Affine stresses: the partition of unity and Kalai's reconstruction conjectures

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Abstract

Kalai conjectured that if P is a simplicial d-polytope that has no missing faces of dimension $d-1$, then the graph of P and the space of affine 2-stresses of P determine P up to affine equivalence. We propose a higher-dimensional generalization of this conjecture: if $2 \leq i \leq d/2$ and P is a simplicial d-polytope that has no missing faces of dimension $\geq d-i+1$, then the space of affine i-stresses of P determines the space of affine 1-stresses of P. We prove this conjecture for (1) k-stacked d-polytopes with $2 \leq i \leq k \leq d/2-1$, (2) d-polytopes that have no missing faces of dimension $\geq d-2i+2$, and (3) flag PL $(d-1)$-spheres with generic embeddings (for all $2 \leq i \leq d/2$). We also discuss several related results and conjectures. For instance, we show that if P is a simplicial d-polytope that has no missing faces of dimension $\geq d-2i+2$, then the $(i-1)$-skeleton of P and the set of sign vectors of affine i-stresses of P determine the combinatorial type of P. Along the way, we establish the partition of unity of affine stresses: the spaces of affine stresses of any PL sphere (with a generic embedding) and the space of affine 1-stresses of any strongly connected complex (with a mildly generic embedding) can be expressed as the sums of affine stress spaces of vertex stars. This is analogous to Adiprasito's partition of unity of linear stresses for Cohen-Macaulay complexes.

1 Introduction

One of the central problems in the theory of face numbers of simplicial complexes is how the information about the local structure of a complex (i.e., properties of the links, or equivalently of the stars) can be used to provide the information about the entire complex. Results of this nature include, among others, McMullen's integral formula that expresses the (sums of the) h-numbers of a pure simplicial complex in terms of the h-numbers of vertex links [19, 31], Kalai's observation that a simplicial sphere of dimension at least four is stacked if and only if all of its links are stacked [13], and Bagchi and Datta's μ- and σ-numbers and their applications [8, 23]. The most recent major development on this front is Adiprasito's *partition of unity* [2, 4] that allows us to express

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linear stress spaces of a Cohen–Macaulay complex (w.r.t. certain embeddings) as the sums of linear stress spaces of vertex stars. This is a fundamental result that has already served as an ingredient in several exciting recent breakthroughs, see, for instance, [1, 2, 4].

The first goal of this paper is to establish several partition-of-unity type results for *affine* stresses. We defer all definitions to the next sections and for now merely mention that linear stress spaces of a simplicial complex Δ can be thought of as Weil dual of (the graded components of) an artinian reduction of the Stanley–Reisner ring of Δ. Similarly, affine stress spaces are Weil dual of an artinian reduction modded out by one additional linear form: typically, the sum of the variables.

Assume Δ is a $(d-1)$-dimensional simplicial complex (e.g., a simplicial sphere) with vertex set V. Specifying an artinian reduction of the Stanley–Reisner ring of Δ is equivalent to choosing a d-*embedding* of Δ — a map p from V to $\mathbb{R}^d$. There are two most common types of embeddings used in the literature. If Δ is the boundary complex of a convex polytope P, then one can take p to be the natural embedding given by the position vectors of vertices of P. The second model is to consider embeddings satisfying certain genericity assumptions. The mildest assumption is to require that the images of vertices of every facet are linearly or affinely independent. This, however, is insufficient in many settings, especially those related to Lefschetz properties. So, instead, one considers a (very) generic embedding, namely, any map such that the multiset of coordinates of vertices is algebraically independent over $\mathbb{Q}$. Both models are extensively used in geometric combinatorics. For instance, both models play a prominent role in the celebrated g-theorem — the theorem that characterizes f-vectors of simplicial spheres. This theorem was first proved for the case of simplicial polytopes by Billera and Lee [9] (sufficiency) and Stanley [29] (necessity); additional proofs of necessity were found by McMullen [20, 21] and then by Fleming and Karu [11]. A recent breakthrough by Adiprasito [2], Papadakis and Petrotou [28], Adiprasito, Papadakis, and Petrotou [3], and Karu and Xiao [15] settles the case of spheres; all available proofs of this case use generic embeddings.

In this paper, we establish the partition of unity of affine stresses in the following two cases:

- The spaces of affine 1-stresses of strongly connected simplicial complexes (of dimension ≥ 2) w.r.t. embeddings satisfying the property that the vertices of any two adjacent facets are affinely independent; see Theorem 4.1. This includes the class of simplicial polytopes with natural embeddings and the class of normal pseudomanifolds with generic embeddings.

- The spaces of affine i-stresses of PL $(d-1)$-spheres with generic embeddings (for all $2 \leq i \leq (d-1)/2$); see Theorem 4.4.

The proof of the second result is based on the g-theorem for spheres; specifically, we rely on the fact that Artinian reductions of the Stanley–Reisner rings (over $\mathbb{R}$) of simplicial spheres w.r.t. generic systems of parameters satisfy the hard Lefschetz property, see, for instance, [15, Theorem 1.3].

The space of affine 1-stresses of a simplicial complex Δ of dimension $d-1$ w.r.t. an embedding p coincides with the space of affine dependencies of the p-images of vertices. In particular, when Δ is the boundary complex of a simplicial d-polytope P with its natural embedding, the space of affine 1-stresses of Δ contains the same information as the Gale diagram of P. Consequently, one may think of the spaces of affine i-stresses of polytopes as higher-dimensional analogs of Gale diagrams. Motivated by this connection, Kalai proposed the following conjectures:

Conjecture 1.1. *Let $d \geq 4$, $2 \leq i \leq d/2$, and let P be a simplicial d-polytope. Then the $(i-1)$-skeleton of P (as an abstract simplicial complex) and the space of affine i-stresses of P determine the combinatorial type of P.*

Conjecture 1.2. *Let $d \geq 4$, let $2 \leq i \leq d/2$, and let P be a simplicial d-polytope.*

1. *If P has no missing $(d-1)$-faces, then the graph of P and the space of affine 2-stresses of P determine P up to affine equivalence.*
2. *More generally, if P has no missing faces of dimension $\geq d-i+1$, then the space of affine i-stresses of P determines P up to affine equivalence.*

Conjectures 1.1 and the first part of Conjecture 1.2 are due to Gil Kalai: Conjecture 1.1 was posited in [14] and Conjecture 1.2(1) was privately communicated to us and recorded in [26]. The second part of Conjecture 1.2 is a generalization of the first part: in addition to being stated for a general i, knowing the $(i-1)$-skeleton is not part of the assumptions. (It is still an open problem whether any simplicial d-polytope P with no missing faces of dimension $\geq d-i+1$ has the property that every $(i-1)$-face of P participates in an affine i-stress on P; see Conjecture 3.3 below.)

That the graph and the space of affine 2-stresses determine the combinatorial type of a d-polytope P (for $d \geq 4$) was verified in [26]. One does not even need to know the entire space of affine 2-stresses: knowing the sign vectors of affine 2-stresses is enough. Furthermore, Cruickshank, Jackson, and Tanigawa [10] recently proved the first part of Conjecture 1.2 for polytopes whose vertices have *generic* coordinates.

Here, we use our partition of unity theorems to establish several results around these conjectures. Most notably, we prove the following:

- If $d \geq 4$ and P is any simplicial d-polytope that has no missing faces of dimension $\geq d-2$, then the space of affine 2-stresses of P determines the affine type of P; see Theorem 5.1.
- Moreover, if $2 \leq i \leq d/2$ and P is any simplicial d-polytope that has no missing faces of dimension $\geq d-2i+2$, then the space of affine i-stresses of P determines the affine type of P (see Theorem 6.3) while the $(i-1)$-skeleton of P and the set of sign vectors of affine i-stresses determine the combinatorial type of P (see Theorem 6.10).
- If $1 \leq i \leq k \leq d/2-1$ and P is a k-stacked d-polytope that has no missing faces of dimension $\geq d-i+1$, then the space of affine i-stresses of P determines the affine type of P. In fact, if Δ is a k-stacked $(d-1)$-sphere that has no missing faces of dimension $\geq d-i+1$ and p is a d-embedding of Δ that satisfies certain mild genericity assumptions, then the space of affine i-stresses of Δ determines the space of affine 1-stresses of Δ; see Theorem 7.1.
- Yet another result in this series is that if $2 \leq i \leq d/2$ and Δ is a *flag* PL sphere with a generic embedding, then the space of affine i-stresses of Δ determines the space of affine j-stresses of Δ for *all* $1 \leq j < i$; see Theorem 6.4.

It is worth emphasizing that p and Δ are not part of the data. The point of the above results is that as long as p and Δ satisfy certain assumptions, we can recover the space of affine 1-stresses of Δ (and sometimes of affine j-stresses) solely from the space of affine i-stresses of Δ. In particular, in all of the above cases, knowing the space of affine i-stresses of (Δ, p) is enough to determine p itself up to an invertible affine transformation.

Similarly to [10, 26], to treat the case of $i = 2$, we mainly use the language and tools from the rigidity theory of frameworks. To treat the case of general i, we work with (higher) affine and linear stress spaces. One idea behind the proofs is that if a simplicial d-polytope P has no large missing faces and τ is an $(i-j-1)$-face of P, then iteratively taking partial derivatives with respect to the variables corresponding to all vertices of τ provides a surjection from the space of affine i-stresses of P to the space of affine j-stresses of the star of τ. The partition of unity then allows us to show that when P has no large missing faces, the space of affine dependencies of vertices of P is determined by the space of affine i-stresses of P.

In view of our results, it is tempting to posit one more conjecture whose $j = 1$ case recovers Conjecture 1.2. For a more precise version of this conjecture, see Conjecture 3.6; for an analogous result on linear stresses, see part 2 of Theorem 3.4.

Conjecture 1.3. *Let $1 \leq j < i \leq d/2$. Let P be a simplicial d-polytope whose boundary complex has no missing faces of dimension $\geq d - i + 1$. Then the space of affine i-stresses of P, $\mathcal{S}_i^a(P)$, determines the space of affine j-stresses of P, $\mathcal{S}_j^a(P)$.*

It is also tempting to posit an analogous conjecture for affine stresses of simplicial spheres w.r.t. generic embeddings. The restriction on missing faces in Conjectures 1.2 and 1.3 is unavoidable. Indeed, if P is an $(i-1)$-stacked polytope that is not $(j-1)$-stacked, then $\mathcal{S}_i^a(P)$ is the zero space while $\mathcal{S}_j^a(P)$ is a non-zero space, and so Conjecture 1.3 does not hold in this case. Similarly, one can slightly perturb the vertices of such P to obtain another polytope P' with the property that P' and P are combinatorially but not affinely equivalent; this is despite the fact that $\mathcal{S}_i^a(P) = 0 = \mathcal{S}_i^a(P')$.

The rest of the paper is structured as follows. In Section 2, we review several definitions and results related to polytopes and simplicial complexes, such as simplicial spheres and normal pseudomanifods; we also discuss combinatorial properties of these objects. In Section 3, we provide a brief introduction to the theory of stress spaces followed by the discussion of the partition of unity of linear stresses and related results. Then in Section 4, we establish the partition of unity of affine stresses for two important classes of complexes, see Theorems 4.1 and 4.4. These tools allow us to prove Kalai's conjectures and their extensions in several cases. Specifically, Section 5 is devoted to reconstructing affine types of complexes that have no missing faces of dimension $\geq d-2$ from the spaces of affine 2-stresses; see Theorem 5.1. Sections 6 and 7 focus on reconstructing affine and combinatorial types from higher affine stresses; see Theorems 6.3, 6.4, 6.10, and 7.1. Along the way, we provide various applications of the tools developed in the paper, most notably of Lemma 6.2. One such application is Theorem 6.7 that establishes lower bounds on the g-numbers of flag $(d-1)$-spheres in terms of d.

2 Preliminaries on polytopes, spheres, and pseudomanifolds

A *polytope* $P \subseteq \mathbb{R}^d$ is the convex hull of a finite set of points in $\mathbb{R}^d$. The *dimension* of P is the dimension of the affine span of P. For brevity, we say that P is a *d-polytope* if P is d-dimensional. If the vertices of P are affinely independent, then P is called a (geometric) *simplex*.

Assume $P \subseteq \mathbb{R}^d$ is a d-polytope. A hyperplane $H \subseteq \mathbb{R}^d$ is a *supporting hyperplane* of P if P is contained in one of the two closed half-spaces determined by H. A *(proper) face of P* is the intersection of P with any supporting hyperplane of P. A face of a polytope is by itself a polytope and each polytope has only finitely many faces. We say that P is *simplicial* if all of its (proper) faces are simplices.

If v is a vertex of P, then the *vertex figure of* P *at* v, denoted P/v, is the polytope obtained by intersecting P with a hyperplane H that has v on one side and all other vertices of P on the other side. (While the resulting polytope does depend on our choice of H, its combinatorial type does not.) In general, if F is a face of P, then the *quotient of* P *by* F, P/F, is obtained from P by iteratively taking vertex figures at the vertices in F.

An (abstract) *simplicial complex* Δ with vertex set $V = V(\Delta)$ is a non-empty collection of subsets of V that is closed under inclusion and contains all singletons: $\{v\} \in \Delta$ for all $v \in V$. The elements of Δ are called *faces* of Δ. A face F of Δ is an *i-face* or a face of *dimension* i if $|F| = i+1$. For instance, 0-faces are *vertices* and 1-faces are *edges*. (To simplify notation, for faces that are vertices and edges, we write v instead of $\{v\}$ and uv instead of $\{u, v\}$.) The dimension of Δ is $\max\{\dim F : F \in \Delta\}$. A set $F \subseteq V$ is a *missing face* of Δ if F is not a face of Δ, but every proper subset σ of F is a face of Δ. The *dimension* of a missing face F is defined as $|F| - 1$. A complex Δ is *flag* if all missing faces of Δ are 1-dimensional.

Two important examples of simplicial complexes on vertex set V are the (abstract) $(|V| - 1)$-simplex $\overline{V} := \{\tau : \tau \subseteq V\}$ and the boundary complex of $\overline{V}$, $\partial\overline{V}$. The latter complex consists of all faces of $\overline{V}$ but V itself.

When studying a simplicial complex Δ, one often considers the following subcomplexes of Δ. The subcomplex of Δ *induced by* $W \subseteq V(\Delta)$ consists of all faces of Δ that are contained in W. The *i-skeleton* of Δ, $\mathrm{Skel}_i(\Delta)$, is the set of all faces of Δ of dimension at most i. The 1-skeleton of Δ is also called the *graph* of Δ. If F is a face of Δ, then the *antistar* of F is $\Delta - F := \{\tau \in \Delta : F \not\subseteq \tau\}$. Furthermore, the *star of* F and the *link of* F *in* Δ are defined by:

$$\mathrm{st}(F) = \mathrm{st}(F, \Delta) := \{\sigma \in \Delta \ : \ \sigma \cup F \in \Delta\} \text{ and } \mathrm{lk}(F) = \mathrm{lk}(F, \Delta) := \{\sigma \in \mathrm{st}(F) \ : \ \sigma \cap F = \emptyset\}.$$

If Γ and Δ are simplicial complexes on disjoint vertex sets, their *join* is the simplicial complex $\Gamma * \Delta = \{\sigma \cup \tau \ : \ \sigma \in \Gamma \text{ and } \tau \in \Delta\}$. When $\Gamma = \{\emptyset, u\}$ consists of a single vertex, we write $\Gamma * \Delta$ as $u * \Delta$; this complex is the *cone* over Δ with apex u. Thus, for a vertex v of Δ, $\mathrm{st}(v, \Delta) = v * \mathrm{lk}(v, \Delta)$.

Each simplicial complex Δ admits a *geometric realization* $\|\Delta\|$ that contains a geometric i-simplex for each i-face of Δ. Conversely, each geometric simplicial complex D corresponds to an abstract simplicial complex whose faces are vertex sets of faces of D. For instance, any simplicial d-polytope P gives rise to an abstract simplicial complex ∂P called the *boundary complex* of P: the faces of ∂P are the vertex sets of all (proper) faces of P. With this definition in hand, it is easy to see that for a vertex v of P, the boundary complex of P/v is $\mathrm{lk}(v, \partial P)$ and, similarly, for any face F, the boundary complex of P/F, is $\mathrm{lk}(F, \partial P)$.

We say that Δ is a *PL $(d-1)$-sphere* if it is PL homeomorphic to the boundary complex of a d-simplex. Similarly, a *PL d-ball* is a simplicial complex PL homeomorphic to a d-simplex. The PL spheres (balls) belong to a larger class of complexes called *simplicial spheres (balls)*: Δ is a simplicial $(d-1)$-sphere (*simplicial d-ball*, respectively) if $\|\Delta\|$ is homeomorphic to a $(d-1)$-sphere (d-ball, respectively). It is worth noting that while all simplicial 3-spheres are PL, there are many non-PL $(d-1)$-spheres for $d \geq 6$.

A simplicial complex Δ is called *pure* if all *facets* (i.e., maximal under inclusion faces) of Δ have the same dimension. If Δ is a pure $(d-1)$-dimensional simplicial complex, then $(d-2)$-faces of Δ are called *ridges*; two facets of such Δ are *adjacent* if they share a common ridge. A pure complex

Δ is *strongly connected* if every two facets of Δ can be connected by a sequence of pairwise adjacent facets of Δ.

Let Δ be a pure simplicial complex. We say that Δ is a *pseudomanifold without boundary* if every ridge of Δ is in exactly two facets. Similarly, Δ is a *pseudomanifold with boundary* if every ridge is in at most two facets and there exists a ridge that is contained in only one facet. The *boundary of* Δ, $\partial\Delta$, is defined as the subcomplex of Δ generated by all the ridges that are contained in only one facet. A pseudomanifold (with or without boundary) is called *normal* if the link of every face of codimension at least two is connected. Any normal pseudomanifold (with or without boundary) is strongly connected. Examples of normal pseudomanifolds without boundary include simplicial spheres and closed simplicial manifolds. Furthermore, examples of normal pseudomanifolds with boundary include antistars of nonempty faces of normal pseudomanifolds without boundary.

Let Δ be a normal pseudomanifold with boundary. A face G of Δ is called a *minimal interior face* of Δ if $G \notin \partial\Delta$ but $\partial\overline{G}$ is a subcomplex of $\partial\Delta$. We denote by $\mathcal{I}(\Delta)$ the collection of all minimal interior faces of Δ. We will need the following elementary lemma.

Lemma 2.1. *Let Δ be a normal pseudomanifold without boundary and F a face of Δ. If σ is a minimal interior face of $\Delta - F$, then there exists $H \subseteq F$ such that $\sigma \cup H$ is a missing face of Δ.*

Proof: Throughout this proof, the links are computed in Δ and Δ is suppressed from notation. We write σ as $\sigma'' \cup \sigma'$ where $\sigma'' = \sigma \cap F$ and $\sigma' = \sigma \backslash \sigma''$. Since $\partial(\Delta - F) = \partial\overline{F} * \mathrm{lk}(F)$ and since σ is a minimal interior face of $\Delta - F$, it follows that σ has the following properties: (1) $\sigma \in \Delta$ but σ is not a subset of F (and so $\sigma' \neq \emptyset$); (2) σ is not a face of $\partial\overline{F} * \mathrm{lk}(F)$, but $\partial\overline{\sigma}$ is a subcomplex of $\partial\overline{F} * \mathrm{lk}(F)$. We conclude that σ' is a missing face of $\mathrm{lk}\, F$. Thus, $\sigma' \cup F$ is not a face of Δ but $\partial\overline{\sigma'} * \overline{F}$ is a subcomplex of Δ. Now, since σ is a face of Δ but $\sigma \cup F$ is not a face, there must exist a minimal under inclusion subset H of $F \backslash \sigma''$ such that $\sigma \cup H$ is not a face of Δ; in particular, H is nonempty. The set $\sigma \cup H$ is then a desired missing face of Δ. ☐

We close this subsection with a few combinatorial properties related to spheres. Let Δ be a PL $(d-1)$-sphere. If Δ contains an induced subcomplex $\overline{A} * \partial\overline{B}$, where A is a j-subset of $V(\Delta)$ and B is a $(d-j+1)$-subset of $V(\Delta)$, then we can perform a *bistellar flip* on Δ by replacing $\overline{A} * \partial\overline{B}$ with $\partial\overline{A} * \overline{B}$. The resulting complex Δ' is another PL $(d-1)$-sphere. We call this operation a *j-flip.* In particular, the vertex sets of Δ and Δ' are identical except in the cases of $j = 1$ and $j = d$: in the former case, Δ has one more vertex (the vertex of A) and in the latter case, Δ' has one more vertex (the vertex of B). The following theorem of Pachner [27] gives an alternative definition of PL spheres.

Theorem 2.2. *Any PL $(d-1)$-sphere can be obtained from the boundary complex of a d-simplex by a sequence of bistellar flips.*

A $(d-1)$-dimensional simplicial complex Δ is *shellable* if its facets can be linearly ordered as $F_1, F_2, \dots, F_k$ in such a way that for all $2 \leq i \leq k$, the subcomplex $\overline{F_i} \cap (\cup_{j<i}\overline{F_j})$ is pure $(d-2)$-dimensional. Such an ordering of facets is called a *shelling* of Δ. Equivalently, $F_1, F_2, \dots, F_k$ is a shelling of Δ if for all $i \leq k$, the collection of faces of $\overline{F_i}$ that are not faces of $\cup_{j<i}\overline{F_j}$ has a unique minimal element. This unique minimal face is called the *restriction face of* F_i and is denoted by $r(F_i)$. We say that F_i is a *shelling step of type m* if $r(F_i)$ is of size m.

If Δ is a $(d-1)$-dimensional simplicial complex or a d-polytope, we define $f_i(\Delta)$ as the number of i-dimensional faces of Δ, where $-1 \leq i \leq d-1$. The *h-numbers* of Δ are obtained from the

f-numbers by the following linear transformation:

$$h_j(\Delta) = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{d-j} f_{i-1}(\Delta), \quad \text{for all } 0 \leq j \leq d.$$

We also let $g_0(\Delta) = 1$ and $g_j(\Delta) = h_j(\Delta) - h_{j-1}(\Delta)$ for $1 \leq j \leq \lceil d/2 \rceil$. The h-numbers and g-numbers of boundary complexes of simplicial polytopes have various interesting interpretations. For now, we only mention the following classical result; see [37, Chapter 8].

Theorem 2.3. *The boundary complex of a simplicial polytope is shellable. Furthermore, the h-number h_i is exactly the number of shelling steps of type i.*

3 Paving the way: the spaces of linear and affine stresses

3.1 Stresses and h- and g-numbers

We start by reviewing several notions and results related to linear and affine stresses. For more details we refer the reader to [16, 17] and [32, 33].

Let Δ be a simplicial complex on the vertex set $V = V(\Delta)$. A map $p : V(\Delta) \to \mathbb{R}^d$ is called a *d-embedding* of Δ. In particular, if Δ is a graph, then (Δ, p) is called a *d-framework*. For $W \subseteq V(\Delta)$, write $p(W) = \{p(v) : v \in W\}$ (considered as a multiset if there are repetitions).

Let $X = X(V) = \{x_v : v \in V\}$ be a set of variables with one variable for each vertex and let $\mathbb{R}[X]$ be the polynomial ring over the real numbers in variables X. Denote by $\mathcal{M}_i(V)$ the set of all squarefree monomials of degree i in $X(V)$. Each variable x_v acts on $\mathbb{R}[X]$ by $\frac{\partial}{\partial x_v}$; for brevity, we will denote this operator by ∂_{x_v}. More generally, if $\mu = x_{v_1} \cdots x_{v_s} \in \mathbb{R}[X]$ is a monomial, then define $\partial_\mu : \mathbb{R}[X] \to \mathbb{R}[X]$ by $\rho \mapsto \partial_{x_{v_s}} \cdots \partial_{x_{v_1}}(\rho)$, and if $\ell(X) = \sum_{v \in V} \ell_v x_v$ is a linear form in $\mathbb{R}[X]$, then define

$$\partial_{\ell(X)} : \mathbb{R}[X] \to \mathbb{R}[X] \quad \text{by} \quad \rho \mapsto \sum_{v \in V} \ell_v \cdot \partial_{x_v} \rho = \sum_{v \in V} \ell_v \frac{\partial \rho}{\partial x_v}.$$

Given a d-embedding p of Δ, consider the $(d+1) \times |V|$ matrix whose columns are labeled by the vertices of Δ and the column corresponding to $v \in V$ consists of the vector $p(v)$ augmented by a one in the last position. The i-th row of this matrix, $\boldsymbol{\theta}_i = [\theta_{iv}]_{v \in V}$, gives rise to a linear form $\theta_i = \sum_{v \in V} \theta_{iv} x_v$. In particular, $\theta_{d+1} = \sum_{v \in V} x_v$. We denote by $\Theta(p)$ or simply by Θ the sequence $(\theta_1, \ldots, \theta_d, \theta_{d+1})$ of these forms.

For a monomial $\mu \in \mathbb{R}[X]$, the *support* of μ is $\mathrm{supp}(\mu) = \{v \in V : x_v | \mu\}$. A homogeneous polynomial $\lambda = \lambda(X) = \sum_\mu \lambda_\mu \mu \in \mathbb{R}[X]$ of degree k is called a *linear k-stress* on (Δ, p) if it satisfies the following conditions:

- Every (non-zero) term $\lambda_\mu \mu$ of λ is supported on a face of Δ: $\mathrm{supp}(\mu) \in \Delta$, and
- $\partial_{\theta_i} \lambda = 0$ for all $i = 1, \ldots, d$.

A linear k-stress λ on (Δ, p) that also satisfies $\partial_{\theta_{d+1}} \lambda = 0$ is called an *affine k-stress.* In particular, $\sum_{v \in V} a_v x_v$ is an affine 1-stress on (Δ, p) if and only if $(a_v : v \in V)$ is an affine dependence of the point configuration $\{p(v) : v \in V\}$. It is immediate from the definitions that the sets of linear k-stresses and affine k-stresses on Δ form vector spaces (over $\mathbb{R}$), denoted by $\mathcal{S}^\ell_k(\Delta, p)$ and $\mathcal{S}^a_k(\Delta, p)$. Furthermore, for $1 \leq k \leq \lceil d/2 \rceil$, $\mathcal{S}^a_k(\Delta, p)$ is the kernel of $\partial_{\theta_{d+1}} : \mathcal{S}^\ell_k(\Delta, p) \to \mathcal{S}^\ell_{k-1}(\Delta, p)$.

It is known (see [17]) that the dimensions of $\mathcal{S}^\ell_k(\Delta, p)$ and $\mathcal{S}^a_k(\Delta, p)$ coincide with the dimensions of the k-th graded components of $\mathbb{R}[\Delta]/(\theta_1, \dots, \theta_d)$ and $\mathbb{R}[\Delta]/(\theta_1, \dots, \theta_d, \theta_{d+1})$, respectively; here $\mathbb{R}[\Delta]$ is the Stanley–Reisner ring of Δ. In particular, if Δ is a Cohen–Macaulay complex of dimension $d-1$ (e.g., a simplicial ball or sphere) and p is a d-embedding such that for every facet F of Δ, the multiset $p(F)$ is linearly independent, then $\dim \mathcal{S}^\ell_k(\Delta, p) = h_k(\Delta)$ for all $0 \leq k \leq d$; see [30].

As was mentioned in the introduction, we will mainly work with simplicial polytopes and simplicial spheres using natural embeddings in the former case and generic embeddings in the latter. Specifically, if P is a simplicial d-polytope, we let p be the natural d-embedding of $\Delta = \partial P$ given by the position vectors of vertices of P. If Δ is a simplicial $(d-1)$-sphere, then we consider a d-embedding p with the property that the multiset of coordinates of the points $p(v)$, $v \in V(\Delta)$, is algebraically independent over $\mathbb{Q}$. Such an embedding is called a *generic* embedding of Δ. The following result is a crucial step in the proof of the g-theorem; it provides arguably the most important interpretation of the g-numbers of simplicial polytopes [29, 20, 21, 11] and spheres [2, 28, 3, 15].

Theorem 3.1. *Let* (Δ, p) *be either the boundary complex of a simplicial* d*-polytope with its natural embedding* p*, or a simplicial* $(d-1)$*-sphere with a generic embedding* p*, and let* $1 \leq i \leq \lceil d/2 \rceil$*. Then* θ_{d+1} *is a Lefschetz element, that is, the linear map* $\partial_{\theta_{d+1}} : \mathcal{S}^\ell_i(\Delta, p) \to \mathcal{S}^\ell_{i-1}(\Delta, p)$ *is surjective. (In fact, if* $d = 2i - 1$*, it is an isomorphism.) In particular,*

$$\dim \mathcal{S}^a_i(\Delta, p) = \dim \mathcal{S}^\ell_i(\Delta, p) - \dim \mathcal{S}^\ell_{i-1}(\Delta, p) = g_i(\Delta).$$

3.2 The cone lemma and supports of affine stresses

Let $\lambda = \sum_\mu \lambda_\mu \mu$ be either a k-linear or a k-affine stress on (Δ, p). We say that λ is supported on a subcomplex Γ of Δ if every monomial of λ is supported on a face of Γ. For instance, $\partial_{x_v}\lambda$ is a $(k-1)$-stress supported on $\mathrm{st}(v, \Delta)$. Let $F \in \Delta$ be a $(k-1)$-face. To simplify notation, we write $x_F := \prod_{v \in F} x_v$ and $\lambda_F := \lambda_{x_F}$, and call λ_F the *weight of* F *in* λ or the *weight assigned to* F *by* λ. If $\lambda_F \neq 0$, we say that F *participates in* λ or that F is *in the support of* λ. We refer to $(\lambda_F : F \in \Delta, |F| = k)$ as the *squarefree part* of λ. It is known that a k-stress is uniquely determined by its squarefree part, see [17].

One of very useful results on linear and affine stress spaces is the cone lemma [17, 33]. The version below discusses affine stresses. Let Δ be any $(d-2)$-dimensional simplicial complex (not necessarily a sphere) with a $(d-1)$-embedding p' and let Γ be the cone over Δ with a d-embedding p. The f-numbers of Γ can be easily expressed in terms of the f-numbers of Δ; for instance, the h- and g-vectors of Γ and Δ coincide. This naturally leads to the question of how the stress spaces $\mathcal{S}^a_i(\Delta, p')$ and $\mathcal{S}^a_i(\Gamma, p)$ are related (for appropriately chosen p and p'). The following lemma [26, Lemma 3.2], originally due to Lee (see [17, Theorem 7]), provides an answer.

Lemma 3.2. *Let* Δ *be a simplicial complex with an embedding* p'*. Let* $\Gamma = v * \Delta$ *be the cone over* Δ *with an embedding* p *such that* $p(v)$ *is the origin in* $\mathbb{R}^d$ *and for all* $u \in V(\Delta)$*,* $p(u) = \begin{bmatrix} a_u p'(u) \\ a_u \end{bmatrix}$ *for some nonzero* $a_u \in \mathbb{R}$*. Then*

1. *there exists an isomorphism* $\phi_k : \mathcal{S}^a_k(\Gamma, p) \to \mathcal{S}^a_k(\Delta, p')$*.*
2. *Furthermore, any affine* k*-stress* ω' *on* (Δ, p') *lifts to an affine* k*-stress* ω *on* (Γ, p) *with the property that for every* $(k-1)$*-face* $F \in \Delta$*,* $\omega'_F = \big(\prod_{u \in F} a_u\big)\omega_F$*.*

A few remarks are in order. The space of affine stresses is unaffected by Euclidean motions and scalings. Thus, we can always assume that the p-image of the cone vertex is the origin. Furthermore, the above lemma applies to vertex links and stars of a simplicial d-polytope P, $\big((\mathrm{lk}(v,\partial P),p'),(\mathrm{st}(v,\partial P),p)\big)$. Here p is the natural embedding of ∂P and p' is the natural embedding of $\partial(P/v)$ in a hyperplane H that separates $p(v)$ from the rest of the vertices of P. Using Euclidean motions and scalings, we can assume that $p(v)$ is the origin and H is given by the equation $x_d = 1$. Hence part 2 of Lemma 3.2 implies that for every $(k-1)$-face $F \in \mathrm{lk}(v,\partial P)$, ω'_F and ω_F have the same sign. We refer to [26, Corollary 3.3] for a more precise and general statement.

We end this subsection with a conjecture on supports of affine stresses. In [36], it is shown that if $d \geq 4$, Δ is a simplicial $(d-1)$-sphere that has no missing faces of dimension $\geq d-1$, and p is a *generic* d-embedding of Δ, then every edge participates in some affine 2-stress on (Δ,p). Hence in this case, the graph of Δ is determined by the space of affine 2-stresses. This motivates the following conjecture:

Conjecture 3.3. *Let $2 \leq i \leq d/2$. Let Δ be the boundary complex of a simplicial d-polytope with its natural embedding p, or a simplicial $(d-1)$-sphere (or more generally, a normal $(d-1)$-pseudomanifold without boundary) with a generic embedding p. In both cases, assume also that Δ has no missing faces of dimension $\geq d-i+1$. Then every $(i-1)$-face of Δ participates in some affine i-stress on (Δ,p). In particular, the space of affine i-stresses determines the $(i-1)$-skeleton of Δ.*

3.3 The partition of unity of linear stresses

Our original motivation for this paper came in part from the following result on linear stresses of spheres. Recall that $\mathcal{M}_i(V)$ the set of all squarefree monomials of degree i in $X(V)$.

Theorem 3.4. *Let Δ be a simplicial $(d-1)$-sphere and let p be a d-embedding of Δ such that the p-images of vertices of any facet $G \in \Delta$ are linearly independent. Then the following holds.*

1. *For all $1 \leq i \leq d-1$ and $k \leq d-i$,*

$$\mathcal{S}^\ell_i(\Delta,p) = \sum_{v \in V(\Delta)} \mathcal{S}^\ell_i(\mathrm{st}(v),p) = \sum_{F\in\Delta,\, |F|=k} \mathcal{S}^\ell_i(\mathrm{st}(F),p).$$

2. *Furthermore, for all $1 \leq j < i \leq d$,*

$$\begin{aligned} \mathcal{S}^\ell_j(\Delta,p) &= \mathrm{span}\left\{\partial_\mu\omega \,:\, \omega \in \mathcal{S}^\ell_i(\Delta,p),\, \mu \in \mathcal{M}_{i-j}(V(\Delta))\right\} \\ &= \mathrm{span}\left\{\partial_{x_F}\omega \,:\, \omega \in \mathcal{S}^\ell_i(\Delta,p),\, F \in \Delta,\, |F| = i-j\right\}. \end{aligned}$$

For shellable spheres, this result is due to Lee [17, Theorem 16]: Lee only proved part 2 of the statement, but since for a stress ω on the entire complex, $\partial_{x_F}\omega$ is a stress supported on the star of F (it is 0 if F is not a face), part 1 is an immediate consequence of part 2. For general simplicial spheres (in fact, for general Cohen–Macaulay complexes), part 1 was proved by Adiprasito [2, Lemma 3.4], see also [1, 4]. In words, part 1 asserts that any linear i-stress on Δ can be written as the sum of linear i-stresses supported on the stars. This property is known as the *partition of unity of linear stresses.* Since we could not find the proof of part 2 in the literature, we provide it here for completeness.

Proof of part 2: Let Δ be a simplicial $(d-1)$-sphere and let F be an $(i-j-1)$-face of Δ. Then $\mathcal{S}^{\ell}_i(\Delta - F, p)$ is a subspace of $\mathcal{S}^{\ell}_i(\Delta, p)$, and this subspace is precisely the kernel of the map $\partial_{x_F} : \mathcal{S}^{\ell}_i(\Delta, p) \to \mathcal{S}^{\ell}_j(\mathrm{st}(F), p)$. Hence we have the following exact sequence:

$$0 \to \mathcal{S}^{\ell}_i(\Delta - F, p) \to \mathcal{S}^{\ell}_i(\Delta, p) \xrightarrow{\partial_{x_F}} \mathcal{S}^{\ell}_j(\mathrm{st}(F), p).$$

Since $\Delta - F$, Δ, and $\mathrm{st}(F)$ are $(d-1)$-dimensional Cohen–Macaulay complexes, the dimensions of the three spaces in the sequence are $h_i(\Delta - F)$, $h_i(\Delta)$, and $h_j(\mathrm{st}(F)) = h_j(\mathrm{lk}(F))$, respectively. Since for every k, $f_{k-1}(\Delta) = f_{k-1}(\Delta - F) + f_{k-(i-j)-1}(\mathrm{lk}(F))$, it follows easily that $h_i(\Delta) = h_i(\Delta - F) + h_j(\mathrm{lk}(F))$ (cf. [7, Lemma 4.1]). We conclude that the right-most map in this sequence, $\partial_{x_F} : \mathcal{S}^{\ell}_i(\Delta, p) \to \mathcal{S}^{\ell}_j(\mathrm{st}(F), p)$, must be onto. Thus

$$\mathrm{span}\left\{\partial_{x_F}\omega : \omega \in \mathcal{S}^{\ell}_i(\Delta, p),\ F \in \Delta,\ |F| = i - j\right\} = \sum_{F\in\Delta,\, |F|=i-j} \mathcal{S}^{\ell}_j(\mathrm{st}(F), p) = \mathcal{S}^{\ell}_j(\Delta, p),$$

where the last step is by part 1. The result follows. ☐

The cone lemma yields the following variation of Theorem 3.4. We will use it in Section 7.

Corollary 3.5. *Let Δ be a simplicial $(k-1)$-sphere, let $\overline{F}$ be a $(d-k-1)$-simplex, and let $\Gamma = \overline{F} * \Delta$. Let p be a d-embedding of Γ such that $p(H)$ is linearly independent for every facet $H \in \Gamma$. Then for all $1 \le i \le k-1$ and $t \le k - i$, $\mathcal{S}^{\ell}_i(\Gamma, p) = \sum_{G\in\Delta,\, |G|=t} \mathcal{S}^{\ell}_i(\mathrm{st}(G, \Gamma), p)$. Furthermore, for all $1 \le j < i \le k$, $\mathcal{S}^{\ell}_j(\Gamma, p) = \mathrm{span}\left\{\partial_{\mu}\omega : \omega \in \mathcal{S}^{\ell}_i(\Gamma, p),\ \mu \in \mathcal{M}_{i-j}(V(\Delta))\right\}$.*

Proof: Theorem 3.4 and the cone lemma for linear stresses [17] imply that

$$\mathcal{S}^{\ell}_i(\Gamma, p) = \mathcal{S}^{\ell}_i(\overline{F} * \Delta, p) = \sum_{G\in\Delta,\, |G|=t} \mathcal{S}^{\ell}_i(\overline{F} * \mathrm{st}(G, \Delta), p) = \sum_{G\in\Delta,\, |G|=t} \mathcal{S}^{\ell}_i(\mathrm{st}(G, \Gamma), p).$$

For the second statement, observe that if $G \in \Delta$, then $\Gamma - G = \overline{F} * (\Delta - G)$ is a $(d-1)$-dimensional Cohen–Macaulay complex. The rest of the proof is identical to that of part 2 of Theorem 3.4. ☐

Part 2 of Theorem 3.4 provides a structural result on spaces of linear stresses of spheres. This result along with Conjectures 1.2 and 1.3 suggest that an analogous statement might hold for affine stresses. To this end, we propose the following conjecture.

Conjecture 3.6. *Let (Δ, p) be either the boundary complex of a simplicial d-polytope with its natural embedding p, or a simplicial $(d-1)$-sphere with a generic embedding p. If Δ has no missing faces of dimension $\ge d - i + 1$, then for all $1 \le j < i \le d/2$,*

$$\mathcal{S}^{a}_j(\Delta, p) = \mathrm{span}\left\{\partial_{\mu}\omega : \omega \in \mathcal{S}^{a}_i(\Delta, p),\ \mu \in \mathcal{M}_{i-j}(V(\Delta))\right\}.$$

4 The partition of unity of affine stresses

The goal of this section is to establish two versions of the partition of unity of affine stresses. Since we only work with the stress spaces on (Δ, p) and never explicitly use the associated Θ (as in Section 3), to avoid confusion, from now on, we will write c instead of θ_{d+1} to denote $\sum_{v\in V(\Delta)} x_v$.

Theorem 4.1. *Let $d \geq 3$. Let Δ be a strongly connected $(d-1)$-dimensional complex with an embedding p in $\mathbb{R}^d$ such that the p-images of vertices of any two adjacent facets are affinely independent. Then $\mathcal{S}_1^a(\Delta, p) = \sum_{G \in \Delta, \dim G = d-3} \mathcal{S}_1^a(\mathrm{st}(G), p)$. In particular, $\mathcal{S}_1^a(\Delta, p) = \sum_{v \in V(\Delta)} \mathcal{S}_1^a(\mathrm{st}(v), p)$.*

Proof: If Δ has at most $d+1$ vertices, then $\mathcal{S}_1^a(\Delta, p)$ and $\mathcal{S}_1^a(\mathrm{st}(G), p)$ are the zero spaces for all faces $G \in \Delta$, and the claim holds. Thus, assume that Δ has at least $d+2$ vertices, and hence, at least three facets. Pick an ordering $F_1, F_2, \ldots, F_m$ of the facets of Δ that satisfies the following property: for every $i \geq 1$, F_{i+1} is adjacent to at least one facet with a smaller index. Such an ordering exists because Δ is strongly connected. Label the vertices of $F_1 \cup F_2$ by $v_1, \ldots, v_{d+1}$. Let $3 \leq i_1 < i_2 < \cdots < i_{g_1(\Delta)} \leq m$ be the indices such that for all $1 \leq j \leq g_1(\Delta)$, the size of $V(\cup_{k=1}^{i_j - 1} F_k)$ is smaller than the size of $V(\cup_{k=1}^{i_j} F_k)$; label the new vertex introduced at step i_j by v_{d+1+j}.

Let $1 \leq j \leq g_1(\Delta)$. By the defining property of our ordering, there exists $t < i_j$ such that F_t is adjacent to F_{i_j}. If $t = 1$, we take $s = 2$ and notice that $F_s = F_2$ is adjacent to $F_t = F_1$. If $t > 1$, then again by the defining property of our ordering, there exists $s < t$ such that F_s and F_t are adjacent. In either case, the complex $\overline{F_s} \cup \overline{F_t} \cup \overline{F_{i_j}}$ contains $d+2$ vertices, among them $v_{d+1+j} \in F_{i_j} \backslash (F_s \cup F_t)$. Thus $(\overline{F_s} \cup \overline{F_t} \cup \overline{F_{i_j}}, p)$ supports a non-trivial affine 1-stress ω_j. Also, since F_s and F_t are adjacent, the p-images of the $d+1$ vertices of $F_s \cup F_t$ are affinely independent. It follows that ω_j assigns a nonzero weight to v_{d+1+j} and zero weights to all v_k for $k > d+1+j$. Therefore, the affine 1-stresses $\omega_1, \ldots, \omega_{g_1(\Delta)}$ are linearly independent, and hence span $\mathcal{S}_1^a(\Delta)$. The result follows since ω_j is supported on the star of $G_j := F_s \cap F_t \cap F_{i_j}$ which is a face of dimension $\geq d-3$. $\square$

Applying the above theorem to the boundary complex of a simplicial polytope P with its natural embedding, we obtain the following

Corollary 4.2. *Let $d \geq 3$ and let Δ be the boundary complex of a simplicial d-polytope P with its natural embedding p. Then*

$$\mathcal{S}_1^a(\Delta, p) = \sum_{\substack{G \in \Delta \\ \dim G = d-3}} \mathcal{S}_1^a(\mathrm{st}(G), p), \quad \textit{and hence,} \quad \mathcal{S}_1^a(\Delta, p) = \sum_{v \in V(\Delta)} \mathcal{S}_1^a(\mathrm{st}(v), p).$$

Remark 4.3. Let Δ be a normal $(d-1)$-pseudomanifold with boundary and let p be a d-embedding of Δ such that the p-images of vertices of any facet of Δ are linearly independent. Assume also that Δ is not a simplex. Then using the same ideas as in the proof of Theorem 4.1, one easily shows that $\mathcal{S}_1^\ell(\Delta, p) = \sum_R \mathcal{S}_1^\ell(\mathrm{st}(R), p)$, where the sum is over interior ridges of Δ. Since every interior ridge contains a minimal interior face, we also obtain that $\mathcal{S}_1^\ell(\Delta, p) = \sum_{F \in \mathcal{I}(\Delta)} \mathcal{S}_1^\ell(\mathrm{st}(F), p)$.

Our second result is about the partition of unity of higher affine stresses.

Theorem 4.4. *Let $2 \leq i \leq (d-1)/2$ and let Δ be a PL $(d-1)$-sphere with a generic d-embedding p. Then $\mathcal{S}_i^a(\Delta, p) = \sum_{v \in V(\Delta)} \mathcal{S}_i^a(\mathrm{st}(v), p)$.*

Proof: Recall that by Pachner's theorem, the PL $(d-1)$-sphere Δ can be obtained from the boundary complex of a d-simplex by a sequence of bistellar flips. Hence, to prove our statement, it suffices to show that if Δ' is obtained from Δ by a j-flip (for any $1 \leq j \leq d$) and Δ satisfies the statement of the theorem, then so does Δ'. For the rest of the proof, we let p be a generic embedding of $V(\Delta \cup \Delta')$ and we suppress p from our notation.

By Theorem 13 in [17], for $1 \leq k \leq d$,

$$\begin{cases} \mathcal{S}^\ell_k(\Delta') \oplus \mathcal{S}^\ell_k(\partial(\overline{A \cup B})) = \mathcal{S}^\ell_k(\Delta) & \text{if } j < d-j+1 \quad \text{and} \quad j \leq k \leq d-j, \\ \mathcal{S}^\ell_k(\Delta') = \mathcal{S}^\ell_k(\partial(\overline{A \cup B})) \oplus \mathcal{S}^\ell_k(\Delta) & \text{if } j > d-j+1 \quad \text{and} \quad d-j+1 \leq k \leq j-1, \\ \mathcal{S}^\ell_k(\Delta') = \mathcal{S}^\ell_k(\Delta) & \text{otherwise.} \end{cases}$$

For a simplicial complex Γ with a face G and a vertex u, we write $\omega_{G,\Gamma}$ to denote an affine i-stress on Γ with G in the support and we write $\omega_{u,G,\Gamma}$ to denote an affine i-stress on $\mathrm{st}(u, \Gamma)$ with G in the support. Recall our convention that $c := \sum_v x_v$. Since $d \geq 2i$ and since Δ and Δ' are PL $(d-1)$-spheres with generic embeddings, Theorem 3.1 implies that the following maps are surjective:

$$\mathcal{S}^\ell_i(\Delta) \xrightarrow{\partial c} \mathcal{S}^\ell_{i-1}(\Delta), \quad \mathcal{S}^\ell_i(\Delta') \xrightarrow{\partial c} \mathcal{S}^\ell_{i-1}(\Delta').$$

Furthermore, the space $\mathcal{S}^\ell_m(\partial(\overline{A \cup B}))$ is 1-dimensional for all $0 \leq m \leq d$, and so $\partial c : \mathcal{S}^\ell_i(\partial(\overline{A \cup B})) \to \mathcal{S}^\ell_{i-1}(\partial(\overline{A \cup B}))$ is an isomorphism. Consequently,

$$\begin{cases} \mathcal{S}^a_i(\Delta') \oplus \mathrm{span}\{\omega_{A,\Delta}\} = \mathcal{S}^a_i(\Delta) & \text{if } i = j, \\ \mathcal{S}^a_i(\Delta') = \mathcal{S}^a_i(\Delta) \oplus \mathrm{span}\{\omega_{B,\Delta'}\} & \text{if } i = d-j+1, \\ \mathcal{S}^a_i(\Delta') = \mathcal{S}^a_i(\Delta) & \text{otherwise.} \end{cases}$$

For a vertex $v \in A$ and $j \neq 1$, $\mathrm{lk}(v, \Delta')$ is obtained from $\mathrm{lk}(v, \Delta)$ by a $(j-1)$-flip $(\overline{A \backslash v}) * \partial \overline{B} \mapsto \partial(\overline{A \backslash v}) * \overline{B}$. For a vertex $v \in B$ and $j \neq d$, $\mathrm{lk}(v, \Delta')$ is obtained from $\mathrm{lk}(v, \Delta)$ by a j-flip $\overline{A} * \partial(\overline{B \backslash v}) \mapsto \partial \overline{A} * (\overline{B \backslash v})$. Finally for every $v \notin A \cup B$, $\mathrm{lk}(v, \Delta) = \mathrm{lk}(v, \Delta')$. Since vertex links of Δ and Δ' are PL $(d-2)$-spheres and $d-1 \geq 2i$, the same argument as above (combined with the cone lemma) implies that

$$\begin{cases} \mathcal{S}^a_i(\mathrm{st}(v, \Delta')) \oplus \mathrm{span}\{\omega_{v,A\backslash v,\Delta}\} = \mathcal{S}^a_i(\mathrm{st}(v, \Delta)) & \text{if } v \in A \quad \text{and} \quad i = j-1, \\ \mathcal{S}^a_i(\mathrm{st}(v, \Delta')) \oplus \mathrm{span}\{\omega_{v,A,\Delta}\} = \mathcal{S}^a_i(\mathrm{st}(v, \Delta)) & \text{if } v \in B \quad \text{and} \quad i = j, \\ \mathcal{S}^a_i(\mathrm{st}(v, \Delta')) = \mathcal{S}^a_i(\mathrm{st}(v, \Delta)) \oplus \mathrm{span}\{\omega_{v,B,\Delta'}\} & \text{if } v \in A \quad \text{and} \quad i = d-j+1, \\ \mathcal{S}^a_i(\mathrm{st}(v, \Delta')) = \mathcal{S}^a_i(\mathrm{st}(v, \Delta)) \oplus \mathrm{span}\{\omega_{v,B\backslash v,\Delta'}\} & \text{if } v \in B \quad \text{and} \quad i = d-j, \\ \mathcal{S}^a_i(\mathrm{st}(v, \Delta)) = \mathcal{S}^a_i(\mathrm{st}(v, \Delta')) & \text{otherwise.} \end{cases}$$

Assume that $A = \{u_1, \dots, u_j\}$ and $B = \{v_1, \dots, v_{d-j+1}\}$. Since the spaces of affine i-stresses ($i \geq 2$) of the entire complex and of the vertex stars are not affected by a facet subdivision or its inverse, we further assume that $j \neq 1$ and $j \neq d$, and so $V(\Delta) = V(\Delta')$. We consider the following five cases:

Case 1: $i \neq j-1, j, d-j, d-j+1$. In this case, by our assumptions on Δ,

$$\mathcal{S}^a_i(\Delta') = \mathcal{S}^a_i(\Delta) = \sum_{v \in V(\Delta)} \mathcal{S}^a_i(\mathrm{st}(v, \Delta)) = \sum_{v \in V(\Delta)} \mathcal{S}^a_i(\mathrm{st}(v, \Delta')).$$

Case 2: $i = j$. To start, we claim that for any $2 \leq k \leq d-i+1$,

$$\mathcal{S}^a_i(\mathrm{st}(v_1, \Delta)) + \mathcal{S}^a_i(\mathrm{st}(v_k, \Delta)) = \mathcal{S}^a_i(\mathrm{st}(v_1, \Delta)) + \mathcal{S}^a_i(\mathrm{st}(v_k, \Delta')).$$

Recall that for $1 \leq k \leq d-i+1$, $\mathcal{S}^a_i(\mathrm{st}(v_k, \Delta')) \oplus \mathrm{span}\{\omega_k\} = \mathcal{S}^a_i(\mathrm{st}(v_k, \Delta))$, where ω_k is any affine i-stress on $\mathrm{st}(v_k, \Delta)$ with A in the support. We will now show that for any fixed $2 \leq k \leq d-i+1$, we can choose ω_1 and ω_k to be the same stress. If $d \geq 2i+2$ and $k \geq 2$, then $\mathrm{lk}(v_1v_k, \Delta)$ and $\mathrm{lk}(v_1v_k, \Delta')$ are spheres of dimension $d-3 \geq 2i-1$ and since their $(i-1)$-skeleta defer only in $\{A\}$, that is, $\mathrm{Skel}_{i-1}(\mathrm{lk}(v_1v_k, \Delta)) = \mathrm{Skel}_{i-1}(\mathrm{lk}(v_1v_k, \Delta') \cup \{A\})$, Theorem 3.1 (combined with the cone lemma) implies that there exists an affine i-stress ω on $\mathrm{st}(v_1v_k, \Delta)$ that has A in its support. We take ω_1 and ω_k to be that ω. Similarly, if $d = 2i+1$, then $\mathrm{lk}(v_1v_k, \Delta) \cup \{B \backslash v_1v_k\}$ and $\mathrm{lk}(v_1v_k, \Delta') \cup \{A\}$ share the same $(i-1)$-skeleton. Hence, by applying Theorem 3.1, we conclude that there is a nonzero element ω of $\mathcal{S}_i^a\big(\mathrm{st}(v_1v_k, \Delta) \cup \{B \backslash v_1v_k\}\big) = \mathcal{S}_i^a\big(\mathrm{st}(v_1v_k, \Delta') \cup \{A\}\big)$ that has A in its support. Since both $\mathrm{st}(v_1, \Delta)$ and $\mathrm{st}(v_k, \Delta)$ contain the subcomplex $\mathrm{st}(v_1v_k, \Delta) \cup \{B \backslash v_1v_k\}$, we can again take ω_1 and ω_k to be that ω. Hence

$$\begin{aligned}
&\mathcal{S}_i^a(\mathrm{st}(v_1, \Delta)) + \mathcal{S}_i^a(\mathrm{st}(v_k, \Delta)) \\
=&\Big(\mathcal{S}_i^a(\mathrm{st}(v_1, \Delta')) \oplus \mathrm{span}\{\omega\}\Big) + \Big(\mathcal{S}_i^a(\mathrm{st}(v_k, \Delta')) \oplus \mathrm{span}\{\omega\}\Big) = \mathcal{S}_i^a(\mathrm{st}(v_1, \Delta)) + \mathcal{S}_i^a(\mathrm{st}(v_k, \Delta')),
\end{aligned}$$

as desired. Since the above equation holds for all $2 \leq k \leq d-i+1$, we infer that

$$\begin{aligned}
\sum_{k=1}^{d-i+1} \mathcal{S}_i^a(\mathrm{st}(v_k, \Delta)) &= \sum_{k=2}^{d-i+1} \left[\mathcal{S}_i^a(\mathrm{st}(v_1, \Delta)) + \mathcal{S}_i^a(\mathrm{st}(v_k, \Delta))\right] = \sum_{k=2}^{d-i+1} \left[\mathcal{S}_i^a(\mathrm{st}(v_1, \Delta)) + \mathcal{S}_i^a(\mathrm{st}(v_k, \Delta'))\right] \\
&= \mathcal{S}_i^a(\mathrm{st}(v_1, \Delta)) + \sum_{k=2}^{d-i+1} \mathcal{S}_i^a(\mathrm{st}(v_k, \Delta')).
\end{aligned}$$

On the other hand, since for all $z \notin B$, $\mathcal{S}_i^a(\mathrm{st}(z, \Delta') = \mathcal{S}_i^a(\mathrm{st}(z, \Delta))$, we also obtain that

$$\begin{aligned}
\mathcal{S}_i^a(\Delta') \oplus \mathrm{span}\{\omega_{A,\Delta}\} &= \mathcal{S}_i^a(\Delta) = \sum_{z \in V(\Delta)} \mathcal{S}_i^a(\mathrm{st}(z, \Delta)) \\
= \mathcal{S}_i^a(\mathrm{st}(v_1, \Delta)) + \sum_{z \in V(\Delta),\, z \neq v_1} \mathcal{S}_i^a(\mathrm{st}(z, \Delta')) &= \left(\sum_{z \in V(\Delta)} \mathcal{S}_i^a(\mathrm{st}(z, \Delta'))\right) \oplus \mathrm{span}\{\omega_1\}.
\end{aligned}$$

Since $\omega_{A,\Delta}$ is any affine i-stress on Δ with A in the support, we can take $\omega_{A,\Delta} = \omega_1$. Finally, since $\mathcal{S}_i^a(\Delta') \supseteq \sum_{z \in V(\Delta)} \mathcal{S}_i^a(\mathrm{st}(z, \Delta'))$, the above equation yields that $\mathcal{S}_i^a(\Delta') = \sum_{z \in V(\Delta)} \mathcal{S}_i^a(\mathrm{st}(z, \Delta'))$, and so Δ' satisfies the partition of unity.

Case 3: $i = d - j + 1$**.** Then

$$\begin{aligned}
&\mathcal{S}_i^a(\Delta') = \mathcal{S}_i^a(\Delta) + \mathrm{span}\{\omega_{u_1,B,\Delta'}, \dots, \omega_{u_{d-i+1},B,\Delta'}\} \\
=& \sum_{k=1}^{d-i+1} \big(\mathcal{S}_i^a(\mathrm{st}(u_k, \Delta)) + \mathrm{span}\{\omega_{u_k,B,\Delta'}\}\big) + \sum_{z \in V(\Delta) \backslash A} \mathcal{S}_i^a(\mathrm{st}(z, \Delta)) = \sum_{z \in V(\Delta)} \mathcal{S}_i^a(\mathrm{st}(z, \Delta')).
\end{aligned}$$

Case 4: $i = j - 1$**.** In this case $\mathrm{Skel}_{i-1}(\Delta') = \mathrm{Skel}_{i-1}(\Delta)$ and hence $\mathcal{S}_i^a(\Delta') = \mathcal{S}_i^a(\Delta)$. For $1 \leq k \leq i+1$, $\mathcal{S}_i^a(\mathrm{st}(u_k, \Delta')) \oplus \mathrm{span}\{\omega_{u_k, A \backslash u_k, \Delta}\} = \mathcal{S}_i^a(\mathrm{st}(u_k, \Delta))$. The subcomplex $\mathrm{st}(u_kv_1, \Delta') \cup \{A \backslash u_k\}$ supports a nontrivial affine i-stress ω_k with $A \backslash u_k$ in its support. Since $\mathrm{st}(u_kv_1, \Delta') \cup \{A \backslash u_k\}$ is contained in both $\mathrm{st}(u_k, \Delta)$ and $\mathrm{st}(v_1, \Delta')$, it follows that

$$\mathcal{S}_i^a(\mathrm{st}(u_k, \Delta)) \subseteq \mathcal{S}_i^a(\mathrm{st}(u_k, \Delta')) + \mathcal{S}_i^a(\mathrm{st}(v_1, \Delta')).$$

On the other hand, $\mathcal{S}^a_i(\mathrm{st}(z,\Delta')) = \mathcal{S}^a_i(\mathrm{st}(z,\Delta))$ for $z \notin A \cup B$. Furthermore, $\mathcal{S}^a_i(\mathrm{st}(v_k,\Delta)) \subseteq \mathcal{S}^a_i(\mathrm{st}(v_k,\Delta'))$ for all $1 \leq k \leq d-i$ and $d \geq 2i+1$. Hence we conclude that

$$\mathcal{S}^a_i(\Delta') = \mathcal{S}^a_i(\Delta) = \sum_{z \in V(\Delta)} \mathcal{S}^a_i(\mathrm{st}(z,\Delta)) \subseteq \sum_{z \in V(\Delta')} \mathcal{S}^a_i(\mathrm{st}(z,\Delta')).$$

The partition of unity in this case follows since $\mathcal{S}^a_i(\Delta') \supseteq \sum_{z \in V(\Delta')} \mathcal{S}^a_i(\mathrm{st}(z,\Delta'))$ always holds.

Case 5: $i = d - j$**.** The proof is similar to case 4. The only difference is that we switch the roles of u_k and v_k, and obtain that $\mathcal{S}^a_i(\mathrm{st}(u_k,\Delta)) \subseteq \mathcal{S}^a_i(\mathrm{st}(u_k,\Delta'))$ for $1 \leq k \leq j = d-i$, and $\mathcal{S}^a_i(\mathrm{st}(v_k,\Delta)) \subseteq \mathcal{S}^a_i(\mathrm{st}(v_k,\Delta')) + \mathcal{S}^a_i(\mathrm{st}(u_1,\Delta'))$ for $1 \leq k \leq d-j+1$. ☐

In view of Theorem 4.4, we ask if the partition of unity of affine stresses also holds for simplicial polytopes with natural embeddings and (non-PL) simplicial spheres with generic embeddings.[1]

Conjecture 4.5. *Let $2 \leq i \leq (d-1)/2$. Let (Δ, p) be either the boundary complex of a simplicial d-polytope with its natural embedding p, or a simplicial $(d-1)$-sphere with a generic embedding p. Then $\mathcal{S}^a_i(\Delta,p) = \sum_{v \in V(\Delta)} \mathcal{S}^a_i(\mathrm{st}(v),p)$.*

5 A warm-up: reconstructing from affine 2-stresses

In this section we prove the following theorem (cf. part 1 of Conjecture 1.2).

Theorem 5.1. *Let $d \geq 4$. Let Δ be either (i) a normal $(d-1)$-pseudomanifold without boundary with a generic embedding p, or (ii) the boundary complex ∂P of a simplicial d-polytope P with its natural embedding p. In both cases, assume also that Δ has no missing faces of dimension $\geq d-2$. Then $\mathrm{span}\{\partial_{x_v}\omega : \omega \in \mathcal{S}^a_2(\Delta,p),\, v \in V(\Delta)\} = \mathcal{S}^a_1(\Delta,p)$, and so the space $\mathcal{S}^a_2(\Delta,p)$ determines the space $\mathcal{S}^a_1(\Delta,p)$. In particular, $\mathcal{S}^a_2(\Delta,p)$ determines the positions of vertices of Δ up to affine equivalence.*

We remark that for the case of normal pseudomanifolds with generic embeddings, Theorem 8.3 in [10] establishes a more general result: it shows that $\mathcal{S}^a_2(\Delta,p)$ determines $\mathcal{S}^a_1(\Delta,p)$ as long as Δ has no missing faces of dimension $d-1$. The result on polytopes with natural embeddings is new. Our proof relies on the partition of unity of affine 1-stresses (Theorem 4.1) as well as on, by now, standard tools from the rigidity theory of frameworks. Specifically, we rely on the cone and gluing lemmas [18, Section 6] and on works of Fogelsanger [12] and Whiteley [34]; we refer the reader to [5, 6, 13, 18] for an introduction to the rigidity theory of frameworks and all undefined terminology.

The key to our proof of Theorem 5.1 is the following lemma. Some special cases of this lemma are known, see the proof of [25, Proposition 2.10]. We recall that a d-framework (Γ,p) is infinitesimally d-rigid if and only if $\dim \mathcal{S}^a_2(\Gamma,p) = g_2(\Gamma)$; see [5, 6].

Lemma 5.2. *Let d and (Δ,p) be as in Theorem 5.1. Then for every nonempty face $F \in \Delta$, $(\Delta - F, p)$ is an infinitesimally d-rigid framework. In particular, $\dim \mathcal{S}^a_2(\Delta - F,p) = g_2(\Delta - F)$.*

Proof: Throughout this proof, all stars and links are computed in Δ. The complex $\Delta - F$ is a normal pseudomanifold with boundary. First we claim that every minimal interior face of $\Delta - F$

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[1] After the preprint was posted on the arxiv, Adiprasito and Murai (independently) commented that Conjecture 4.5 can likely be proved by using the spectral sequence argument. We leave this observation for future work.

has dimension $\leq d-4$. Indeed, if $\sigma \in \mathcal{I}(\Delta - F)$, then by Lemma 2.1, there exists $H \subseteq F$ such that $\sigma \cup H$ is a missing face of Δ. Thus, $d-3 \geq \dim(\sigma \cup H) > \dim(\sigma)$, and so $\dim \sigma \leq d-4$ as claimed. This implies that for every $\sigma \in \mathcal{I}(\Delta - F)$, $(\mathrm{st}(\sigma), p)$ is infinitesimally d-rigid: in the case that Δ is a normal pseudomanifold, this follows from Fogelsanger's result [12] and the cone lemma, while in the case that Δ is the boundary complex of a polytope, this follows from Whiteley's result [34] and the cone lemma.

Define $K = \bigcup\{\mathrm{st}(\sigma) : \sigma \in \mathcal{I}(\Delta - F)\}$. Clearly, $K \subseteq \Delta - F$. On the other hand, every facet T of $\Delta - F$ is an interior face. Hence T contains a minimal interior face. It follows that $\overline{T} \subseteq K$, and so by purity, $K = \Delta - F$. We conclude that $(\Delta - F, p)$ can be expressed as the union of infinitesimally d-rigid frameworks.

The claim that $(\Delta - F, p)$ is infinitesimally d-rigid now follows from repeated applications of the gluing lemma. Indeed, observe that if σ and τ are elements of $\mathcal{I}(\Delta - F)$ such that $\sigma \cup \tau \in \Delta$, then $\mathrm{st}(\sigma) \cap \mathrm{st}(\tau) \supseteq \mathrm{st}(\sigma \cup \tau)$, where $\mathrm{st}(\sigma \cup \tau)$ contains d vertices of a facet. These d vertices are affinely independent in $\mathbb{R}^d$, and so by the gluing lemma, $(\mathrm{st}(\sigma) \cup \mathrm{st}(\tau), p)$ is infinitesimally d-rigid. Thus, to complete the proof, it remains to show that the following graph $\mathcal{G}$ is connected: the vertices of $\mathcal{G}$ correspond to elements of $\mathcal{I}(\Delta - F)$, and we put an edge between σ and τ if $\sigma \cup \tau \in \Delta$. To see that $\mathcal{G}$ is connected, let σ and τ be elements of $\mathcal{I}(\Delta - F)$. Then there exist facets H_σ and H_τ of $\Delta - F$ that contain σ and τ, respectively. Since $\Delta - F$ is strongly connected, we can walk from H_σ to H_τ along a path of facets in $\Delta - F$: $H_\sigma = H^0, H^1, \ldots, H^\ell = H_\tau$, such that $H^i \cap H^{i+1}$ is a common ridge of both H^i and H^{i+1}. Then $H^i \cap H^{i+1}$ contains a minimal interior face; denote it by σ_{i+1}. This gives us a sequence $S = (\sigma_0 := \sigma, \sigma_1, \ldots, \sigma_\ell, \sigma_{\ell+1} := \tau)$ of elements of $\mathcal{I}(\Delta - F)$, where for every i, σ_i and σ_{i+1} are contained in the facet H^i. Thus, either $\sigma_i = \sigma_{i+1}$, or σ_i and σ_{i+1} are connected by an edge in $\mathcal{G}$. In other words, S is a walk from σ to τ in G, and so $\mathcal{G}$ is connected. □

The proof of Theorem 5.1 now follows in the same spirit as the proof of part 2 of Theorem 3.4.

Proof of Theorem 5.1: It suffices to show that $\{\partial_{x_v} w : w \in \mathcal{S}^a_2(\Delta, p),\ v \in V(\Delta)\} = \mathcal{S}^a_1(\Delta, p)$. First observe that the sequence

$$0 \to \mathcal{S}^a_2(\Delta - v, p) \to \mathcal{S}^a_2(\Delta, p) \overset{\partial_{x_v}}{\to} \mathcal{S}^a_1(\mathrm{st}(v), p)$$

is exact. Now, whether Δ is a normal pseudomanifold or the boundary of a polytope, $\dim \mathcal{S}^a_2(\Delta, p) = g_2(\Delta)$ and $\dim \mathcal{S}^a_1(\mathrm{st}(v), p) = g_1(\mathrm{lk}(v))$. Also, by Lemma 5.2, $\dim \mathcal{S}^a_2(\Delta - v, p) = g_2(\Delta - v)$. Since $g_2(\Delta) = g_2(\Delta - v) + g_1(\mathrm{lk}(v))$, we conclude that the map $\partial_{x_v} : \mathcal{S}^a_2(\Delta, p) \to \mathcal{S}^a_1(\mathrm{st}(v), p)$ is onto, and hence that $\{\partial_{x_v} w : w \in \mathcal{S}^a_2(\Delta),\ v \in V(\Delta)\} = \sum_{v \in V(\Delta)} \mathcal{S}^a_1(\mathrm{st}(v), p)$. Our claim then follows from Theorem 4.1.

The "in particular" part also follows since the space of affine dependencies of the multiset $p(V(\Delta))$ that affinely spans $\mathbb{R}^d$, determines the multiset itself up to affine equivalence. □

We end this section with a corollary to our results. The second part verifies a special case of Conjecture 3.3.

Corollary 5.3. *Let d and (Δ, p) be as in Theorem 5.1.*

1. *For any vertex $v \in V(\Delta)$, $g_2(\Delta) \geq g_1(\mathrm{lk}(v))$.*

2. *Every edge of Δ participates in some affine 2-stress on Δ.*

Proof: Part 1 follows from the facts that $g_2(\Delta) = \dim \mathcal{S}_2^a(\Delta, p)$, $g_1(\mathrm{lk}(v)) = \dim \mathcal{S}_1^a(\mathrm{st}(v), p)$, and $\partial_{x_v} : \mathcal{S}_2^a(\Delta, p) \to \mathcal{S}_1^a(\mathrm{st}(v), p)$ is onto. For part 2, we apply Lemma 5.2 to an edge F. Since the graph of Δ is the graph of $\Delta - F$ plus the edge F, it follows that $\dim \mathcal{S}_2^a(\Delta - F, p) = g_2(\Delta - F) = g_2(\Delta) - 1$. Hence F must participate in some affine 2-stress on (Δ, p). $\square$

6 Reconstructing from higher affine stresses

In this section we prove several results related to Conjectures 1.1, 1.2(2), 3.3, and 3.6 for polytopes without large missing faces and for flag PL spheres. We also briefly touch on the g-numbers of flag spheres.

6.1 Polytopes and flag spheres

We begin with establishing the partition of unity of spaces of linear stresses of antistars.

Lemma 6.1. *Let $d \geq 4$. Let Δ be the boundary complex of a simplicial d-polytope P with its natural embedding p, or a flag $(d-1)$-sphere with a generic embedding p. Let $\tau \in \Delta$. Then for $1 \leq i \leq d-1$, $\mathcal{S}_i^\ell(\Delta - \tau, p) = \sum_{H \in \mathcal{I}(\Delta - \tau)} \mathcal{S}_i^\ell\big(\mathrm{st}(H, \Delta), p\big)$.*

Proof: The case of flag spheres is proved in [4, Theorem 50] (in this case, all minimal interior faces of $\Delta - \tau$ are vertices). Thus, assume that $\Delta = \partial P$. The proof of this case is essentially the same as that of [17, Theorem 16]. Since P is a polytope, there is a line shelling of Δ that lists the facets of the star of τ last. Consequently, there exists a shelling of $\Delta - \tau$. Consider such a shelling and let $F_1, \ldots, F_k$ be the facets at the shelling steps of type i; here $k = h_i(\Delta - \tau)$. For $1 \leq j \leq k$, let $G_j = F_j \backslash r(F_j)$ and let Δ_j be the subcomplex of $\Delta - \tau$ generated by F_j and all the facets that were added before F_j (in the shelling order). We claim that $\mathrm{st}(G_j, \Delta)$ is a subcomplex of Δ_j; in particular, G_j is an interior face of $\Delta - \tau$. Indeed, $\mathrm{lk}(G_j, \Delta)$ is an $(i-1)$-sphere and in the induced shelling of this sphere, the step that adds F_j corresponds to the shelling step of type i (that adds $r(F_j)$). This means that after this step, all facets of the link are in the complex. Now, since $\mathrm{st}(G_j, \Delta)$ and $\mathrm{st}(G_j, \Delta) - r(F_j)$ are Cohen–Macaulay complexes with $h_i(\mathrm{st}(G_j, \Delta)) = 1$ and $h_i\big(\mathrm{st}(G_j, \Delta) - r(F_j)\big) = 0$, it follows that there is a linear i-stress ω_j supported on $\mathrm{st}(G_j, \Delta) \subseteq \Delta_j$ that assigns a nonzero weight to $r(F_j)$. Also, since ω_j is supported on Δ_j, it assigns zero weights to all $r(F_s)$ with $s > j$. We conclude that $\{\omega_j : 1 \leq j \leq k\}$ is a linearly independent set of stresses. Furthermore, since $k = h_i(\Delta - \tau)$, this set is a basis of $\mathcal{S}_i^\ell(\Delta - \tau, p)$. The result follows since G_j is an interior face of $\Delta - \tau$ and hence G_j contains a minimal interior face H_j of $\Delta - \tau$. Therefore, $\mathcal{S}_i^\ell(\Delta - \tau, p) = \sum_j \mathcal{S}_i^\ell\big(\mathrm{st}(G_j, \Delta), p\big) = \sum_{H \in \mathcal{I}(\Delta - \tau)} \mathcal{S}_i^\ell\big(\mathrm{st}(H, \Delta), p\big)$. $\square$

Lemma 6.1 along with Theorem 3.1 imply the following higher-dimensional analogs of Lemma 5.2 and part 1 of Corollary 5.3. This result will be used in essentially all proofs of this section.

Lemma 6.2. *Let $d \geq 4$ and $1 \leq j \leq i \leq d/2$. Let Δ be the boundary complex of a simplicial d-polytope P with its natural embedding p, or a flag $(d-1)$-sphere with a generic embedding p. In the case that $\Delta = \partial P$, assume further that all missing faces of Δ have dimension $\leq d - 2i + 1$. Then for any nonempty face τ of Δ of dimension $\leq j - 1$,*

1. *the map $\partial_c : \mathcal{S}_j^\ell(\Delta - \tau, p) \to \mathcal{S}_{j-1}^\ell(\Delta - \tau, p)$ is onto. In particular, $\dim \mathcal{S}_j^a(\Delta - \tau, p) = g_j(\Delta - \tau)$;*

2. the map $\partial_{x_\tau}: \mathcal{S}^a_j(\Delta,p) \to \mathcal{S}^a_{j-|\tau|}(\mathrm{st}(\tau,\Delta),p)$ *is onto, and hence* $g_j(\Delta) \geq g_{j-|\tau|}(\mathrm{lk}(\tau))$*.*

Proof: Throughout the proof, all stars and links are computed in Δ. Let σ be a minimal interior face of $\Delta - \tau$. Since all missing faces of Δ have dimension $\leq d-2i+1$, it follows from Lemma 2.1 that $\dim\sigma \leq d-2i$, and so $\mathrm{lk}(\sigma)$ is at least $(2i-2)$-dimensional. By the cone lemma and by Theorem 3.1, we obtain that the map $\partial_c : \mathcal{S}^\ell_j(\mathrm{st}(\sigma),p) \to \mathcal{S}^\ell_{j-1}(\mathrm{st}(\sigma),p)$ is onto for all $j \leq i$ and all minimal interior faces σ of $\Delta - \tau$. This together with Lemma 6.1 implies that $\partial_c : \mathcal{S}^\ell_j(\Delta-\tau,p) \to \mathcal{S}^\ell_{j-1}(\Delta-\tau,p)$ is also onto. Finally, since $\Delta-\tau$ is Cohen–Macaulay, $\dim\mathcal{S}^\ell_j(\Delta-\tau) = h_j(\Delta-\tau)$ for all j. We conclude that $\dim\mathcal{S}^a_j(\Delta-\tau) = g_j(\Delta-\tau)$ for all $j \leq i$.

As in the proof of Theorem 3.4, we have the following exact sequence:

$$0 \to \mathcal{S}^a_j(\Delta-\tau,p) \to \mathcal{S}^a_j(\Delta,p) \xrightarrow{\partial x_\tau} \mathcal{S}^a_{j-|\tau|}(\mathrm{st}(\tau),p).$$

By part 1 and Theorem 3.1, the dimensions of the three spaces in the sequence are $g_j(\Delta-\tau)$, $g_j(\Delta)$, and $g_{j-1}(\mathrm{lk}(\tau))$, respectively. Since for any sphere Δ and $\tau \in \Delta$, $g_j(\Delta) = g_j(\Delta-\tau) + g_{j-|\tau|}(\mathrm{lk}(\tau))$, it follows that the right-most map in this sequence, $\partial_{x_\tau} : \mathcal{S}^a_j(\Delta,p) \to \mathcal{S}^a_{j-|\tau|}(\mathrm{st}(\tau),p)$, must be onto. Consequently, $g_j(\Delta) \geq g_{j-|\tau|}(\mathrm{lk}(\tau))$. This completes the proof of both parts. □

With Lemmas 6.1 and 6.2 at our disposal, we are ready to verify a special case of Conjecture 1.2 and a special case of Conjecture 3.6.

Theorem 6.3. *Let* $1 \leq k \leq d/2$ *and let* Δ *be the boundary complex of a simplicial* d*-polytope* P *with its natural embedding* p*. Assume that all missing faces of* Δ *have dimension* $\leq d-2k+1$*. Then for each* $i \leq k$*,* $\mathrm{span}\{\partial_\mu\omega : \omega \in \mathcal{S}^a_i(\Delta,p),\ \mu \in \mathcal{M}_{i-1}(V(\Delta))\} = \mathcal{S}^a_1(\Delta,p)$*, and so the space* $\mathcal{S}^a_i(\Delta,p)$ *determines the space* $\mathcal{S}^a_1(\Delta,p)$*. In particular, the space of affine* i*-stresses determines* P *up to affine equivalence.*

Theorem 6.4. *Let* $1 \leq j < i \leq d/2$*. Let* (Δ,p) *be a PL* $(d-1)$*-sphere with a generic embedding* p*. If* Δ *is flag, then* $\mathrm{span}\{\partial_\mu\omega : \omega \in \mathcal{S}^a_i(\Delta,p),\ \mu \in \mathcal{M}_{i-j}(V(\Delta))\} = \mathcal{S}^a_j(\Delta,p)$*, and so* $\mathcal{S}^a_i(\Delta,p)$ *determines* $\mathcal{S}^a_j(\Delta,p)$*.*

Proof: By Lemma 6.2, for any $(i-j-1)$-face τ, the map $\partial_{x_\tau} : \mathcal{S}^a_i(\Delta,p) \to \mathcal{S}^a_j(\mathrm{st}(\tau),p)$ is onto. In the case that $\Delta = \partial P$ and $j = 1$, the result then follows from Corollary 4.2, while in the case that Δ is a flag PL sphere and $1 \leq j < i$, the result follows from Theorem 4.4. □

Remark 6.5. Let $2 \leq j < i \leq d/2$ and let Δ be the boundary complex of a simplicial d-polytope P with its natural embedding p. Assume that all missing faces of Δ have dimension $\leq d-2i+1$. The same proof as above shows that if Conjecture 4.5 holds for affine j-stresses on (Δ,p), then Conjecture 3.6 holds for (Δ,p) and the pair (i,j), namely, $\mathcal{S}^a_i(\Delta,p)$ determines $\mathcal{S}^a_j(\Delta,p)$.

Using Lemma 6.2, we establish the following special case of Conjecture 3.3 (cf. Corollary 5.3).

Corollary 6.6. *Let* $2 \leq i \leq d/2$*. Let* Δ *be the boundary complex of a simplicial* d*-polytope with its natural embedding* p*, or a flag* $(d-1)$*-sphere with a generic embedding* p*. In the case that* $\Delta = \partial P$*, assume also that all missing faces of* Δ *have dimension* $\leq d-2i+1$*. Then every* $(i-1)$*-face of* Δ *participates in some affine* i*-stress on* Δ*.*

Proof: Let τ be an $(i-1)$-face of Δ. By part 2 of Lemma 6.2, the map $\partial x_\tau : \mathcal{S}^a_i(\Delta,p) \to \mathcal{S}^a_0(\mathrm{st}(\tau,\Delta),p) \cong \mathbb{R}$ is onto. Any preimage of 1 is then an affine i-stress that has τ in its support.
□

6.2 An interlude: g-numbers of flag spheres

The techniques developed in the previous subsection will be useful in obtaining lower bounds. The *octahedral* $(d-1)$*-sphere* is the boundary complex of the d-dimensional cross-polytope $\mathcal{C}_d^*$. As an abstract simplicial complex, $\partial\mathcal{C}_d^*$ is the join of d copies of the 0-sphere. In particular, octahedral spheres are flag and $h_i(\partial\mathcal{C}_d^*) = \binom{d}{i}$ for all $0 \leq i \leq d$. Meshulam [22] proved that in the class of all flag $(d-1)$-spheres, the octahedral sphere simultaneously minimizes all the f-numbers. This result was strengthened by Athanasiadis [7] who showed that, in fact, it simultaneously minimizes all the h-numbers. Here we prove that it even simultaneously minimizes all the g-numbers. This was conjectured in [35] where the case of $i = 2$ was established.

Theorem 6.7. *Let Δ be a flag $(d-1)$-sphere. Then for every $1 \leq i \leq d/2$, $g_i(\Delta) \geq \binom{d}{i} - \binom{d}{i-1}$, and equality holds if and only if Δ is the octahedral sphere.*

Proof: If $\Delta = \partial\mathcal{C}_d^*$, then $g_i(\Delta) = \binom{d}{i} - \binom{d}{i-1}$ for all $1 \leq i \leq d/2$. To prove the inequality and show that $\partial\mathcal{C}_d^*$ is the only minimizer, we use induction on d. The claim is known to hold for $i = 1$. Thus assume that $i > 1$ and let v be a vertex of Δ. If $d = 2i$, then by part 2 of Lemma 6.2 and the inductive hypothesis,

$$\begin{aligned} g_i(\Delta) \geq g_{i-1}(\mathrm{lk}(v)) &\geq \binom{2i-1}{i-1} - \binom{2i-1}{i-2} \\ = \binom{2i-1}{i-1} + \binom{2i-1}{i} &- \binom{2i-1}{i-1} - \binom{2i-1}{i-2} = \binom{2i}{i} - \binom{2i}{i-1}. \end{aligned}$$

If $d > 2i$, then let v' be an interior vertex of $\Delta - v$ (it exists since Δ is flag). Since every affine i-stress on $\mathrm{st}(v')$ is also an affine i-stress on $\Delta - v$, by Lemma 6.2 and the inductive hypothesis,

$$\begin{aligned} g_i(\Delta) &= g_i(\Delta - v) + g_{i-1}(\mathrm{lk}(v)) \geq g_i(\mathrm{lk}(v')) + g_{i-1}(\mathrm{lk}(v)) \\ &\geq \binom{d-1}{i} - \binom{d-1}{i-1} + \binom{d-1}{i-1} - \binom{d-1}{i-2} = \binom{d}{i} - \binom{d}{i-1}. \end{aligned}$$

In both cases, if equality holds, then the link of every vertex v is octahedral. Thus, every vertex v has degree $2d-2$, and so $h_2(\Delta) = f_1(\Delta) - (d-1)f_0(\Delta) + \binom{d}{2} = \binom{d}{2}$. Since $g_2(\Delta) \geq \binom{d}{2} - \binom{d}{1}$, it follows that $f_0(\Delta) \leq 2d$. As Δ is flag, we must have $f_0(\Delta) = 2d$, and so Δ itself is octahedral. $\square$

6.3 Sign vectors of affine stresses

In this subsection we discuss another conjecture related to Kalai's Conjecture 1.1. The statement of this conjecture is based on the notion of sign vectors.

Definition 6.8. Let Δ be the boundary complex of a simplicial d-polytope P with its natural embedding p. For an affine i-stress λ on (Δ, p) and an $(i-1)$-face G of Δ, let

$$\mathrm{sign}(\lambda_G) = \begin{cases} + & \text{if } \lambda_G > 0 \\ - & \text{if } \lambda_G < 0 \\ 0 & \text{if } \lambda_G = 0. \end{cases}$$

Define $\mathcal{V}_i(P) = \{(\mathrm{sign}(\lambda_G))_{G\in\Delta,\, |G|=i} : \lambda \in \mathcal{S}_i^a(\Delta, p)\}$. Thus $\mathcal{V}_i(P)$ is the collection of sign vectors of the squarefree parts of i-stresses on P.

In view of results from [26], the following strengthening of Conjecture 1.1 was proposed there:

Conjecture 6.9. *Let $2 \leq i \leq d/2$. Let $P \subset \mathbb{R}^d$ be a simplicial d-polytope. The $(i-1)$-skeleton of ∂P and the set $\mathcal{V}_i(P)$ determine the combinatorial type of P (i.e., they determine the entire abstract simplicial complex ∂P.)*

The goal of this section is to verify Conjecture 6.9 in the following special case.

Theorem 6.10. *Let $2 \leq i \leq d/2$. Let $P \subset \mathbb{R}^d$ be a simplicial d-polytope whose boundary complex has only missing faces of dimension $\leq d-2i+1$. Then the $(i-1)$-skeleton of ∂P and the set $\mathcal{V}_i(P)$ determine the entire complex ∂P.*

The following lemma will be handy.

Lemma 6.11. *Let $2 \leq i \leq d/2$. Let P be a simplicial d-polytope, let F be a $(j-1)$-face of ∂P with $j \leq i-1$, and let Q be the quotient polytope P/F. Assume also that all missing faces of ∂P have dimension $\leq d-2i+1$. Then every affine $(i-j)$-stress ω' on Q can be lifted to an affine i-stress ω on P with the following property: for each $(i-j-1)$-face τ of ∂Q, $\operatorname{sign}(\omega'_\tau) = \operatorname{sign}(\omega_{F\cup\tau})$.*

Proof: We work with the boundary complex of P, ∂P, with its natural embedding p, and the boundary complex of Q, ∂Q, with its natural embedding q; in particular, $\partial Q = \operatorname{lk}(F, \partial P)$. Consider the sequence

$$\mathcal{S}^a_i(\partial P, p) \xrightarrow{\partial_{x_F}} \mathcal{S}^a_{i-j}\big(\operatorname{st}(F, \partial P), p\big) \xrightarrow{\phi_{i-j}} \mathcal{S}^a_{i-j}\big(\operatorname{lk}(F, \partial P), q\big),$$

where ϕ_{i-j} is the map from Lemma 3.2. The map ∂_{x_F} is surjective by Lemma 6.2, while the map ϕ_{i-j} is an isomorphism by Lemma 3.2. Furthermore, by the remark following Lemma 3.2, if ω' is an affine $(i-j)$-stress on $(\operatorname{lk}(F, \partial P), q)$ and $\omega'' := (\phi_{i-j})^{-1}(\omega')$, then for every $(i-j-1)$-face τ of $\operatorname{lk}(F, \partial P)$, ω'_τ and ω''_τ have the same signs. The result follows by letting ω be any element of $\mathcal{S}^a_i(\partial P, p)$ such that $\partial_{x_F}(\omega) = \omega''$ and noting that $\omega_{F\cup\tau} = \omega''_\tau$. ☐

By Lemma 4.5 in [26], to prove Theorem 6.10, it suffices to establish the following result, which is interesting in its own right. This result concludes this section.

Theorem 6.12. *Let $i \geq 1$ and $d \leq 2i$. Let ∂P be the boundary complex of a simplicial d-polytope P with its natural embedding p. Assume that all missing faces of ∂P have dimension $\leq d-2i+1$. Let M be a missing face of ∂P of size $\geq i+1$ and let $F \subset M$ be any subset of size $i-1$. Then there exists an affine i-stress λ on $(\partial P, p)$ with the following property: for every $(i-1)$-face $G = F \cup v$ of ∂P, $\lambda_G > 0$ if $v \in M \backslash F$ while $\lambda_G \leq 0$ if $v \notin M$.*

Proof: If $i = 1$, then $d - 2i + 1 < d$, so P is a nonsimplex polytope of dimension $d \geq 2$ and $F = \emptyset$. Since M is a missing face, it follows that the intersection $\operatorname{conv}(p(M)) \cap \operatorname{conv}(V(P)\backslash p(M))$ is nonempty and that it is contained in the relative interior of $\operatorname{conv}(p(M))$. Thus, there exists $\lambda \in \mathcal{S}^a_1(\partial P, p)$ such that $\lambda_v > 0$ if the vertex v is in M and $\lambda_v \leq 0$ if $v \notin M$. This completes the proof of the $i = 1$ case.

We now prove the statement for $i > 1$. Let $Q := P/F$ be the quotient polytope and let q be the natural embedding of ∂Q. Then Q is a $(d-i+1)$-polytope and $M' := M\backslash F$ is a missing face of ∂Q. Since all missing faces of ∂P have dimension $\leq d-2i+1 < d-i$, so do all missing faces of ∂Q. In particular, Q is a nonsimplex polytope. Applying the first paragraph to the triple $(Q, M', \emptyset)$, we find an affine stress $\lambda' \in \mathcal{S}^a_1(\partial Q, q) = \mathcal{S}^a_1(\mathrm{lk}(F, \partial P), q)$ such that for every vertex v of ∂Q, $\lambda'_v > 0$ if $v \in M'$ and $\lambda'_v \leq 0$ otherwise. Lemma 6.11 then guarantees the existence of a stress $\lambda \in \mathcal{S}^a_i(\partial P, p)$ such that for every $(i-1)$-face $F \cup v$ of ∂P, $\lambda_{F\cup v} > 0$ if $v \in M' = M \backslash F$ and $\lambda_{F \cup v} \leq 0$ if $v \notin M$. This completes the proof. □

7 k-stacked spheres

To close the paper, we prove Conjecture 1.2(2) for the case of k-stacked polytopes and spheres.

Let $0 \leq k \leq d$. A simplicial d-ball B is called *k-stacked* if all interior faces of B are of dimension $\geq d-k$; in other words, B is k-stacked if every face of B of dimension $\leq d-k-1$ is a face of ∂B. A simplicial $(d-1)$-sphere Δ is *k-stacked* if there exists a k-stacked d-ball B such that $\Delta = \partial B$; in this case, $\mathrm{Skel}_{d-k-1}(\Delta) = \mathrm{Skel}_{d-k-1}(B)$. We say that a simplicial polytope P is k-stacked if ∂P is a k-stacked sphere. The significance of k-stacked spheres is explained by the Generalized Lower Bound Theorem: if $0 \leq k \leq d/2 - 1$, then a simplicial $(d-1)$-sphere Δ satisfies $g_{k+1}(\Delta) = 0$ if and only if Δ is k-stacked. This result for the boundary complexes of simplicial polytopes is due to Murai and Nevo [24] (see also [1]); the general case follows from Murai–Nevo's results and the g-theorem (see Theorem 3.1).

Murai and Nevo also proved that if $0 \leq k \leq d/2-1$ and Δ is a k-stacked $(d-1)$-sphere, then a k-stacked d-ball whose boundary is equal to Δ is *unique*. This ball is given by

$$T(\Delta) := \{F \subseteq V(\Delta) : \mathrm{Skel}_{d-k-1}(\overline{F}) \subseteq \Delta\};$$

see [24, Theorem 2.3]. Furthermore, if P is a k-stacked d-polytope then $T(\partial P)$ provides a *geometric triangulation* of P; see [24, Theorem 1.2]. In particular, the p-images of vertices of any d-face of $T(\partial P)$ are affinely independent.

Assume that $0 \leq k \leq d/2-1$ and that Δ is a k-stacked $(d-1)$-sphere with a d-embedding p. The complex $T(\Delta)$ is d-dimensional; hence, to talk about stress spaces of $T(\Delta)$, we need to specify a map $\tilde{p} : V(T(\Delta)) = V(\Delta) \to \mathbb{R}^{d+1}$. We define such $\tilde{p}$ by $\tilde{p}(v) := (p(v), 1)$. The important thing to notice is that since $\mathrm{Skel}_{d-k-1}(T(\Delta)) = \mathrm{Skel}_{d-k-1}(\Delta)$, it follows from the definition of $\tilde{p}$ that $\mathcal{S}^a_j(\Delta, p) = \mathcal{S}^\ell_j(T(\Delta), \tilde{p})$ for all $j \leq d/2$.

Another thing to notice is that by definition of $T(\Delta)$, $\mathcal{I}(T(\Delta))$ is precisely the set of missing faces of Δ of dimension $\geq d-k$ and

$$T(\Delta) = \bigcup_{F \in \mathcal{I}(T(\Delta))} \mathrm{st}(F, T(\Delta)) = \bigcup_{F \in \mathcal{I}(T(\Delta))} \overline{F} * S_F,$$

where $S(F) = \mathrm{lk}(F, T(\Delta))$ is a sphere of dimension $d - |F| \leq k-1$; in particular, $S_F \subseteq \Delta$.

We are now ready to prove the following case of Conjecture 1.2.

Theorem 7.1. *Let $1 \leq i \leq k \leq d/2-1$. Let Δ be a k-stacked $(d-1)$-sphere that has no missing faces of dimension $\geq d-i+1$. Let p be a d-embedding of Δ such that the p-images of vertices of any d-face of $T(\Delta)$ are affinely independent. Then $\mathcal{S}^a_1(\Delta, p) = \mathrm{span}\,\{\partial_\mu \omega : \omega \in \mathcal{S}^a_i(\Delta, p),\ \mu \in \mathcal{M}_{i-1}(V(\Delta))\}$, and so $\mathcal{S}^a_i(\Delta, p)$ determines $\mathcal{S}^a_1(\Delta, p)$. In particular, if P is a k-stacked d-polytope, then the space of affine i-stresses of P determines P up to affine equivalence.*

Proof: Since $\mathcal{S}^a_j(\Delta, p) = \mathcal{S}^\ell_j(T(\Delta), \tilde{p})$ for all $j \le d/2$, it suffices to show that $\mathcal{S}^\ell_i(T(\Delta), \tilde{p})$ determines $\mathcal{S}^\ell_1(T(\Delta), \tilde{p})$. Also, since the p-images of vertices of any d-face of $T(\Delta)$ are affinely independent, the $\tilde{p}$-images of these vertices are linearly independent. Finally, by our assumptions, the dimension of each F appearing in the decomposition $T(\Delta) = \bigcup_{F \in I(T(\Delta))} \overline{F} * S_F$ is $\le d-i$, and so each sphere S_F in this decomposition has dimension $\ge i-1$. Thus Remark 4.3 applies to $T(\Delta)$ while Corollary 3.5 applies to each $\overline{F} * S_F$. We obtain that

$$\begin{aligned}
& \operatorname{span}\Big\{\partial_\mu \omega : \omega \in \mathcal{S}^\ell_i(T(\Delta), \tilde{p}),\, \mu \in \mathcal{M}_{i-1}(V(\Delta))\Big\} \\
\supseteq \quad & \sum_{F \in \mathcal{I}(T(\Delta))} \operatorname{span}\Big\{\partial_{x_G}\omega : \omega \in \mathcal{S}^\ell_i(\overline{F} * S_F, \tilde{p}),\, G \in S_F,\, |G| = i-1\Big\} \\
\overset{(*)}{=} \quad & \sum_{F \in \mathcal{I}(T(\Delta))} \mathcal{S}^\ell_1\left(\overline{F} * S_F, \tilde{p}\right) \\
= \quad & \sum_{F \in \mathcal{I}(T(\Delta))} \mathcal{S}^\ell_1\left(\operatorname{st}(F, T(\Delta)), \tilde{p}\right) \overset{(\dagger)}{=} \mathcal{S}^\ell_1\left(T(\Delta), \tilde{p}\right).
\end{aligned}$$

Here $(*)$ is by Corollary 3.5 and $(\dagger)$ is by Remark 4.3. The result follows. ☐

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