Reading assignment: Read carefully the lecture notes. In addition, (i) read pages 230-235, through the end of Corollary 9.13 (the p-Test), (ii) look at Example 2.27 on pages 39–40 of the proof of the divergence of the Harmonic Series, and glance at pages 236-239.

Written assignment: Solve the following problems. Please don’t forget to follow the Homework Guidelines (see our web-page).

1. Determine whether each of the following series converges or diverges. If the series converges, give its sum. Give complete and careful proofs of your answers.

(a) \[ \sum_{k=1}^{\infty} \left( \frac{3}{4} \right)^k \]

(b) \[ \sum_{k=2}^{\infty} \frac{k-3}{4k+5} \]

(c) \[ \sum_{k=2}^{\infty} \frac{4k+1}{5k} \]

(d) \[ \sum_{k=1}^{\infty} a_k \], where \( a_k = \begin{cases} k^2 & \text{if } 1 \leq k \leq 5 \\ 0 & \text{if } k > 5 \end{cases} \)

(Be clear about the sequence of partial sums.)

(e) \[ \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \]

HINT: Note that \( \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \).

2. Let \( \{a_k\} \) and \( \{b_k\} \) be sequences of real numbers. In each of the following, your proof should contain no ellipses [\ldots\].

(a) Let \( c \in \mathbb{R} \). Use the Distributive Axiom and induction on \( n \) to prove that, for every \( n \in \mathbb{N} \), \( \sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k \).

(b) Use the Commutative and Associative Axioms and induction on \( n \) to prove that, for every \( n \in \mathbb{N} \), \( \sum_{k=1}^{n} (a_k + b_k) = \left( \sum_{k=1}^{n} a_k \right) + \left( \sum_{k=1}^{n} b_k \right) \).

(c) Suppose that \( c \) and \( d \) are real numbers and that \( \sum_{k=1}^{\infty} a_k \) and \( \sum_{k=1}^{\infty} b_k \) both converge. Give a careful proof (using parts (a) and (b), partial sums, and Limit Properties) that \( \sum_{k=1}^{\infty} (ca_k + db_k) = c \sum_{k=1}^{\infty} a_k + d \sum_{k=1}^{\infty} b_k \).

3. Determine whether each statement is true or false. Justify your answer.

(a) If \( \sum_{k=1}^{\infty} a_k \) converges, then \( \sum_{k=1}^{\infty} \frac{1}{1 + a_k} \) must diverge.

(b) If \( a_k \geq 0 \) for all \( k \in \mathbb{N} \) and \( \sum_{k=1}^{\infty} a_k \) converges, then \( \sum_{k=1}^{\infty} \frac{a_k}{1 + a_k} \) must converge.
(c) If \( \sum_{k=1}^{\infty} a_k^2 \) converges, then \( \sum_{k=1}^{\infty} a_k \) must converge.

(d) If \( 0 \leq a_k \leq b_k \) for all \( k \in \mathbb{N} \) and \( \sum_{k=1}^{\infty} b_k \) diverges, then \( \sum_{k=1}^{\infty} a_k \) must also diverge.

4. Determine whether each series converges or diverges. Justify your answer.

(a) \( \sum_{k=1}^{\infty} \frac{1}{3^k + k - 1} \)

(b) \( \sum_{k=1}^{\infty} \frac{k}{k^2 - k + 2} \)

5. In class, we proved that the Geometric Series \( \sum r^k \) converges for \( |r| < 1 \) and diverges otherwise. This proof depended on the convergence or divergence of the sequence \( \{r^n\} \). We dealt with the cases in which \( |r| < 1, r = 1, \) and \( r = -1 \). The following problem will take care of the remaining cases: \( r > 1 \) and \( r < -1 \).

**Definition:** Let \( \{a_n\} \) be a sequence of real numbers. We say that \( \lim_{n \to \infty} a_n = \infty \) if, for every \( M > 0 \), there is an \( N \in \mathbb{N} \) such that, if \( n \geq N \), then \( a_n > M \).

(a) Suppose \( \{a_n\} \) is a sequence with \( a_n > 0 \) for all \( n \in \mathbb{N} \). Prove that \( \lim_{n \to \infty} a_n = \infty \) if and only if \( \lim_{n \to \infty} \frac{1}{a_n} = 0 \).

(b) We’ve proved that, if \( |c| < 1 \), then \( \lim_{n \to \infty} c^n = 0 \). Use this fact and part (a) to prove that if \( r > 1 \), then \( \lim_{n \to \infty} r^n = \infty \).

(c) In each of the following, suppose \( r < -1 \).

i. Prove that \( r^n \) does not converge to any real number.

**Hint:** assume by contradiction that \( \{r^n\} \) converges to \( m \). What can we then conclude about \( \{|r|^n\} \)? Is this possible in view of part (b)?

ii. Prove that \( \lim_{n \to \infty} r^n \neq \infty \).

iii. Modify the definition above for \( \lim_{n \to \infty} a_n = \infty \) to give a definition for the statement \( \lim_{n \to \infty} a_n = -\infty \). Then prove that \( \lim_{n \to \infty} r^n \neq -\infty \).

(d) Summarize the results of this exercise by giving a complete description of the behavior of the sequence \( \{r^n\} \). That is, list the values of \( r \) for which the sequence converges (include the limit of the sequence in your description) and the values of \( r \) for which the sequence diverges.