Reading assignment: Read carefully the lecture notes. In addition, (i) read the portion of §2.2 on Boundedness and Convergent Sequences, but skip everything after the heading Sequential Denseness of the Reals; (ii) read §2.3 and start reading §2.4. (You can skip Theorem 2.32 and everything after Theorem 2.33. This is the theorem we'll call the Bolzano-Weierstrass Theorem.)

Written assignment: Solve the following problems. Please don't forget to follow the Homework Guidelines (see our web-page).

1. The goal of this exercise is to prove the Root Property for Limits: let $\{a_n\}$ be a sequence of **non-negative** numbers that converges to a. Then $\{\sqrt{a_n}\}$ converges to $\sqrt{a}$. That is, prove that if $\{a_n\}$ is a convergent sequence of non-negative numbers, then $\lim \sqrt{a_n} = \sqrt{\lim a_n}$. To do so, use the following steps:

   (a) Prove that, under the hypothesis of the statement, a must be non-negative.

   (b) Prove the result in the case that $a = 0$.

   (c) Prove the result in the case that $a > 0$. The Comparison Lemma and the fact that $a_n - a = (\sqrt{a_n} - \sqrt{a})(\sqrt{a_n} + \sqrt{a})$ might be of help.

   If you want to challenge yourself, use the same steps along with the Difference of Powers Formula to show that under the hypothesis of our problem, $\{a_n^{1/m}\}$ converges to $a^{1/m}$ for every natural number m.

2. Let $a_n = n^{1/n} - 1$.

   (a) Check that for each $n \in \mathbb{N}$, $a_n \geq 0$ and $n = (a_n + 1)^n$.

   (b) Use part (a) and the Binomial Formula to prove that for every $n \in \mathbb{N}$,

   $$n \geq \frac{n(n-1)}{2} \cdot a_n^2, \quad \text{and thus,} \quad a_n \leq \sqrt{\frac{2}{n-1}} \text{ for all } n > 1.$$

   (c) Use part (b) to prove that $\lim n^{1/n} = 1$.

3. Determine whether the statement is TRUE or FALSE. Justify your answer with a proof or counterexample.

   (a) If $\{a_n\}$ converges to a and $c < a_n < d$ for all $n \in \mathbb{N}$, then $c < a < d$.

   (b) The sequence $n - 5$ converges.

   (c) If $\{a_n\}$ is bounded, then $\{a_n\}$ converges.

4. Determine whether the sequence is monotonic. Prove you are correct.

   (a) $\{n^2\}$

   (b) $\left\{\dfrac{1}{\sqrt{n+2}}\right\}$

   (c) $\left\{1 + \dfrac{(-1)^n}{n}\right\}$

   (d) $\left\{\dfrac{n}{2^n}\right\}$

5. Suppose $\{a_n\}$ is monotone. Prove that $\{a_n\}$ converges if and only if $\{a_n^2\}$ converges.