1. State whether each of the following is TRUE or FALSE. Justify your answer by giving a proof or a counterexample.
   (a) If a convergent sequence is bounded, then it is monotone.
   (b) If $\{a_n\}$ and $\{a_n b_n\}$ are convergent sequences, then $\{b_n\}$ is convergent.
   (c) If $\lim\limits_{k\to\infty} a_k = 0$, then $\displaystyle\sum_{k=1}^{\infty}(-1)^k a_k$ converges.
   (d) If the functions $g : \mathbb{R} \to \mathbb{R}$ and $f + g : \mathbb{R} \to \mathbb{R}$ are continuous, then so is the function $f : \mathbb{R} \to \mathbb{R}$.

2. (a) Find the limit of the sequence $\sqrt{n+1} - \sqrt{n}$ or show that it does not exist. Prove that you are correct. (You can use limit properties.)
   (b) Suppose that $\{a_n\}$ is a convergent sequence, and let the sequence $\{b_n\}$ be defined by $b_n = a_{n+10000} - a_n$ (for each $n \in \mathbb{N}$). Prove that $\{b_n\}$ converges to 0.

3. Determine whether the series converges absolutely, converges conditionally or diverges. Justify your answer.
   (a) $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{2k^2+1}{4k^2-1}$
   (b) $\displaystyle\sum_{k=1}^{\infty}\frac{1+(-1)^k}{k^2+5}$
   (c) $\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1} \frac{k^2+1}{k^3}$

4. For which **integers** a does the series $\displaystyle\sum_{k=1}^{\infty}\frac{k^k}{k!}a^k$ converge? For which **integers** a does it diverge?

5. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} \frac{x^2-5}{x+3} & \text{if } x \neq 2 \\ 0 & \text{if } x = 2. \end{cases}$ Determine $\lim\limits_{x\to 2} f(x)$ and use the sequence denition of a limit to prove that your response is correct. (You can also use limit properties.)

6. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x+1 & \text{if } x \leq 1 \\ 3-2x^2 & \text{if } x > 1. \end{cases}$ Determine all the values of x_0 at which f is continuous. Prove that you are correct. (You can use the sequence or the ϵ-δ definition of continuity. You can also use limit properties.)

7. For each $n \in \mathbb{N}$, define $f_n : [0, \infty) \to \mathbb{R}$ by $f_n(x) = \frac{nx}{1+nx}$.
   (a) Find the function f to which $\{f_n\}$ converges pointwise on $[0, \infty)$.
   (b) Show that if $r > 0$, the convergence is uniform on $[r, \infty)$.
   (c) Show that the convergence is *not* uniform on $(0, \infty)$.

Unless asked specifically for its proof, you may use without proof any of the following results on the final. You should prove any other result that you wish to use.

• All of the axioms and elementary properties of real numbers, and elementary properties of absolute value

• The Triangle Inequality, the Archimedean Property, and the Rational and Irrational Density Theorems

• The Binomial Formula, the Difference of Powers Formula, and the Geometric Sum Formula (for finite sums and convergent geometric series)

• The Comparison Lemma for Convergence of sequences; the Sum, Product, Quotient, and Power Properties for Limits; the Sandwich Theorem.

• The Monotone Convergence Theorem, the Nested Interval Theorem, the Bolzano–Weierstrass Theorem, the Cauchy Criterion for convergence of sequences

• **The Values of the Following Limits:**

- $\lim_{n\to\infty} \frac{1}{n}$
- $\lim_{n\to\infty} r^n$
- $\lim_{n\to\infty} n^{1/n}$ and $\lim_{n\to\infty} x^{1/n}$ for $x > 0$
- $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$

• **Tests for Series Convergence:**

- The Term Test
- The Comparison Test
- The Ratio Test
- The Root Test
- The Alternating Series Test

• **The Convergence/Divergence of Particular Series:**

- The Harmonic Series and Alternating Harmonic Series
- Geometric Series
- p-series

• Continuity of Sums, Products, Quotients, and Compositions of Continuous Functions

• The Comparison Lemma for Uniform Convergence

• **The following Theorems:**

1. $\{x_n\}$ converges to 0 of and only if $\{|x_n|\}$ converges to 0.
2. If $\{x_n\}$ converges to x and $a < x$, then there is an $N \in \mathbb{N}$ such that, if $n \geq N$, then $a < x_n$. If $\{x_n\}$ converges to x and $x < b$, then there is an $N \in \mathbb{N}$ such that, if $n \geq N$, then $x_n < b$.
3. If a sequence converges, then it is bounded.
4. If $\sum a_k$ converges absolutely, then $\sum a_k$ converges.
5. $\lim_{n\to\infty} a_n = \infty$ if and only if $\lim_{n\to\infty} \frac{1}{a_n} = 0$.
6. If $\{f_n\}$ converges uniformly to f on D, then $\{f_n\}$ converges pointwise to f on D.
7. The uniform limit of a sequence of continuous functions is continuous.
8. If $\{f_n\}$ converges uniformly to f on D and $\{x_n\}$ is a sequence in D, then
$$\lim_{n\to\infty} [f_n(x_n) - f(x_n)] = 0.$$