Reading assignment: Read carefully the lecture notes. In addition, read §3.1, §3.5 (pages 70-72 only, through the end of the proof of Theorem 3.20), and §3.7; glance at Sections 9.2 and 9.3.

Written assignment: Solve the following problems. Please don’t forget to follow the Homework Guidelines (see our web-page).

1. Determine whether the statement is true or false. Justify your answer with a proof or a counterexample.
   (a) If \( f + g : \mathbb{R} \to \mathbb{R} \) is continuous, then \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) are continuous.
   (b) If \( f^2 : \mathbb{R} \to \mathbb{R} \) is continuous, then \( f : \mathbb{R} \to \mathbb{R} \) is continuous.
   (c) If \( f + g : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) are continuous, then \( f : \mathbb{R} \to \mathbb{R} \) is continuous.

2. (a) Define \( f : \mathbb{R} \to \mathbb{R} \) by
   \[
   f(x) = |x| = \begin{cases} 
   x & \text{if } x \geq 0 \\
   -x & \text{if } x < 0.
   \end{cases}
   
   Use the sequence definition of continuity to show that \( f \) is continuous on \( \mathbb{R} \).

   (b) Define \( f : [0, 5] \to \mathbb{R} \) by
   \[
   f(x) = \begin{cases} 
   7 & \text{if } 0 \leq x \leq 4 \\
   3x & \text{if } 4 < x \leq 5.
   \end{cases}
   
   At what points is the function \( f \) continuous? Use the sequence definition of continuity to justify your answer.

3. Define \( f : \mathbb{R} \to \mathbb{R} \) by
   \[
   f(x) = \begin{cases} 
   x & \text{if } x \leq 1 \\
   4 - 2x & \text{if } x > 1.
   \end{cases}
   
   Use the \( \varepsilon-\delta \) definition of continuity to show that \( f \) is continuous at every \( x_0 \in \mathbb{R} \) except \( x_0 = 1 \).

4. We say that a function \( f : \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous if there exists a non-negative constant \( C \) such that, for every \( x_1, x_2 \in \mathbb{R} \),
   \[
   |f(x_1) - f(x_2)| \leq C|x_1 - x_2|.
   
   Use the \( \varepsilon-\delta \) definition of continuity to show that if \( f \) is Lipschitz continuous, then \( f \) is continuous on \( \mathbb{R} \).

5. Define \( f : \mathbb{R} \to \mathbb{R} \) by
   \[
   f(x) = \begin{cases} 
   3x - 6 & \text{if } x \neq 0 \\
   1 & \text{if } x = 0.
   \end{cases}
   
   Determine \( \lim_{x \to 0} f(x) \) and use the sequence definition of a limit to prove that your response is correct.

6. Prove that \( \lim_{x \to 0} \frac{|x|}{x} \neq 1 \), \( \lim_{x \to 0} \frac{|x|}{x} \neq -1 \), and \( \lim_{x \to 0} \frac{|x|}{x} \neq 0. \)