Polytopes **Homework #1** **due Friday, 4/8/16**

1. Prove the following application of Radon's theorem to combinatorics: Let $F_1, \ldots, F_m$ be subsets of $[n] := \{1, 2, \ldots, n\}$, where $m \geq n + 2$. Then there exist two nonempty disjoint sets $I \subset [m]$ and $J \subset [m]$ such that $\cup_{i \in I} F_i = \cup_{j \in J} F_j$.
   Hint: Associate with each set F_i its incidence vector $v_i \in \mathbb{R}^n$. Use Radon's theorem.

2. Prove Carathéodory's theorem: the convex hull $\operatorname{conv}(M)$ of a set $M \subseteq \mathbb{R}^d$ is the union of all convex hulls of subsets of M containing at most $d+1$ elements. (**Hint:** Assume $x = \lambda_1 x_1 + \cdots + \cdots + \lambda_r x_r \in \operatorname{conv}(M)$, where r is as small as possible. If $r \geq d+2$, then use the same ideas as in the proof of Radon's theorem to express x as a convex combination of a proper subset of $\{x_1, \ldots, x_r\}$.)

3. Prove Helly's theorem: if $m \geq d+1$ and every $d+1$ of the convex sets $K_1, \ldots, K_m$ in $\mathbb{R}^d$ have a nonempty intersection, then $\cap_{i=1}^m K_i \neq \emptyset$.
   (**Hint:** for $m = d+1$ there is nothing to prove. Apply induction on m and use Radon's theorem.)

4. Show that the convex hull of a compact set is compact.
   Hint: One possible solution is to use Carathéodory's theorem to express $\operatorname{conv}(K)$ as a continuous image of a compact set.

5. Show that if $C = \operatorname{conv}(X) \subset \mathbb{R}^d$, then $C^* = \cap_{x \in X} D_0(x)^-$.

6. Show that for a convex set C, C^* is bounded if and only if $0 \in \operatorname{int}(C)$.
   (**Hint:** Separation theorem.)