1. Show that if $P$ is a 3-dimensional simple polytope, then $f_1(P) = \frac{3}{2} f_0(P)$ and $f_2(P) = 2 + \frac{1}{2} f_0(P)$.

2. (a) Show that for a $d$-dimensional simplex $\Sigma$, $h_i(\Sigma) = 1$ for all $i = 0, 1, \ldots, d$.

   (b) Show that for the $d$-dimensional cube $I = \{(x_1, \ldots, x_d) \mid 0 \leq x_i \leq 1\}$, $h_i(I) = \binom{d}{i}$ for all $i = 0, 1, \ldots, d$.

3. Let $P$ be a $d$-dimensional simplicial polytope. Define $h_i(P) := h_i(P^*)$ for $i = 0, 1, \ldots, d$, and consider two polynomials
   \[ f(P, x) = \sum_{j=0}^{d} f_{j-1}(P) \cdot x^j \quad \text{and} \quad h(P, x) = \sum_{i=0}^{d} h_i(P) \cdot x^i. \]

   (Recall that $f_{-1}(P) = 1$.)

   (a) Prove that
   \[ \sum_{i=0}^{d} h_i(P) x^{d-i} = \sum_{j=0}^{d} f_{j-1}(P) (x-1)^{d-j}. \]

   (Later on we’ll use this equation to define the $h$-numbers of $(d-1)$-dimensional simplicial complexes.) Conclude that $f_{j-1}(P) = \sum_{i=0}^{j} \binom{d-i}{d-j} h_i(P)$ for $j = 0, \ldots, d$ and $h_i(P) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}(P)$ for $i = 0, \ldots, d$.

   (b) Verify that $h(P, x) = x^d h(P, \frac{1}{x}) = (1-x)^d f(P, \frac{x}{1-x})$.

   (c) Show that $h_i(C(d, n)) = \binom{n-d-1+i}{d-i}$ for all $0 \leq i \leq \lfloor d/2 \rfloor$. (Here $C(d, n)$ is the cyclic polytope.)

   **Hint:** Use the fact that the polynomials $f(C(d, n), x)$ and $(1 + x)^n$ agree in terms of degree $\leq d/2$, and hence the same holds for polynomials $h(C(d, n), x)$ and $(1 - x)^d (1 + \frac{x}{1-x})^n$.

4. (a) Let $P$ be a $d$-dimensional simplicial polytope and let $Q$ be a simplicial polytope obtained by building a (shallow) pyramid over a single facet of $P$. Show that $h_i(Q) = h_i(P) + 1$ for all $i = 1, \ldots, d-1$ (and of course, $h_0(Q) = h_d(P) = h_0(Q) = h_d(Q) = 1$).

   **Hint:** What is the change in the $f$-numbers?

   (b) A simplicial $d$-dimensional polytope is called **stacked** if it can be obtained from a $d$-dimensional simplex by repeatedly building shallow pyramids over facets. What is the $h$-vector of a stacked polytope with $n$ vertices?