Polytopes **Homework #4** **due Friday, 4/29/16**

1. Let P be the d-dimensional cube. Show that if $\mathcal{O}$ is an acyclic orientation of $G(P)$ that has more than one sink then there is a 2-dimensional face of P on which $\mathcal{O}$ induces more than one sink.

   (In other words, to check whether $\mathcal{O}$ is Kalai's "good orientation" on $G(P)$, it is enough to inspect 2-dimensional faces only! This result, in fact, holds for ALL simple polytopes, not just cubes, and in this generality is due to Joswig, Kaibel, and Körner.)

   Hint: by induction on d one can assume that the two global sinks are antipodal vertices t_1 and t_2 of P. Consider a sink of the subgraph $G(A) \subset G(P)$ induced by $A := \text{vert}\,(P) - \{t_1, t_2\}$.

2. Show that a subset $\mathcal{C}$ of facets of the d-dimensional cube C_d determines a shellable subcomplex of ∂C_d (and, in fact, a beginning segment of a shelling of ∂C_d) if and only if (i) $\mathcal{C}$ contains all facets of C_d or no facets (i.e., $\mathcal{C}$ is empty), or (ii) $\mathcal{C}$ contains at least one facet such that the opposite facet is not in $\mathcal{C}$.

   Hint: induct on d. (Note: a facet of a cube is a cube!) In the case when $\mathcal{C}$ satisfies the above condition, produce a shelling of ∂C_d in which the facets of $\mathcal{C}$ come first.

3. For a simple d-polytope $P \in \mathbb{R}^d$ with n vertices, a numbering of the vertices of P by $1, 2, \ldots, n$ is called *completely unimodal* if every k-face ($2 \leq k \leq d$) of P has a unique local minimum. (In other words, a numbering of vertices is completely unimodal iff orienting the edges from a larger vertex to a smaller one defines Kalai's "good orientation" on $G(P)$.)

   Show that completely unimodal numberings of a simple polytope P exactly correspond to the shelling orders of the (simplicial) polar polytope P^*.

   Hint: By dualizing the "shelling condition" for simplicial complexes, observe that $\widehat{v_1}, \ldots, \widehat{v_n}$ is a shelling of P^* iff for every $2 \leq j \leq n$ and $1 \leq i < j$

   > if v_i and v_j share a common face F of P, then there exists $l < j$ such that v_j and v_l are connected by an edge in F.