BIPARTITE MINORS

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Abstract. We introduce a notion of bipartite minors and prove a bipartite analog of Wagner's theorem: a bipartite graph is planar if and only if it does not contain $K_{3,3}$ as a bipartite minor. Similarly, we provide a forbidden minor characterization for outerplanar graphs and forests. We then establish a recursive characterization of bipartite $(2,2)$-Laman graphs — a certain family of graphs that contains all maximal bipartite planar graphs.

1. Introduction

Wagner's celebrated theorem [5], [2, Theorem 4.4.6] provides a characterization of planar graphs in terms of minors: a graph G is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor. Unfortunately, a minor of a bipartite graph is not always bipartite as contracting edges destroys 2-colorability. Here, we introduce a notion of a *bipartite minor*: an operation that applies to bipartite graphs and outputs bipartite graphs. We then prove a bipartite analog of Wagner's theorem: a bipartite graph is planar if and only if it does not contain $K_{3,3}$ as a bipartite minor. Similarly, we provide a forbidden bipartite minor characterization for bipartite outerplanar graphs and forests.

All the graphs considered in this note are simple graphs. A graph with vertex set V and edge set E is denoted by $G = (V, E)$. We denote the edge connecting vertices i and j by ij. A graph is *bipartite* if there exists a bipartition (or bicoloring in red and blue) of the vertex set V of G, $V = A \uplus B$, in such a way that no two vertices from the same part are connected by an edge. When discussing bipartite graphs, we fix such a bipartition and write $G = (A \uplus B, E)$; we refer to A and B as *parts* or *sides* of G.

As bipartite planar graphs with $n \geq 3$ vertices have at most $2n - 4$ edges, and as all their subgraphs are also bipartite and planar, and hence satisfy the same restriction on the number of edges, it is natural to consider the family of maximal bipartite graphs possessing this property. Specifically, we say that a bipartite graph $G = (A \biguplus B, E)$ with $|A| \geq 2$ and $|B| \geq 2$ is $(2,2)$-*Laman* if (i) G has exactly $2(|A| + |B|) - 4$ edges, and (ii) every subgraph H of G with at least 3 vertices has at most $2|V(H)| - 4$ edges. Note that the family of $(2,2)$-Laman graphs is strictly larger than that of maximal bipartite planar graphs: indeed, taking $n \geq 2$ copies of $K_{3,3}$ minus an edge, and gluing all these

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copies together along the two vertices of the missing edge, produces a graph on $4n + 2$ vertices with $8n$ edges; this graph is $(2, 2)$-Laman, but it is not planar.

Our second main result is a recursive characterization of $(2, 2)$-Laman graphs. We remark that the name $(2, 2)$-Laman is motivated by Laman's theorem [4] from rigidity theory of graphs, and its relation to a recent theory of rigidity for bipartite graphs can be found in [3]. As such, this paper is a part of a project to understand notions of minors and graph-rigidity for bipartite graphs as well as to understand higher-dimensional generalizations.

The rest of this note is organized as follows: in Section 2 we define bipartite minors and prove the bipartite analog of Wagner's theorem and analogous theorems for bipartite outerplanar graphs and forests (deferring treatment of some of the cases to the Appendix). Then in Section 3 we discuss $(2, 2)$-Laman graphs.

2. Wagner's theorem for bipartite graphs

We start by defining a couple of basic operations on (bipartite) graphs. If $G = (V, E)$ is a graph and v is a vertex of G, then $G - v$ denotes the induced subgraph of G on the vertex set $V - \{v\}$. If $G = (V = A \uplus B, E)$ is a bipartite graph and u, v are two vertices from the same part, then the *contraction* of u with v is a graph G' on the vertex set $V - \{u\}$ obtained from G by identifying u with v and deleting the extra copy from each double edge that was created. Observe that G' is also bipartite.

Recall that if G is a graph and C is a cycle of G, then C is *non-separating* if the removal of the vertices of C from G does not increase the number of connected components. A cycle C is *induced* (or chordless) if each two nonadjacent vertices of C are not connected by an edge in G. Induced non-separating cycles are known in the literature as *peripheral* cycles.

We now come to the main definition of this section.

Definition 2.1. Let G be bipartite graph. We say that a graph H is a *bipartite minor of* G, denoted $H <_b G$, if there is a sequence of graphs $G = G_0, G_1, \ldots, G_t = H$ where for each i, G_{i+1} is obtained from G_i by either deletion (of a vertex or an edge) or *admissible contraction*. A contraction of a vertex u with a vertex v in G_i is called *admissible* if u and v have a common neighbor in G_i, and at least one of these common neighbors, say w, is such that the path (u, w, v) is a part of a peripheral cycle in G_i.

For instance, applying an admissible contraction to an 8-cycle results in a 6-cycle plus an edge attached to this cycle at one vertex. Note that since each admissible contraction identifies two vertices that have a common neighbor, these two vertices are from the same part of the graph. Thus all bipartite minors are bipartite graphs. The importance of the notion of bipartite minors is explained by the following result that can be considered as a bipartite analog of Wagner's theorem.

Theorem 2.2. *A bipartite graph G is planar if and only if G does not contain $K_{3,3}$ as a bipartite minor.*

Proof. First assume that G is planar. We may assume that G is connected. To verify that G does not contain $K_{3,3}$ as a bipartite minor, it suffices to show that deletions and admissible contractions preserve planarity. This is clear for deletions. For admissible contractions, consider an embedding of G in a 2-sphere $\mathbb{S}^2$, and let C be a peripheral cycle of G that contains a path (u, w, v). By the Jordan-Schönflies theorem, the complement of the image of C in $\mathbb{S}^2$ consists of two components, each homeomorphic to an open 2-ball. As C is peripheral, one of these components contains no vertices/edges of G, and hence is a face of the embedding of G. Contracting u with v "inside this face" produces an embedding of the resulting graph in $\mathbb{S}^2$.

Assume now that G is not planar. We must show that G contains $K_{3,3}$ as a bipartite minor. By Kuratowski's theorem [2, Theorem 4.4.6], G contains a subgraph H that is a subdivision of either K_5 or $K_{3,3}$. Hence, it only remains to show that $K_{3,3} <_b H$. We first treat the case where an edge e of the original K_5 (or $K_{3,3}$) is subdivided at least twice. Let C be a peripheral cycle of the original K_5 (or $K_{3,3}$) that contains e (there exists such C — a 3-cycle for K_5 and a 4-cycle for $K_{3,3}$), let C' be the subdivision of C in H, and let (a, b, a') be a path of length two in H that is contained in e. Then (a, b, a') is a part of a peripheral cycle C' in H, and hence contracting a' with a is an admissible contraction in H. Performing this contraction and then deleting b, we obtain a new bipartite subdivision H' of K_5 (or $K_{3,3}$) that subdivides e with two fewer interior vertices than H, but agrees with H on all other edges of K_5 ($K_{3,3}$, respectively). Thus, we can assume that each edge of the original K_5 (or $K_{3,3}$) is subdivided at most once. By symmetry, this reduces the problem of finding $K_{3,3}$ as a bipartite minor of H to finding $K_{3,3}$ as a bipartite minor of the nine bipartite graphs described below. These cases are treated in the Appendix.

There are three bipartite graphs which are subdivisions of K_5 to consider, denoted by $G_{(i)}$ for $i = 5, 4, 2$, where $G_{(i)}$ is the graph obtained from K_5 by coloring i of its vertices red, the other $5 - i$ blue, then subdividing each monochromatic edge once, and coloring the subdivision vertex red/blue so that its color is opposite to that of the vertices of the original monochromatic edge. For example, $G_{(5)}$ is the barycentric subdivision of K_5, endowed with a 2-coloring. Note that no edge of K_5 that connects two vertices of opposite colors is subdivided.

There are six bipartite graphs which are subdivisions of $K_{3,3}$ to consider, denoted by $G_{(i,j)}$ and defined as follows. Let X and Y be the two sides of $K_{3,3}$. Then $G_{(i,j)}$ is the graph obtained from $K_{3,3}$ by first (i) coloring red exactly i vertices from X and j vertices from Y, and coloring blue the other $6 - i - j$ vertices; then (ii) subdividing each monochromatic edge once, and coloring the subdivision vertex red/blue as before, so that a proper 2-coloring is obtained. Note that as before, no edge of $K_{3,3}$ that connects two vertices of opposite colors is subdivided. Up to symmetry, the six graphs we need to consider are $G_{(3,3)}, G_{(3,2)}, G_{(3,1)}, G_{(3,0)}, G_{(2,2)}, G_{(2,1)}$. (Observe that $G_{(3,0)} = K_{3,3}$, and so this case is trivial.) □

Remark 2.3. It is worth noting that the barycentric subdivision of K_5 (which is a bipartite graph) does not contain subgraphs homeomorphic to $K_{3,3}$.

A graph is *outerplanar* if it can be embedded in the plane in such a way that all of the vertices lie on the outer boundary. Equivalently, a graph G is outerplanar if adding a new vertex to G and connecting it to all vertices of G results in a planar graph; we denote this graph by $\widehat{G}$. Outerplanar graphs are characterized by not having as a minor K_4 and $K_{2,3}$. Here is a bipartite analog of this result for bipartite minors; the proof is similar to the proof of Theorem 2.2.

Theorem 2.4. *A bipartite graph G is outerplanar if and only if G does not contain $K_{2,3}$ as a bipartite minor.*

Proof. First assume that G is outerplanar. To verify that G does not contain $K_{2,3}$ as a bipartite minor, it suffices to show that deletions and admissible contractions preserve outerplanarity. This is clear for deletions. To deal with admissible contractions, consider the graph $\widehat{G}$ defined right before the statement of the theorem. Then $\widehat{G}$ is planar, although not bipartite, and if G' is obtained from G by an admissible contraction of a' with a, then $\widehat{G'} = (\widehat{G})'$. (Note that a peripheral cycle of G is also a peripheral cycle of $\widehat{G}$, so the same contraction is admissible in $\widehat{G}$.) As $(\widehat{G})'$ is planar (the same argument as in the proof of Theorem 2.2 applies), we infer that G' is outerplanar.

Next assume that G is not outerplanar. By a result of Chartrand and Harary [1], G contains a subdivision of either K_4 or $K_{2,3}$. Let H be such a subgraph of G. As in the proof of Theorem 2.2, we may assume that each edge of the original K_4 ($K_{2,3}$, respectively) is subdivided at most once. Thus, it suffices to show that $K_{2,3}$ is a bipartite minor of each of the following nine bipartite graphs. Given the coloring below, we can find $K_{2,3}$ with three red vertices and two blue ones as a bipartite minor.

For subdivisions of K_4, we need to consider $H = H_{(i)}$ for $i = 4, 3, 2$, where in $H_{(i)}$ exactly i of the original vertices of K_4 are red (the other $4-i$ vertices are blue). For subdivisions of $K_{2,3}$, let X and Y be the two sides of $K_{2,3}$, with $|X| = 2$ and $|Y| = 3$, and for the bipartite subdivisions $H_{(i,j)}$ as above, with exactly i red vertices from X and j red vertices from Y, we need to consider H being one of $H_{(2,3)}, H_{(1,3)}, H_{(0,3)}, H_{(2,2)}, H_{(1,2)}, H_{(2,1)}$. We leave the verification of these nine cases to the readers. □

Similarly, the following results hold (we omit their easy proofs). We define an *apex planar* graph (respectively, *apex outerplanar* graph) to be any graph G that has a vertex v such that $G - v$ is planar (respectively, outerplanar).

Theorem 2.5. *A bipartite graph G is a forest if and only if G does not contain $K_{2,2}$ as a bipartite minor.*

Theorem 2.6. *Deletions and admissible contractions preserve apex planarity; in other words, the family of bipartite apex planar graphs is closed under bipartite minors. Similarly, the family of bipartite apex outerplanar graphs is closed under bipartite minors.*

However, there are examples showing that neither linkless embeddability nor embeddability into closed surfaces other than spheres is preserved under bipartite minors.

A natural question at this point to consider is then

Problem 2.7. *Let G and H be bipartite graphs. We say that $H \leq_b G$ if H is a bipartite minor of G. Is this relation a well-quasi-ordering on the set of bipartite graphs?*

3. (2, 2)-Laman graphs

We now turn our discussion to $(2,2)$-Laman graphs. Note that if $G = (V = A \uplus B, E)$ is $(2,2)$-Laman, then every vertex of G has degree at least two: indeed, if v were a vertex of degree one, then $G - v$ would have $2(|A| + |B| - 1) - 3$ edges instead of at most $2(|A| + |B| - 1) - 4$ edges allowed by the definition of $(2,2)$-Laman graphs. Moreover, if G is $(2,2)$-Laman and v is a vertex of degree two, then either G is $K_{2,2}$ or $G - v$ is also $(2,2)$-Laman. Finally, since G has fewer than $2|V|$ edges, there is a vertex of G that has degree at most three. Hence, we can assume that G is a graph with minimal degree three. The following theorem can thus be considered as a recursive characterization of $(2,2)$-Laman graphs.

Theorem 3.1. *Let $G = (V, E)$ be a bipartite $(2,2)$-Laman graph with minimal degree three. Then every vertex v of degree three has two neighbors x, y with the property that there exists a vertex p that is adjacent to y and not adjacent to x, and such that the graph $G' = (G - v) \cup xp$ is $(2,2)$-Laman.*

Proof. If $X \subseteq V$, we write $E(X) = |E(G[X])|$ — the cardinality of the edge set of the subgraph of G induced by X. A subset X of V is *critical* if $|X| \geq 3$ and $E(X) = 2|X| - 4$; equivalently, if $|X| \geq 3$ and $G[X]$ is $(2,2)$-Laman or $K_{2,1}$.

Let (A, B) be a bipartition of V. In what follows vertices called a_i belong to A, and vertices called b_i belong to B. Suppose that $a_0 \in A$ is a vertex of degree 3 and let b_1, b_2, b_3 be the neighbors of a_0. We prove the theorem in several steps, which we number below by (i*).

(0*) *Every subset $X \subseteq V$ with $|X| \geq 2$ such that $G[X]$ is not an edge satisfies $E(X) \leq 2|X| - 4$.*

Proof. This is immediate from the definition of $(2,2)$-Laman graphs. □

(1*) *At least two neighbors of a_0 have non-neighbors in A.*

Proof. The subgraph induced on $A \cup \{b_1, b_2, b_3\}$ has $|A| + 3$ vertices and hence at most $2|A| + 2$ edges. Thus if b_1, b_2 are adjacent to all of A then b_3 has degree at most two, a contradiction. □

Let b_1, b_2 be as guaranteed in (1*). Let $Z \subseteq V$ be maximal with $2|Z| - 4$ edges, containing b_1, b_2 and not containing a_0 (possibly $Z = \{b_1, b_2\}$).

(2*) *The element b_3 is not in Z and has at most one neighbor in Z. In particular, b_3 has a neighbor in $A \setminus (Z \cup \{a_0\})$, and so the latter set is nonempty.*

Proof. If $b_3 \in Z$ then $Z \cup \{a_0\}$ violates Laman condition (ii), a contradiction, and so $b_3 \notin Z$. Now $G[Z \cup \{a_0\}]$ is $(2,2)$-Laman, hence the Laman condition (ii) for $Z \cup \{a_0, b_3\}$ shows that b_3 has at most one neighbor in Z. As $\deg(b_3) \geq 3$ the rest of (2*) follows. □

Denote by $M(b_i)$ the set of non-neighbors of b_i in $A \setminus (Z \cup \{a_0\})$, and w.l.o.g. assume $|M(b_1)| \geq |M(b_2)|$. Note that the maximality of Z implies that

(3*) *Every vertex in $A \setminus (Z \cup \{a_0\})$ has at most one neighbor in Z.*

In particular, we will use the following:

(4*) *No element in $A \setminus (Z \cup \{a_0\})$ is a neighbor of both b_1 and b_2.*

(5*) *The set A_1 of neighbors of either b_2 or b_3 in $M(b_1)$ is nonempty.*

Proof. Either there exists $p \in M(b_1) \setminus M(b_2)$ (that is, p is a neighbor of b_2 but not of b_1), in which case we are done, or $M(b_1) = M(b_2)$. In the latter case $M(b_1) = A \setminus (Z \cup \{a_0\})$ by (4*). Hence by (2*), there is a vertex $p \in M(b_1)$ that is a neighbor of b_3. The statement follows. □

We need to prove that there is $p \in A_1$ such that no critical set contains p and b_1 and not a_0. Assume the contrary. Then for every $p \in A_1$ there is some critical set X_p containing p and b_1, and not containing a_0. We will reach a contradiction to (5*). We may assume that the sets X_p are maximal with these properties.

(6*) *If $p \in A_1$ then $b_2 \notin X_p$.*

Proof. Assume by contradiction $b_2 \in X_p$. By maximality of Z and as $p \notin Z$, it is enough to show that $X_p \cup Z$ is critical. Indeed,

$$E(Z \cup X_p) \geq E(Z) + E(X_p) - E(Z \cap X_p) =$$
$$2(|Z| + |X_p|) - 8 - E(Z \cap X_p) \geq 2(|Z| + |X_p|) - 8 - (2|Z \cap X_p| - 4) \geq$$
$$2|Z \cup X_p| - 4,$$

where the middle inequality is by (0*), as $\{b_1, b_2\} \subseteq Z \cap X_p$ are two vertices on the same side. □

(7*) *For every $p \in A_1$, b_1 has at least two neighbors in X_p.*

Proof. Since X_p is critical containing two non-adjacent vertices from opposite sides (namely p and b_1), we deduce that $|X_p| \geq 4$. Then $E(X_p \setminus b_1) \leq 2(|X_p| - 1) - 4$ by Laman condition (ii), and so b_1 has at least two neighbors in X_p. □

Let $X_0 \subseteq V$ (in our notations $0 \notin V$) consist of b_1 and its neighbors different from a_0. For notational convenience, identify A_1 with $\{1, 2, \ldots, s\}$, and let $Y_s = X_0 \cup X_1 \cup \cdots \cup X_s$. Let $k = |X_0| - 1$, namely the degree of b_1 in $G[V \setminus a_0]$ (hence $k \geq 2$).

(8*) $E(Y_s) \geq 2|Y_s| - 4 - (k-2)$.

Proof. Let $Y_i = \cup_{0 \leq j \leq i} X_j$. We show by induction on i that $E(Y_i) \geq 2|Y_i| - 4 - (k-2)$. The case $i = 0$ trivially holds with equality. For $i \geq 1$, $|X_0 \cap X_i| \geq 3$ by (7*), so applying Laman condition (ii) to $Y_{i-1} \cap X_i$ we see that

$$E(Y_i) \geq E(Y_{i-1}) + E(X_i) - E(Y_{i-1} \cap X_i) \geq$$
$$(2|Y_{i-1}| - 4 - (k-2)) + (2|X_i| - 4) - (2|Y_{i-1} \cap X_i| - 4) =$$
$$2|Y_{i-1} \cup X_i| - 4 - (k-2),$$

as desired. □

Next we will show that $Z \cup Y_s$ is critical.

(9*) *$Z \cap Y_s \subseteq X_0$, and thus $Z \cap Y_s = Z \cap X_0$.*

Proof. Assume the contrary. Then there is i such that $Z \cap X_i$ has a non-neighbor of b_1. Hence by (0*), $E(Z \cap X_i) \leq 2|Z \cap X_i| - 4$. Thus, our usual yoga shows $E(Z \cup X_i) \geq 2|Z \cup X_i| - 4$. By maximality of X_i, we conclude that $Z \subseteq X_i$, and so $b_2 \in X_i$. This contradicts (6*). $\square$

(10*) *$Z \cup Y_s$ is critical.*

Proof. Denote by d_i the number of neighbors of b_i in $A \setminus (Z \cup \{a_0\})$. By (9*), $E(Z \cap Y_s) = k - d_1$ and $|Z \cap Y_s| = k - d_1 + 1$. There are d_2 edges from $b_2 \in Z$ into $A_1 \subseteq Y_s \setminus Z$ by (4*). Therefore,

$$E(Z \cup Y_s) \geq E(Z) + E(Y_s) - E(Z \cap Y_s) + d_2 \geq$$
$$(2|Z| - 4) + (2|Y_s| - 4 - (k-2)) - (k - d_1) + d_2 =$$
$$2|Z \cup Y_s| - 4 + (d_2 - d_1),$$

and by $|M(b_1)| \geq |M(b_2)|$ we have $d_2 - d_1 \geq 0$. Thus $Z \cup Y_s$ is critical. $\square$

By maximality of Z, (10*) implies $Y_s \subseteq Z$. Thus $A_1 \subseteq Z$. Hence by the definition of A_1, A_1 must be empty. This contradicts (5*) and completes the proof of the theorem. $\square$

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4. Appendix

For each of the nine graphs described in the proof of Theorem 2.2, we describe the sequences of deletions and admissible contractions yielding $K_{3,3}$ as a bipartite minor. For each contraction we indicate a peripheral cycle showing that the contraction is admissible, called a *witness cycle*.

We start with the subdivisions of K_5. Denote the vertices of K_5 by v_i where $i \in [5]$ (here $[n] = [1, n] = \{1, 2, \cdots, n\}$), and the subdivision vertex of the edge $v_i v_j$ in $G_{(i)}$ by $v_{ij} = v_{ji}$. When referring to vertices after performing contractions on $G_{(i)}$, we use *any* representative from the vertices of $G_{(i)}$; this should cause no confusion.

Case 1: $G_{(5)}$**.** Here all v_i where $i \in [5]$ are red.

(1) Contract v_{15} with v_{13}. Witness cycle: $(v_{15}v_1v_{13}, v_3v_{35}v_5)$.
(2) Contract v_{25} with v_{23}. Witness cycle: $(v_{25}v_2v_{23}, v_3v_{35}v_5)$.
(3) Contract v_{45} with v_{14}. Witness cycle: $(v_{45}v_4v_{14}, v_1v_{15}v_5)$.

(4) Contract v_1 with v_2. Witness cycle: $(v_1v_{12}v_2, v_{25}v_5v_{15})$.
(5) Contract v_3 with v_4. Witness cycle: $(v_4v_{34}v_3, v_{35}v_5v_{54})$.

Call the resulting graph H. The induced subgraph of H on the 3 red vertices v_1, v_3, v_5 and the 3 blue vertices v_{15}, v_{25}, v_{45} is $K_{3,3}$.

Case 2: $G_{(4)}$**.** Here all v_i where $i \in [4]$ are red, v_5 is blue.

(1) Contract v_{34} with v_{23}. Witness cycle: $(v_{34}v_3v_{23}, v_2v_{24}v_4)$.
(2) Contract v_{12} with v_{14}. Witness cycle: $(v_{12}v_1v_{14}, v_4v_{24}v_2)$.
(3) Contract v_{12} with v_{13}. Witness cycle: $(v_{12}v_1v_{13}, v_3v_{32}v_2)$.

Call the resulting graph H. The induced subgraph of H on the 3 red vertices v_2, v_3, v_4 and the 3 blue vertices v_5, v_{34}, v_{12} is $K_{3,3}$.

Case 3: $G_{(2)}$**.** Here all v_i where $i \in [2]$ are red, the other 3 vertices of $G_{(2)}$ are blue.

(1) Contract v_{34} with v_{35}. Witness cycle: $(v_{34}v_3v_{35}, v_5v_{45}v_4)$.

Call the resulting graph H. The induced subgraph of H on the 3 red vertices v_1, v_2, v_{34} and the 3 blue vertices v_3, v_4, v_5 is $K_{3,3}$.

We now turn to the six subdivisions of $K_{3,3}$. Let the sides of $K_{3,3}$ be $X = \{v_1, v_2, v_3\}$ and $Y = \{v_4, v_5, v_6\}$. We use the same notation v_{ij} as in the case of subdivisions of K_5.

Case 1: $G_{(3,3)}$**.** Here all v_i where $i \in [6]$ are red.

(1) Contract v_{15} with v_{35}. Witness cycle: $(v_{15}v_5v_{35}, v_3v_{34}v_4v_{14}v_1)$.
(2) Contract v_{14} with v_{24}. Witness cycle: $(v_{14}v_4v_{24}, v_2v_{26}v_6v_{16}v_1)$.
(3) Contract v_{26} with v_{36}. Witness cycle: $(v_{26}v_6v_{36}, v_3v_{34}v_4v_{24}v_2)$.
(4) Contract v_1 with v_6. Witness cycle: $(v_1v_{16}v_6, v_{36}v_2v_{24})$.
(5) Contract v_2 with v_5. Witness cycle: $(v_2v_{25}v_5, v_{35}v_3v_{34}v_4v_{24})$.
(6) Delete v_{16}, then delete v_{25}.
(7) Contract v_3 with v_4. Witness cycle: $(v_3v_{34}v_4, v_{24}v_1v_{36})$.

Call the resulting graph H. The induced subgraph of H on the 3 red vertices v_1, v_2, v_3 and the 3 blue vertices v_{15}, v_{14}, v_{26} is $K_{3,3}$.

Case 2: $G_{(3,2)}$**.** Here all v_i where $i \in [5]$ are red and v_6 is blue.

(1) Contract v_{15} with v_{35}. Witness cycle: $(v_{15}v_5v_{35}, v_3v_{34}v_4v_{14}v_1)$.
(2) Contract v_{14} with v_{24}. Witness cycle: $(v_{14}v_4v_{24}, v_2v_{25}v_5v_{15}v_1)$.
(3) Contract v_3 with v_4. Witness cycle: $(v_3v_{34}v_4, v_{24}v_2v_{25}v_5v_{15})$.
(4) Contract v_2 with v_5. Witness cycle: $(v_2v_{25}v_5, v_{15}v_1v_6)$.

Call the resulting graph H. The induced subgraph of H on the 3 red vertices v_1, v_2, v_3 and the 3 blue vertices v_{15}, v_{14}, v_6 is $K_{3,3}$.

Case 3: $G_{(3,1)}$**.** Here all v_i where $i \in [4]$ are red and v_5, v_6 are blue.

(1) Contract v_{24} with v_{34}. Witness cycle: $(v_{24}v_4v_{34}, v_3v_5v_2)$.
(2) Contract v_1 with v_4. Witness cycle: $(v_1v_{14}v_4, v_{34}v_3v_5)$.

Call the resulting graph H. The induced subgraph of H on the 3 red vertices v_1, v_2, v_3 and the 3 blue vertices v_{24}, v_5, v_6 is $K_{3,3}$.

Case 4: $G_{(2,2)}$**.** Here all v_i where $i \in [2,5]$ are red and v_1, v_6 are blue.

(1) Contract v_{24} with v_{34}. Witness cycle: $(v_{34}v_4v_{24}, v_2v_{25}v_5v_{53}v_3)$.
(2) Contract v_{25} with v_{35}. Witness cycle: $(v_{25}v_5v_{35}, v_3v_{34}v_2)$.
(3) Contract v_4 with v_5. Witness cycle: $(v_4v_1v_5, v_{25}v_3v_{34})$.
(4) Contract v_1 with v_6. Witness cycle: $(v_1v_{16}v_6, v_2v_{25}v_4)$.

Call the resulting graph H. The induced subgraph of H on the 3 red vertices v_2, v_3, v_4 and the 3 blue vertices v_{24}, v_{25}, v_6 is $K_{3,3}$.

Case 5: $G_{(2,1)}$**.** Here all v_i where $i \in [2,4]$ are red and v_1, v_5, v_6 are blue.

(1) Contract v_{16} with v_{15}. Witness cycle: $(v_{16}v_1v_{15}, v_5v_3v_6)$.
(2) Contract v_{24} with v_{34}. Witness cycle: $(v_{24}v_4v_{34}, v_3v_6v_2)$.
(3) Contract v_1 with v_{34}. Witness cycle: $(v_1v_4v_{34}, v_2v_5v_{15})$.

Call the resulting graph H. The induced subgraph of H on the 3 red vertices v_{16}, v_2, v_3 and the 3 blue vertices v_1, v_5, v_6 is $K_{3,3}$.

Case 6: $G_{(3,0)}$**.** In this case $G_{(3,0)} = H = K_{3,3}$. This completes the proof of Theorem 2.2. □

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