Polytopes **Homework #5** **due Friday, 5/6/16**

The problems below rely on the following definition: for a simplicial complex Δ and a vertex v of Δ, define the **star** of v in Δ and the **link** of v in Δ by

$$\mathrm{st}_\Delta(v) = \{F \in \Delta \mid F \cup \{v\} \in \Delta\} \quad \text{and} \quad \mathrm{lk}_\Delta(v) = \{F - \{v\} \mid v \in F \in \Delta\}, \text{ resp.}$$

(In other words, if Δ is pure, then the facets of $\mathrm{st}_\Delta(v)$ are precisely the facets of Δ that contain v.)

1. Show that if a simplicial complex Δ is shellable, then $\mathrm{st}_\Delta(v)$ and $\mathrm{lk}_\Delta(v)$ are also shellable.

2. Recall that for a simplicial $(d-1)$-dimensional complex Γ, the f- and h-polynomials of Γ are defined by $f(\Gamma, x) := \sum_{i=0}^{d} f_{i-1}(\Gamma) \cdot x^i$ and $h(\Gamma, \lambda) := \sum_{i=0}^{d} h_i(\Gamma) \cdot x^i$.

   (a) Let Γ and Δ be simplicial complexes on disjoint vertex sets V and W. The *join* of Γ and Δ is the simplicial complex $\Gamma * \Delta$ on the vertex set $V \cup W$ whose faces are all sets of the form $F \cup G$ where $F \in \Gamma$ and $G \in \Delta$. Show that

   $$f(\Gamma * \Delta, x) = f(\Gamma, x) \cdot f(\Delta, x) \quad \text{and} \quad h(\Gamma * \Delta, x) = h(\Gamma, \lambda) \cdot h(\Delta, x).$$

   (b) Let Δ be a *pure* simplicial complex of dimension $d-1$ and v a vertex of Δ. Prove that

   $$h(\Delta, x) = \begin{cases} h(\Delta - v, x) + x \cdot h(\mathrm{lk}_\Delta(v), x) & \text{if } \dim(\Delta) = \dim(\Delta - v) \\ h(\Delta - v, x) & \text{otherwise.} \end{cases}$$

   (Here, $\Delta - v := \{F \in \Delta \mid v \notin F\}$ is the subcomplex of Δ consisting of all faces of Δ that do not contain v.)

   Hint: Recall that the h-numbers of a $(d-1)$-dimensional simplicial complex are defined by $\sum h_i(\Delta) x^{d-i} = \sum f_{i-1}(\Delta)(x-1)^{d-i}$, which by Problem #3b) of HW3 is equivalent to $h(\Delta, x) = (1-x)^d f\big(\Delta, \frac{x}{1-x}\big)$.

3. Let Δ be a pure $(d-1)$-dimensional simplicial complex with the vertex set V. Prove that

   $$\sum_{v \in V} h_{j-1}(\mathrm{lk}_\Delta(v)) = j \cdot h_j(\Delta) + (d+1-j) \cdot h_{j-1}(\Delta) \quad \text{for all } 1 \le j \le d.$$

   Hint: express $\sum_{v \in V} f_{i-1}(\mathrm{lk}_\Delta(v))$ in terms of the f-numbers of Δ.

   (Note that we saw a dual version of this formula for the case of simple polytopes when discussing the proof of the Upper Bound Theorem.)

4. A simplicial complex Δ on the vertex set $[n]$ is called *shifted* if it satisfies the following property: for every $F \in \Delta$, $i \in F$, and $1 \le j < i$, the set $(F \setminus \{i\}) \cup \{j\}$ is also a face of Δ. Prove that a shifted simplicial complex is shellable if and only if it is pure.
   Hint: the rev-lex order might be of help.