The problems below rely on the following definition: for a simplicial complex $\Delta$ and a vertex $v$ of $\Delta$, define the **star** of $v$ in $\Delta$ and the **link** of $v$ in $\Delta$ by

$$st_\Delta(v) = \{ F \in \Delta \mid F \cup \{v\} \in \Delta \} \quad \text{and} \quad lk_\Delta(v) = \{ F - \{v\} \mid v \in F \in \Delta \}$$

(In other words, if $\Delta$ is pure, then the facets of $st_\Delta(v)$ are precisely the facets of $\Delta$ that contain $v$.)

1. Show that if a simplicial complex $\Delta$ is shellable, then $st_\Delta(v)$ and $lk_\Delta(v)$ are also shellable.

2. Recall that for a simplicial $(d-1)$-dimensional complex $\Gamma$, the $f$- and $h$-polynomials of $\Gamma$ are defined by

$$f(\Gamma, x) := \sum_{i=0}^{d} f_{i-1}(\Gamma) \cdot x^i \quad \text{and} \quad h(\Gamma, \lambda) := \sum_{i=0}^{d} h_i(\Gamma) \cdot \lambda^i.$$

(a) Let $\Gamma$ and $\Delta$ be simplicial complexes on disjoint vertex sets $V$ and $W$. The **join** of $\Gamma$ and $\Delta$ is the simplicial complex $\Gamma \ast \Delta$ on the vertex set $V \cup W$ whose faces are all sets of the form $F \cup G$ where $F \in \Gamma$ and $G \in \Delta$. Show that

$$f(\Gamma \ast \Delta, x) = f(\Gamma, x) \cdot f(\Delta, x) \quad \text{and} \quad h(\Gamma \ast \Delta, x) = h(\Gamma, \lambda) \cdot h(\Delta, x).$$

(b) Let $\Delta$ be a pure simplicial complex of dimension $d-1$ and $v$ a vertex of $\Delta$. Prove that

$$h(\Delta, x) = \begin{cases} h(\Delta - v, x) + x \cdot h(\Delta(v), x) & \text{if } \dim(\Delta) = \dim(\Delta - v) \\ h(\Delta - v, x) & \text{otherwise} \end{cases}.$$

(Here, $\Delta - v := \{ F \in \Delta \mid v \notin F \}$ is the subcomplex of $\Delta$ consisting of all faces of $\Delta$ that do not contain $v$.)

**Hint:** Recall that the $h$-numbers of a $(d-1)$-dimensional simplicial complex are defined by $\sum h_i(\Delta)x^{d-i} = \sum f_{i-1}(\Delta)(x-1)^{d-i}$, which by Problem #3b) of HW3 is equivalent to $h(\Delta, x) = (1-x)^d f(\Delta, \frac{x}{1-x}).$

3. Let $\Delta$ be a pure $(d-1)$-dimensional simplicial complex with the vertex set $V$. Prove that

$$\sum_{v \in V} h_{j-1}(\Delta(v)) = j \cdot h_j(\Delta) + (d+1-j) \cdot h_{j-1}(\Delta) \quad \text{for all } 1 \leq j \leq d.$$

**Hint:** express $\sum_{v \in V} f_{i-1}(\Delta(v))$ in terms of the $f$-numbers of $\Delta$.

(Note that we saw a dual version of this formula for the case of simple polytopes when discussing the proof of the Upper Bound Theorem.)

4. A simplicial complex $\Delta$ on the vertex set $[n]$ is called **shifted** if it satisfies the following property: for every $F \in \Delta$, $i \in F$, and $1 \leq j < i$, the set $(F \setminus \{i\}) \cup \{j\}$ is also a face of $\Delta$. Prove that a shifted simplicial complex is shellable if and only if it is pure.

**Hint:** the rev-lex order might be of help.