1. Prove the following application of Radon’s theorem to combinatorics: Let $F_1, \ldots, F_m$ be subsets of $[n] := \{1, 2, \ldots, n\}$, where $m \geq n + 2$. Then there exist two nonempty disjoint sets $I \subset [m]$ and $J \subset [m]$ such that $\cup_{i \in I} F_i = \cup_{j \in J} F_j$.

**Hint:** Associate with each set $F_i$ its incidence vector $v_i \in \mathbb{R}^n$. Use Radon’s theorem.

2. Prove Carathéodory’s theorem: the convex hull $\text{conv} (M)$ of a set $M \subseteq \mathbb{R}^d$ is the union of all convex hulls of subsets of $M$ containing at most $d + 1$ elements.

**Hint:** Assume $x = \lambda_1 x_1 + \cdots + \lambda_r x_r \in \text{conv} (M)$, where $r$ is as small as possible. If $r \geq d + 2$, then use the same ideas as in the proof of Radon’s theorem to express $x$ as a convex combination of a proper subset of $\{x_1, \ldots, x_r\}$.

3. Prove Helly’s theorem: if $m \geq d + 1$ and every $d + 1$ of the convex sets $K_1, \ldots, K_m$ in $\mathbb{R}^d$ have a nonempty intersection, then $\cap_{i=1}^m K_i \neq \emptyset$.

**Hint:** for $m = d + 1$ there is nothing to prove. Apply induction on $m$ and use Radon’s theorem.

4. Show that the convex hull of a compact set is compact.

**Hint:** One possible solution is to use Carathéodory’s theorem to express $\text{conv} (K)$ as a continuous image of a compact set.

5. Show that if $C = \text{conv} (X) \subset \mathbb{R}^d$, then $C^* = \cap_{x \in X} D_0(x)^\circ$.

6. Show that for a convex set $C$, $C^*$ is bounded if and only if $0 \in \text{int} (C)$.

**Hint:** Separation theorem.