Polytopes **Homework #6** **due Friday, 5/20/16**

1. Prove the following basic facts about infinitesimal rigidity.

   (a) *Replacement Lemma*: Let $G = \langle V, E\rangle$ be a d-dimensional framework, and let G' be an induced subframework on k joints $v_1, \ldots, v_k$ of G. Assume further that G is infinitesimally rigid. Show that if G' is replaced by a new subframework G'' on joints $v_1, \ldots, v_k$ that is infinitesimally rigid as a framework in the affine space spanned by $v_1, \ldots, v_k$, then the modified framework, $\widetilde{G} = (G \setminus G') \cup G''$, is infinitesimally rigid in the d-space.

   (b) *Cone Lemma*: Let $G = \langle V, E\rangle$ be a framework in $\mathbb{R}^{d+1}$ such that $H := \mathrm{Aff}(V)$ is a *hyperplane* in $\mathbb{R}^{d+1}$. For $u \in \mathbb{R}^{d+1} \setminus H$ define $G^*\{u\} := \langle V', E'\rangle$, where $V' = V \cup \{u\}$ and $E' = E \cup \{uv : v \in V\}$ (i.e., $G^*\{u\}$ is a cone over G with apex u). Show that G is infinitesimally rigid in H if and only if $G^*\{u\}$ is infinitesimally rigid in R^{d+1}.

   Hint for part (b): what is the relationship between the stress spaces of G and G^*?

2. (a) Show that if to a d-dimensional infinitesimally rigid framework with more than d joints a new joint $v \in \mathbb{R}^d$ is added together with d bars incident with v that span $\mathbb{R}^d$, then the new framework is also infinitesimally rigid.

   (b) Show that a 1-dimensional framework with all joints distinct is infinitesimally rigid if and only if the underlying graph is connected. (**Hint:** start with a tree.)

   (c) A framework is called a *triangulated polygon* if its underlying graph can be drawn in a plane as a convex n-gon ($n \geq 3$) with a maximal set of non-crossing edges inside it.

   Show that any framework $G = \langle V, E\rangle$ in the plane which is a triangulated polygon with no triangle of bars collinear is infinitesimally rigid, but deleting any bar from G makes it infinitesimally flexible.

3. Let $G = ([n], E)$ be a graph on n vertices (considered as an abstract graph). A map $\psi : [n] \to \mathbb{R}^3$ is called a *3-embedding* of G into $\mathbb{R}^3$. We identify the set of all 3-embeddings of G with $\mathbb{R}^{3n}$ via $\psi \mapsto (v_1, \ldots, v_n) = (\psi(1), \ldots, \psi(n)) \in \mathbb{R}^3 \times \ldots \times \mathbb{R}^3 \cong \mathbb{R}^{3n}$. Each 3-embedding ψ of G defines a 3-dimensional framework whose joints are $\psi(1), \ldots, \psi(n)$ and whose bars are the elements of E. A 3-embedding of G into $\mathbb{R}^3$ is *infinitesimally rigid (flexible,* resp.) if the corresponding framework is infinitesimally rigid (flexible, resp.).

   Prove the following result due to Gluck (1974): the set of infinitesimally rigid 3-embeddings of the graph of a 3-dimensional simplicial polytope with n vertices is open and dense in $\mathbb{R}^{3n}$.

   Directions: Show first that a 3-embedding ψ (whose joints affinely span $\mathbb{R}^3$) is infinitesimally flexible if and only if the stress matrix $\mathcal{R}(\psi)$ of the corresponding framework satisfies the following polynomial equation: $\sum_B \det(B)^2 = 0$, where the sum is over all $(3n-6)\times(3n-6)$ submatrices of A. (**Hint:** How many edges does G have? What can you say about the rank of A?) Conclude that the set of infinitesimally flexible 3-embeddings of G is an algebraic variety, and hence its complement in $\mathbb{R}^{3n}$ is either empty or is open and dense. Why is the complement non-empty?