d-1)$-dimensional $\mathbf{k}$-homology manifold iff

$$H_i(X, X-\{p\};\mathbf{k}) = \begin{cases} 0 & \text{for } i < d-1, \\ \mathbf{k} & \text{for } i = d-1 \end{cases} \qquad \text{for all } p \in X,$$

and that there are the following relations between various classes of complexes:

simplicial manifolds $\subset$ homology manifolds $\subset$ Buchsbaum complexes.

It is also well-known that every simplicial manifold is orientable over $\mathbf{k} = \mathbb{Z}/2\mathbb{Z}$.

Poincaré duality [8] asserts that if Γ is an orientable $\mathbf{k}$-homology manifold, then

$$\beta_i(\Gamma) = \begin{cases} \beta_{d-1-i}(\Gamma) & \text{if } 0 < i < d-1, \\ \beta_{d-1}(\Gamma) - 1 & \text{if } i = 0, \end{cases}$$

where $\dim(\Gamma) = d-1$ and $\beta_i(\Gamma)$ are the reduced Betti numbers of Γ over $\mathbf{k}$. A combinatorial analog of this result is the fact (due to Klee [19]) that if Γ is a $(d-1)$-dimensional semi-Eulerian complex (e.g., a homology manifold) then its h-numbers satisfy certain linear relations known as Dehn-Sommerville relations:

$$h_{d-i}(\Gamma) - h_i(\Gamma) = (-1)^i \binom{d}{i} \left(\widetilde{\chi}(\Gamma) + (-1)^d\right) \quad \text{for } 0 \le i \le d. \tag{10}$$

Note that if Γ is a $(d-1)$-dimensional *Eulerian complex* (that is, a semi-Eulerian complex with $\widetilde{\chi}(\Gamma) = (-1)^{\dim \Gamma}$, e.g. the boundary complex of a simplicial polytope, or a simplicial sphere, or an odd-dimensional simplicial manifold), then the Dehn-Sommerville relations take an especially convenient form:

$$h_j(\Gamma) = h_{d-j}(\Gamma) \quad \text{for all } 0 \le j \le d. \tag{11}$$

The following result proved in [28, Lemma 5.1] is an easy consequence of Schenzel's theorem (Eq. (2)), Dehn-Sommerville relations (Eq. (10)), and Poincaré duality:

Lemma 4.7. *Let Γ be an orientable $\mathbf{k}$-homology manifold of dimension $2k$. Then*

$$h'_{k+1}(\Gamma) - h'_k(\Gamma) = \binom{2k+1}{k} \left(\beta_k - \beta_{k-1}\right).$$

We now use Theorem 3.9 and Lemma 4.7 to prove the following upper bound.

Theorem 4.8. *Let $\Gamma \subseteq \Lambda(p,m,l)$ be a $2k$-dimensional orientable $\mathbf{k}$-homology manifold on $n = pm + l$ vertices, and let $S = S\left((p-1)^m, \infty^{n-2k-1-m}\right)$. Then*

$$\beta_k(\Gamma) + \frac{1}{2}\beta_{k-1}(\Gamma) + 3\sum_{i=0}^{k-2} \beta_i(\Gamma) \quad \le \quad \frac{F_{k+1}(S) - F_k(S)}{\binom{2k+1}{k}} \quad \text{if } n \ge 6k+3. \text{ Moreover,}$$

$$\beta_k(\Gamma) + 2\sum_{i=0}^{k-1} \beta_i(\Gamma) \quad \le \quad \frac{F_{k+1}(S) - F_k(S)}{\binom{2k+1}{k}} \quad \text{if } n \ge 9k+4.$$

Proof: Write $F_i := F_i(S)$ and $h'_i := h'_i(\Gamma)$. We have

$$\begin{aligned}
&\binom{2k+1}{k}(\beta_k - \beta_{k-1}) = h'_{k+1} - h'_k && \text{(by Lemma 4.7)}\\
&\le \frac{F_{k+1}}{F_k}\left(h'_k - \binom{2k}{k}\beta_{k-1}\right) - h'_k && \text{(by Thm. 3.9 and Lemma 4.2)}\\
&\le \frac{F_{k+1} - F_k}{F_k}\left(F_k - \sum_{i=0}^{k-1} \frac{F_k}{F_i}\binom{2k}{i}\beta_{i-1}\right) - \frac{F_{k+1}}{F_k}\binom{2k}{k}\beta_{k-1} && \text{(by Eq. (8)).}
\end{aligned}$$

Therefore,

$$\beta_k + \left(\frac{F_{k+1}}{F_k} \cdot \frac{\binom{2k}{k}}{\binom{2k+1}{k}} - 1\right)\beta_{k-1} + \sum_{i=0}^{k-1} \frac{F_{k+1} - F_k}{F_i} \cdot \frac{\binom{2k}{i}}{\binom{2k+1}{k}}\beta_{i-1} \le \frac{F_{k+1} - F_k}{\binom{2k+1}{k}}. \tag{12}$$

Since by Lemma 4.3, $F_{i+1}/F_i \ge \binom{n-2k-1}{i+1}/\binom{n-2k-1}{i}$, it follows that for $n \ge 6k+3$ and $i \le k-1$

$$\begin{aligned}
\frac{F_{k+1} - F_k}{F_i} \cdot \frac{\binom{2k}{i}}{\binom{2k+1}{k}} \quad &\ge \quad \frac{\binom{n-2k-1}{k+1} - \binom{n-2k-1}{k}}{\binom{n-2k-1}{i}} \cdot \frac{\binom{2k}{i}}{\binom{2k+1}{k}} = \frac{\binom{n-k-1-i}{2k-i}}{\binom{n-k-1-i}{k}} \cdot \frac{n-4k-2}{2k+1}\\
&\ge \quad \frac{\binom{n-k-1-i}{k+1}}{\binom{n-k-1-i}{k}} = \frac{n-(2k+1+i)}{k+1} \ge \frac{n-3k}{k+1} \ge 3. \qquad (13)
\end{aligned}$$

Similarly,

$$\frac{F_{k+1}}{F_k}\cdot\frac{\binom{2k}{k}}{\binom{2k+1}{k}}-1\geq\frac{n-3k-1}{2k+1}-1\geq\begin{cases}1/2 & \text{if } n\geq 6k+3\\ 2 & \text{if } n\geq 9k+4.\end{cases}\tag{14}$$

Equations (12)-(14) yield the result. □

Remark. If Γ is a $2k$-dimensional orientable $\mathbf{k}$-homology manifold, then

$$(-1)^k(\widetilde{\chi}(\Gamma)-1)=(-1)^k\left(\sum_{i=0}^{2k}(-1)^i\beta_i-1\right)=\beta_k-2\beta_{k-1}+2\sum_{i=0}^{k-1}(-1)^{k-i}\beta_i,$$

where the last step uses Poincaré duality. Since every simplicial manifold is orientable over $\mathbf{k}=\mathbb{Z}/2\mathbb{Z}$, Theorem 1.4 is an immediate corollary of Theorem 4.8. Moreover, equality $(-1)^k(\widetilde{\chi}(\Gamma)-1)=(F_{k+1}(S)-F_k(S))/\binom{2k+1}{k}$ holds iff $\beta_i(\Gamma)=0$ for all $i<k$ and $h'_{k+1}(\Gamma)-h'_k(\Gamma)=\binom{2k+1}{k}\beta_k(\Gamma)=F_{k+1}(S)-F_k(S)$. Since $\partial_{S,j}(h'_{j+1}(\Gamma))\leq h'_j(\Gamma)$, the latter condition is easily seen to be equivalent to $h_j(\Gamma)=h'_j(\Gamma)=F_j(S)=h_j(\mathrm{Skel}_{\,d-1}\Lambda(p,m,l))$ for all $j\leq k+1$, which in turn is equivalent to the fact that the k-dimensional skeleton of Γ coincides with that of $\Lambda(p,m,l)$. In the special case of $p=2$ and $l=0$, we obtain a partial proof (that is, for all $n\geq 6k+4$) of Sparla's conjecture [31] asserting that if Γ is a centrally symmetric $2k$-dimensional manifold on $n=2m$ vertices, then the inequality $(-1)^k\binom{2k+1}{k}(\widetilde{\chi}(\Gamma)-1)\leq 4^{k+1}\binom{\frac{1}{2}(m-1)}{k+1}$ holds, with equality iff $\mathrm{Skel}_{\,k}\Gamma=\mathrm{Skel}_{\,k}(C_m^{\Delta})$.

5. The Upper Bound theorem

In this section we discuss connections between Theorem 1.3 (and 3.9) and the Upper Bound Theorem (abbreviated, UBT). In particular we establish a new relative of the UBT for centrally symmetric manifolds. We also use this opportunity to include a brief survey on the current state of the art related to the UBT.

Recall that the d-dimensional cyclic polytope on n vertices, $C_d(n)$, is defined as the convex hull of n distinct points $\psi(t_1),\ldots,\psi(t_n)$ on the d-dimensional moment curve $\psi(t)=(t,t^2,\ldots,t^d)$. It is well known (see [40]) that the cyclic polytope is a simplicial polytope whose combinatorial type is independent of the choice of $t_1,\ldots,t_n$ and that it is $\lfloor d/2\rfloor$-neighborly (that is, every set of its $r\leq\lfloor d/2\rfloor$ vertices forms a face).

The UBT, conjectured by Motzkin in 1957, asserts that among all d-dimensional (simplicial) polytopes on n vertices, the (boundary of the) cyclic polytope maximizes the number of i-dimensional faces for every $i\leq d-1$. Here is a history of this conjecture in short: Klee [20] proved the UBT for polytopes with a sufficiently large number of vertices, and McMullen [25] verified the assertion of the UBT for all polytopes. (In fact Klee showed that the UBT holds for a much wider class of complexes — Eulerian complexes — provided that they have sufficiently many vertices. He also conjectured that the UBT holds for all Eulerian complexes.) McMullen's result was extended further and shown to hold for all simplicial spheres (Stanley [33]), and for several classes of simplicial manifolds (Novik [28]) and odd-dimensional pseudomanifolds with isolated singularities (Hersh and Novik [12]).

We now sketch some of the main ideas used in the proofs. Since the f-numbers of a $(d-1)$-dimensional complex Γ are nonnegative linear combinations of its h-numbers (see Eq. (1)), to prove that $f_{i-1}(\Gamma)\leq f_{i-1}(C_d(n))$ for all $0\leq i\leq d$ it suffices to show that $h_j(\Gamma)\leq h_j(C_d(n))$ for all $0\leq j\leq d$. The latter quantities,

$h_j(C_d(n)$, can be easily computed for $0 \leq j \leq \lfloor d/2 \rfloor$ from Eq. (1) and the fact that $f_{i-1}(C_d(n)) = \binom{n}{i}$ for $0 \leq i \leq \lfloor d/2 \rfloor$. In conjunction with (11) this yields

$$(15) \qquad h_j(C_d(n)) = h_{d-j}(C_d(n)) = \binom{n-d+j-1}{j} \qquad \text{for all } 0 \leq j \leq \lfloor d/2 \rfloor.$$

Eqs. (11) and (15) imply the proof of the UBT for all Eulerian complexes with sufficiently many vertices since, as Klee showed by a completely combinatorial argument [20], if Γ is a $(d-1)$-dimensional complex on n vertices, then $h_j(\Gamma) \leq \binom{n-d+j-1}{j}$ for all j, provided that n is sufficiently large ($n = O(d^2)$).

McMullen [25] verified that if Γ is a d-dimensional polytope on n vertices then $h_j(\Gamma) \leq h_j(C_d(n))$ for all $0 \leq j \leq d$ by using Eq. (11) and the (then recent) fact due to Bruggesser and Mani that the boundary of every polytope is shellable. Later, Alon and Kalai [5] applied linear algebra methods and extended McMullen's result to all shellable spheres.

Stanley's proof of the UBT for simplicial spheres [33] relied on combinatorics of Cohen-Macaulay complexes. Using the notation of this paper it can be outlined as follows: if Γ is a $(d-1)$-dimensional simplicial sphere on n vertices, then it is a CM complex, and so $h_j(\Gamma) = |B^d(\Gamma, g)_j|$ (see Eq. (5) and Lemma 3.2(b)). Since $B^d_j \subseteq S(\infty^{n-d})_j$, we obtain that $h_j(\Gamma) \leq F_j(S(\infty^{n-d})) = \binom{n-d+j-1}{j} = h_j(C_d(n))$ for all $0 \leq j \leq \lfloor d/2 \rfloor$. Eq. (11) then implies that $h_j(\Gamma) \leq h_j(C_d(n))$ for all $0 \leq j \leq d$ completing the proof.

Recall that if Γ is a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$, then the link of x_r is a subcomplex $\mathrm{lk}\, x_r = \{\mu/x_r : x_r|\mu \in \Gamma\}$. The short simplicial h-vector of Γ, $\widetilde{h}(\Gamma) := \sum_{x_r \in V} h(\mathrm{lk}\, x_r)$, introduced in [12], is used to extend Stanley's result to the class of all odd-dimensional simplicial complexes via the following theorem (implicit in [12]).

Theorem 5.1. *Let Γ be a $(2k+1)$-dimensional pure simplicial complex on n vertices such that for every vertex v of Γ, the link of v is semi-Eulerian and satisfies the inequalities $h_j(\mathrm{lk}\, v) \leq h_j(C_{2k+1}(n-1))$ for all $0 \leq j \leq k+1$. Then $f_{i-1}(\Gamma) \leq f_{i-1}(C_{2k+2}(n))$ for all $1 \leq i \leq 2k+1$.*

Corollary 5.2. *The UBT holds for all odd-dimensional simplicial manifolds.*

A different proof of the UBT for odd-dimensional manifolds using the Schenzel theorem (Theorem 2.6) was given in [28, Thm. 1.4].

While not all even-dimensional simplicial complexes satisfy the UBT (the smallest example violating the UBT is the minimal triangulation of the projective plane — a 2-dimensional simplicial complex with $f_0 = 6$, $f_1 = 15$ and $f_2 = 10$), all even-dimensional manifolds with $\widetilde{\chi} = 1$ (and, more generally, all $2k$-dimensional manifolds satisfying the inequality $\beta_k \leq 2\beta_{k-1} + 2\sum_{i=0}^{k-3} \beta_i$) do. The verification of this result in [28] relied on a special case of Theorem 1.3 (for the trivial group action) [28, Thm. 1.7], Poincaré duality and Dehn-Sommerville relations (10), and on computations very similar to those used in the proof of Theorem 1.5. Unfortunately it was not realized until after the paper was published that the proof of [28, Thm. 1.7] given there required an additional restriction: it worked only for a field $\mathbf{k}$ of characteristic zero. (Indeed, the proof relied on the fact that the generic initial ideal of $I_\Gamma \subset \mathbf{k}[\mathbf{x}]$ is strongly stable — the result known so far only in characteristic zero.) Our new proof (Theorem 1.3) finally allows to remove this restriction and restate the results from [28, Thm. 6.6, 6.7] as follows.

Theorem 5.3. *Let Γ be a $2k$-dimensional orientable* $\mathbf{k}$*-homology manifold on n vertices. If $\beta_k(\Gamma) \leq 2\beta_{k-1}(\Gamma) + 2\sum_{i=0}^{k-3}\beta_i(\Gamma)$, then*

$$f_{i-1}(\Gamma) \leq f_{i-1}(C_{2k+1}(n)) \quad \textit{for all } i \leq 2k+1,$$

and moreover, $h_j(\Gamma) \leq h_j(C_{2k+1}(n))$ for all $0 \leq j \leq k+1$.

Corollary 5.4. *The UBT holds for the following classes of $2k$-dimensional simplicial manifolds:*

1. *manifolds whose Euler characteristic $\widetilde{\chi}$ satisfies $(-1)^k(\widetilde{\chi}-1) \leq 0$;*
2. *manifolds with vanishing middle homology (computed over $\mathbb{Z}/2\mathbb{Z}$).*

Proof: Such complexes satisfy the assumption of Theorem 5.3 (with $\mathbf{k} = \mathbb{Z}/2\mathbb{Z}$). □

Theorem 5.1 together with Theorem 5.3 imply the following result from [12]:

Theorem 5.5. *Let Γ be a pure $(2k+1)$-dimensional simplicial complex on n vertices, such that for every vertex v of Γ, the link of v is an orientable* $\mathbf{k}$*-homology manifold satisfying the following condition*

$$\beta_k(\mathrm{lk}\, v) \leq 2\beta_{k-1}(\mathrm{lk}\, v) + 2\sum_{i=0}^{k-3}\beta_i(\mathrm{lk}\, v).$$

Then $f_{i-1}(\Gamma) \leq f_{i-1}(C_{2k+2}(n))$ for all $i \leq 2k+2$. In particular, the UBT holds for all odd-dimensional Eulerian pseudomanifolds with isolated singularities.

We now discuss a version of the UBT for centrally symmetric (cs for short) simplicial complexes. A cs complex Γ on $2m$ vertices is *nearly k-neighborly* if its $(k-1)$-dimensional skeleton coincides with that of the m-dimensional cross-polytope C_m^{Δ}. (This is analogous to the notion of k-neighborliness: a simplicial complex on n vertices is k-neighborly if its $(k-1)$-dimensional skeleton coincides with that of the $(n-1)$-dimensional simplex.) Thus the lower half of the h-vector of an arbitrary nearly $\lfloor d/2 \rfloor$-neighborly $(d-1)$-dimensional cs complex on $2m$ vertices is given by

$$h_j = h_j(\mathrm{Skel}_{\,d-1}(C_m^{\Delta})) \quad \text{for all } 0 \leq j \leq \lfloor d/2 \rfloor.$$

If in addition such a complex is Eulerian, then by Dehn-Sommerville relations, the lower half of its h-vector determines the upper half, which in turn determines all the f-numbers as certain non-negative linear combinations of $h_j(\mathrm{Skel}_{\,d-1}(C_m^{\Delta}))$, $0 \leq j \leq \lfloor d/2 \rfloor$ (see Eq. (1)). We denote the values of these linear combinations by $CS_i(d,m)$, $0 \leq i \leq d-1$. (These are the face numbers that a nearly cs sphere would have.)

Our knowledge of the UBT for cs complexes can be summarized as follows:

Theorem 5.6. *Let Γ be a $(d-1)$-dimensional cs complex on $2m$ vertices. Assume further that Γ is a simplicial sphere or an odd-dimensional manifold. Then $f_i(\Gamma) \leq CS_i(d,m)$ (for all $0 \leq i \leq d-1$).*

The part of Theorem 5.6 concerning a simplicial sphere is due independently to Adin [2] and Stanley [37], and (exactly as in the non-symmetric case) follows from Dehn-Sommerville relations (11) and the fact that if Γ is a $(d-1)$-dimensional CM cs complex on $2m$ vertices then $h_j(\Gamma) \leq h_j(\mathrm{Skel}_{\,d-1}(C_m^{\Delta}))$ for all j (see Cor. 1.2). The part concerning odd-dimensional manifolds was established in [29]. The proof is very similar to that of Cor. 5.2 above (by observing that the link of every vertex is a subcomplex of C^{Δ}_{m-1}).

The bounds on the (lower half of) h-numbers of cs Buchsbaum complexes from Cor. 4.5 together with Dehn-Sommerville relations (11) immediately imply another extension:

Theorem 5.7. *Let Γ be a $2k$-dimensional cs homology manifold on $n = 2m$ vertices. If $\widetilde{\chi}(\Gamma) = 1$ and $n \geq 5k$, then $f_i(\Gamma) \leq CS_i(2k+1, m)$ (for all $0 \leq i \leq 2k$).*

Thus the cs version of the UBT is analogous to the original UBT except for one thing: while every cyclic d-polytope is $\lfloor d/2 \rfloor$-neighborly (and many other constructions of $\lfloor d/2 \rfloor$-neighborly d-polytopes are known), the neighborliness of cs polytopes is quite restricted ([26], [9]) and very mysterious. In particular, no version of the UBT for cs polytopes is known, nor even conjectured. On the other hand, existence of nearly $\lfloor d/2 \rfloor$-neighborly $(d-1)$-dimensional cs spheres on $2m$ vertices for all d and m is plausible (and would make the above two theorems sharp), albeit still open.

6. Final remarks and conjectures

6.1. **Sufficiency of conditions.** The main results of this paper — Theorems 1.1 and 1.3 — provide certain necessary conditions on the face numbers of Cohen-Macaulay and Buchsbaum complexes endowed with a proper $\mathbb{Z}/p\mathbb{Z}$-action. Another set of necessary conditions on the face numbers of cs Cohen-Macaulay complexes is due to Stanley who showed in [36] that every such $(d-1)$-dimensional complex Γ satisfies $h_j(\Gamma) \geq \binom{d}{j}$ for all $0 \leq j \leq d$. These lower bounds were generalized further by Adin [2] to (not always tight) bounds on the face numbers of Cohen-Macaulay complexes with a free $\mathbb{Z}/p\mathbb{Z}$-action. (For further progress regarding the lower bounds on the h-numbers (toric h-numbers, respectively) of simplicial as well as general polytopes with symmetry the reader is referred to [1, 3, 4, 14, 15].) Since Theorem 1.1 does not imply Stanley's and Adin's inequalities, its conditions cannot be expected to be also sufficient conditions on the face numbers of CM complexes with a proper $\mathbb{Z}/p\mathbb{Z}$-action. On the other hand, the statements of Theorems 1.1 and 1.3 apply to a more general class of complexes (see Theorems 3.3 and 3.9), namely to all CM, Buchsbaum respectively, subcomplexes of $\Lambda(p, m, l)$. It would be interesting to clarify if for that larger family of complexes the conditions of Theorem 3.3, Theorem 3.9 respectively, do provide a complete characterization.

6.2. **Manifolds.** An important subclass of Buchsbaum complexes is the class of simplicial manifolds. Kalai conjectured [28, Section 7] that for a $(d-1)$-dimensional simplicial manifold Γ on n vertices, the condition $\partial_S(h'_{j+1}) \leq h'_j - \binom{d-1}{j}\beta_{j-1}$ $(1 \leq j \leq d-1)$, where $S = S(\infty^{n-d})$, can be strengthened to $\partial_S(h'_{j+1}) \leq h'_j - \binom{d}{j}\beta_{j-1}$. We propose the following generalization of this conjecture.

Conjecture 6.1. *Let Γ be a $(d-1)$-dimensional simplicial manifold on $n = pm+l$ vertices, and let $S = S\left((p-1)^m, \infty^{n-d-m}\right)$. If Γ is a subcomplex of $\Lambda(p, m, l)$ then*

$$\partial_S(h'_{j+1}(\Gamma)) \leq h'_j(\Gamma) - \binom{d}{j}\beta_{j-1}(\Gamma) \quad \textit{for all } 1 \leq j \leq d-1.$$

The same computations as those involved in the proof of Theorem 4.8 show that Conjecture 6.1, if true, would imply the assertion of Theorem 1.4 for every

$2k$-dimensional simplicial manifold $\Gamma \subseteq \Lambda(p,m,l)$ on $n \geq 4k+3$ vertices. Since all $(d-1)$-dimensional cs complexes (except for the boundary complex of the cross polytope C_d^Δ) have at least $2d+2$ vertices, Conjecture 6.1 in particular would imply Sparla's conjecture for all even-dimensional centrally symmetric manifolds.

We also propose the following generalization of Kühnel's and Sparla's conjectures (proved here for $2k$-dimensional manifolds on $n \geq 6k+3$ vertices — Theorem 4.8).

Conjecture 6.2. *Let Γ be a $2k$-dimensional simplicial manifold on $n = pm + l$ vertices, and let $S = S\left((p-1)^m, \infty^{n-2k-1-m}\right)$. If Γ is a subcomplex of $\Lambda(p,m,l)$ then*

$$(-1)^k \binom{2k+1}{k} (\widetilde{\chi}(\Gamma) - 1) \leq F_{k+1}(S) - F_k(S).$$

6.3. **Björner-Kalai's theorem.** Björner and Kalai provide in [7] a complete characterization of pairs of sequences (f, β) for which there exists a simplicial complex Γ with $f(\Gamma) = f$ and $\beta(\Gamma) = \beta$. It would be interesting to extend their result to the class of simplicial complexes endowed with a proper $\mathbb{Z}/p\mathbb{Z}$-action.

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