Polytopes **Homework #3** **due Friday, 4/22/16**

1. Show that if P is a 3-dimensional simple polytope, then $f_1(P) = \frac{3}{2}f_0(P)$ and $f_2(P) = 2 + \frac{1}{2}f_0(P)$.

2. (a) Show that for a d-dimensional simplex Σ, $h_i(\Sigma) = 1$ for all $i = 0, 1, \ldots, d$.

   (b) Show that for the d-dimensional cube $I = \{(x_1, \ldots, x_d) \mid 0 \le x_i \le 1\}$, $h_i(I) = \binom{d}{i}$ for all $i = 0, 1, \ldots, d$.

3. Let P be a d-dimensional simplicial polytope. Define $h_i(P) := h_i(P^*)$ for $i = 0, 1, \ldots, d$, and consider two polynomials

$$f(P, x) = \sum_{j=0}^{d} f_{j-1}(P) \cdot x^j \quad \text{and} \quad h(P, x) = \sum_{i=0}^{d} h_i(P) \cdot x^i.$$

   (Recall that $f_{-1}(P) = 1$.)

   (a) Prove that

$$\sum_{i=0}^{d} h_i(P) x^{d-i} = \sum_{j=0}^{d} f_{j-1}(P)(x-1)^{d-j}.$$

   (Later on we'll use this equation to define the h-numbers of $(d-1)$-dimensional simplicial complexes.) Conclude that $f_{j-1}(P) = \sum_{i=0}^{j} \binom{d-i}{d-j} h_i(P)$ for $j = 0, \ldots, d$ and $h_i(P) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}(P)$ for $i = 0, \ldots, d$.

   (b) Verify that $h(P, x) = x^d h(P, \frac{1}{x}) = (1-x)^d f(P, \frac{x}{1-x})$.

   (c) Show that $h_i(C(d, n)) = \binom{n-d-1+i}{i}$ for all $0 \le i \le \lfloor d/2 \rfloor$. (Here $C(d, n)$ is the cyclic polytope.)
   Hint: Use the fact that the polynomials $f(C(d, n), x)$ and $(1 + x)^n$ agree in terms of degree $\le d/2$, and hence the same holds for polynomials $h(C(d, n), x)$ and $(1 - x)^d (1 + \frac{x}{1-x})^n$.

4. (a) Let P be a d-dimensional simplicial polytope and let Q be a simplicial polytope obtained by building a (shallow) pyramid over a single facet of P. Show that $h_i(Q) = h_i(P) + 1$ for all $i = 1, \ldots, d-1$ (and of course, $h_0(P) = h_d(P) = h_0(Q) = h_d(Q) = 1$).
   Hint: What is the change in the f-numbers?

   (b) A simplicial d-dimensional polytope is called **stacked** if it can be obtained from a d-dimensional simplex by repeatedly building shallow pyramids over facets. What is the h-vector of a stacked polytope with n vertices?