1. Prove the following basic facts about infinitesimal rigidity.

(a) **Replacement Lemma:** Let \( G = \langle V, E \rangle \) be a \( d \)-dimensional framework, and let \( G' \) be an induced subframework on \( k \) joints \( v_1, \ldots, v_k \) of \( G \). Assume further that \( G \) is infinitesimally rigid. Show that if \( G' \) is replaced by a new subframework \( G'' \) on joints \( v_1, \ldots, v_k \) that is infinitesimally rigid as a framework in the affine space spanned by \( v_1, \ldots, v_k \), then the modified framework, \( \tilde{G} = (G \setminus G') \cup G'' \), is infinitesimally rigid in the \( d \)-space.

(b) **Cone Lemma:** Let \( G = \langle V, E \rangle \) be a framework in \( \mathbb{R}^{d+1} \) such that \( H := \text{Aff}(V) \) is a hyperplane in \( \mathbb{R}^{d+1} \). For \( u \in \mathbb{R}^{d+1} \setminus H \) define \( G^* \{ u \} := \langle V', E' \rangle \), where \( V' = V \cup \{ u \} \) and \( E' = E \cup \{ uv : v \in V \} \) (i.e., \( G^* \{ u \} \) is a cone over \( G \) with apex \( u \)). Show that \( G \) is infinitesimally rigid in \( H \) if and only if \( G^* \{ u \} \) is infinitesimally rigid in \( \mathbb{R}^{d+1} \).

**Hint for part (b):** what is the relationship between the stress spaces of \( G \) and \( G^* \)?

2. (a) Show that if to a \( d \)-dimensional infinitesimally rigid framework with more than \( d \) joints a new joint \( v \in \mathbb{R}^d \) is added together with \( d \) bars incident with \( v \) that span \( \mathbb{R}^d \), then the new framework is also infinitesimally rigid.

(b) Show that a 1-dimensional framework with all joints distinct is infinitesimally rigid if and only if the underlying graph is connected. (**Hint:** start with a tree.)

(c) A framework is called a **triangulated polygon** if its underlying graph can be drawn in a plane as a convex \( n \)-gon \((n \geq 3)\) with a maximal set of non-crossing edges inside it. Show that any framework \( G = \langle V, E \rangle \) in the plane which is a triangulated polygon with no triangle of bars collinear is infinitesimally rigid, but deleting any bar from \( G \) makes it infinitesimally flexible.

3. Let \( G = (\{ n \}, E) \) be a graph on \( n \) vertices (considered as an abstract graph). A map \( \psi : \{ n \} \rightarrow \mathbb{R}^3 \) is called a **3-embedding** of \( G \) into \( \mathbb{R}^3 \). We identify the set of all 3-embeddings of \( G \) with \( \mathbb{R}^{3n} \) via \( \psi \mapsto (v_1, \ldots, v_n) = (\psi(1), \ldots, \psi(n)) \in \mathbb{R}^3 \times \cdots \times \mathbb{R}^3 \cong \mathbb{R}^{3n}. \) Each 3-embedding \( \psi \) of \( G \) defines a 3-dimensional framework whose joints are \( \psi(1), \ldots, \psi(n) \) and whose bars are the elements of \( E \). A 3-embedding of \( G \) into \( \mathbb{R}^3 \) is **infinitesimally rigid** (flexible, resp.) if the corresponding framework is infinitesimally rigid (flexible, resp.).

Prove the following result due to Gluck (1974): the set of infinitesimally rigid 3-embeddings of the graph of a 3-dimensional simplicial polytope with \( n \) vertices is open and dense in \( \mathbb{R}^{3n}. \)

**Directions:** Show first that a 3-embedding \( \psi \) (whose joints affinely span \( \mathbb{R}^3 \)) is infinitesimally flexible if and only if the stress matrix \( R(\psi) \) of the corresponding framework satisfies the following polynomial equation: \( \sum_B \det(B)^2 = 0 \), where the sum is over all \( (3n - 6) \times (3n - 6) \) submatrices of \( A \). (**Hint:** How many edges does \( G \) have? What can you say about the rank of \( A' \)?) Conclude that the set of infinitesimally flexible 3-embeddings of \( G \) is an algebraic variety, and hence its complement in \( \mathbb{R}^{3n} \) is either empty or is open and dense. Why is the complement non-empty?