Midterm Practice Problems

1. Prove \( C \subseteq A \text{ and } C \subseteq B \implies (C \subseteq A \cap B) \)

Proof:

Let \( c \in C \). We need to show that \( c \in A \cap B \).

Since \( C \subseteq A \), if \( c \in C \) then \( c \in A \).

Since \( C \subseteq B \), if \( c \in C \) then \( c \in B \).

Since \( c \in A \) and \( c \in B \), \( c \in A \cap B \). \( \square \)

2. \( \forall x, y \in \mathbb{R}, |x+y| \leq |x| + |y| \)

Proof #1 (by cases)

Case 1: Both \( x \) and \( y \) are non-negative, so \( |x| = x, |y| = y \)

Since \( x \geq 0 \) and \( y \geq 0 \) \( \implies x + y \geq 0 \)

So \( |x+y| = x+y = |x| + |y| \).

Case 2: Both \( x \) and \( y \) are negative, so \( |x| = -x, |y| = -y \)

Since \( x < 0 \) and \( y < 0 \) \( \implies x + y < 0 \)

So \( |x+y| = -(x+y) = -x-y = |x| + |y| \).

Case 3: One is non-negative and one is non-positive

The proof is the same in both cases, so we may assume \( x \geq 0, y \leq 0 \). Note that \( |x| = x \) and \( |y| = -y \).

Sub-case 3a): \( x \geq y \), so \( x + y \geq 0 \).

Since \( y \leq 0 \), adding \( x \) to both sides we get \( x+y \leq x \)

Since \( x+y \geq 0 \), \( |x+y| = x+y \leq x \leq |x| + |y| \).

Sub-case 3b): \( x \leq -y \), so \( x + y \leq 0 \), hence \( |x+y| = -(x+y) \).

Since \( x \geq 0 \), \( -y \leq 0 \) so \( -y \leq |y| \)

\( \implies -x-y \leq |x|-y = |x| + |y| \) (using \( y \leq 0 \Rightarrow |y| = -y \))
Hence $|x+y| = -(x+y) = -x - y \leq 1|x| + 1|y|$.  

We proved that in all possible cases $|x+y| \leq |x|+|y|$, $\forall x, y \in \mathbb{R}$.  \[\text{QED.}\]

**Proof #2 (Shorter)**

$\forall a \in \mathbb{R}$, $a \leq |a|$ since $a = |a|$ if $a \geq 0$ and $a < 0$ while $|a| > 0$, if $a < 0$.

So, $\forall x, y \in \mathbb{R}$ it is true that

$2xy \leq 2|xy|$

Adding $x^2 = 1x^2$ and $y^2 = 1y^2$ to both sides:

$x^2 + 2xy + y^2 \leq 1x^2 + 2|x|y + 1y^2$

i.e.

$(x+y)^2 \leq (|x| + |y|)^2$

Taking square root of both sides:

$|x+y| \leq |x| + |y|$

Since $1x| + 1y| > 0$.

Hence $|x+y| \leq 1|x| + 1|y|$.  \[\text{QED.}\]

3. a) One cannot prove a universal statement ("for any") by showing a specific example.

The given argument shows that the statement is true for the case when $A = \{a, b, c, d, e\}$ and $B = \{d, e, f, g, h\}$ only. It needs to be written for any set $A$ and any set $B$.

b) If $C = \emptyset$, then $\emptyset \subseteq A$ so the result holds.

If $C \neq \emptyset$, let $e \in C$. We need to show that
\[ c \in C \Rightarrow c \in A. \]
Since \( C \subseteq A \cup B \), \( c \in C \Rightarrow c \in A \cup B \)
\[ \Rightarrow c \in A \text{ or } c \in B. \]
Since \( B \cap C = \emptyset \), \( c \in B \) (or else \( c \in C \) and \( c \in B \)
\[ \Rightarrow c \in B \cap C \Rightarrow B \cap C \neq \emptyset \) (contradiction).
Hence \( c \in A. \)
\[ \quad \Box \]

\[ \forall n \geq 0, \ 13 \mid 4^{2n+1} + 3^{n+2}. \]

Proof: We'll prove this by induction on \( n \)

Base Case: \( n = 0 \):
\[ 4^{2(0)+1} + 3^{(0)+2} = 4 + 9 = 13 \text{ and } 13 \mid 13. \]
so the result holds true for \( n = 0 \).

Inductive step: Suppose the result is true for some \( k \geq 0 \). That is:
\[ j \in \mathbb{Z}, k \in \mathbb{Z} \text{ such that } 4^{2k+3} + 3^{k+3} = 13j. \]
We need to show that
\[ 13 \mid 4^{2(k+1)+1} + 3^{(k+1)+2}. \]
\[ 4^{2k+3} + 3^{k+3} = \left(4^{2k+1}\right)(4^2) + 3^{k+2}(3) \]
\[ = (4^{2k+1})(13 + 3) + 3^{k+2}(3) \]
\[ = 13 \cdot 4^{2k+1} + 3 \cdot 4^{2k+1} + 3 \cdot 3^{k+2} \]
\[ = 13 \cdot 4^{2k+1} + 3 \left( 4^{2k+1} + 3^{k+2} \right) \]
\[ = 13 \cdot 4^{2k+1} + 3 \cdot 13 j \text{ by induction hypothesis} \]
\[ = 13 \left( 4^{2k+1} + 3j \right). \]
Since \( 2k+1 \geq 0 \), \( 4^{2k+1} + 3j \in \mathbb{Z} \) so \( 13 \mid 4^{2k+3} + 3^{k+3}. \)

Hence, by induction, \( 13 \mid 4^{2n+1} + 3^{n+2} \) \( \forall n \geq 0. \)
\[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}
\]

**Base Case:** \( n = 1 \): \( \sum_{i=1}^{1} i^2 = 1^2 = \frac{1(1+1)(2\cdot1+1)}{6} = 1 \).

So the result holds for \( n = 1 \).

**Inductive Step:**

Suppose \( \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6} \) for some \( k \in \mathbb{Z}^+ \).

Then \( \sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 \)

\[
= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{by inductive hypothesis}
\]

\[
= \frac{k(k+1)(2k+1) + (k+1)^2 \cdot 6}{6}
\]

\[
= \frac{(k+1) \left[ k(2k+1) + 6(k+1) \right]}{6}
\]

\[
= \frac{(k+1) \left[ 2k^2 + 7k + 6 \right]}{6}
\]

\[
= \frac{(k+1)(k+2)(2k+3)}{6}
\]

(Since \( (k+2)(2k+3) = 2k^2 + 3k + 4k + 6 = 2k^2 + 7k + 6 \)).

This shows that the result holds for \( k+1 \).

Hence, by induction \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \) for \( n \geq 1 \).
6. \( \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \quad [(x \leq y) \Rightarrow (x^2 \leq y^2)] \)

   a) This statement is false.
      
      Counterexample: \( x = -7, \ y = 0 \)
      
      Then \( x = -7 \leq 0 = y \) but \( x^2 = 49 \gtrsim y^2 = 0 \).

   b) \( \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \quad [(x \leq y) \land (x^2 \gtrsim y^2)] \)

7. 

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8. (i) The sun shines if and only if either
       the wind does not blow or the rain does not
       shine, or both.

(ii) \( S \land \neg W \land (T \Rightarrow R) \)

(iii) a) False (since \( (T \Rightarrow T) \land (\neg F \land T) \)

\[ \frac{F \land F}{F} \]

b) True (since \( (T \lor F) \leftrightarrow (T \lor F) \)

\[ \frac{T \leftrightarrow T}{T} \]

(c) False (since \( T \land \neg T \land T \)
\[(A-B) \cap B = \emptyset\]

**Proof:** Suppose, by contradiction, that \( \exists x \in (A-B) \cap B \). Then, since \( x \in A - B \), \( x \in A \) and \( x \notin B \). So, since \( x \in (A-B) \cap B \), \( (x \in A \text{ and } x \notin B) \) and \( x \in B \). This implies \( x \in B \text{ and } x \notin B \), which is a contradiction. Hence, there exists no element in \((A-B) \cap B\). \( \square \)

10.

No, the set \( S = \left\{ 1 - \frac{1}{n} \mid n \in \mathbb{Z}^+ \right\} \) does not have a greatest element.

**Proof** Suppose, by contradiction, that \( S \) does have a greatest element \( m_0 = 1 - \frac{1}{n_0} \) for some \( n_0 \in \mathbb{Z}^+ \).

Since \( n_0 + 1 > n_0 \) (true for all real numbers) and both \( n_0 \) and \( n_0 + 1 \) are positive, so dividing by \( n_0(n_0+1) \) does not change the inequality:

\[
\frac{n_0}{n_0(n_0+1)} > \frac{-1}{m_0(n_0+1)}
\]

Multiplying by \( -1 < 0 \) does reverse the inequality:

\[
-\frac{n_0}{n_0(n_0+1)} > -\frac{1}{m_0+1}
\]

Adding \( 1 \) to both sides:

\[
m_0 = 1 - \frac{1}{n_0} < 1 - \frac{1}{n_0+1}
\]

But \( 1 - \frac{1}{n_0+1} \in S \) by definition of \( S \), so \( m_0 \leq 1 - \frac{1}{n_0+1} \) contradicts our assumption that \( m_0 \) is the largest element of \( S \). Hence, there exists no largest element of \( S \). \( \square \)
\[ f(x) = \sqrt{x+7} \]

a) For \( f \) to be defined, \( x+7 \geq 0 \) \( \iff \) \( x \geq -7 \).

So the maximal domain of \( f \) is \( x = [-7, \infty) \).

For codomain we can take \( Y = \mathbb{R} \) (or any subset of \( \mathbb{R} \) that includes \( [0, \infty) \)).

b) \( f \) is injective: \( f(x_1) = f(x_2) \Rightarrow \sqrt{x_1+7} = \sqrt{x_2+7} \)

\[ \Rightarrow (\sqrt{x_1+7})^2 = (\sqrt{x_2+7})^2 \]
\[ \Rightarrow x_1 + 7 = x_2 + 7 \]
\[ \Rightarrow x_1 = x_2. \]

\( f : [-7, \infty) \rightarrow \mathbb{R} \) is not surjective.

If \( y \in \mathbb{R} \), say \( y = -1 \) such that \( -1 \neq \sqrt{x+7} \) for any \( x \in \mathbb{R} \).

c) \( f = g \circ h \) where \( h(x) = x+7 \), \( h : [-7, \infty) \rightarrow [0, \infty) \) and \( g(x) = \sqrt{x} \), \( g : [0, \infty) \rightarrow [0, \infty) \).

d) \( j(x) = \frac{f(x)}{x^2 + c} \) if \( x > 2 \)

is well-defined if \( f(2) = (2)^2 + c \)

\[ \Rightarrow \sqrt{2+7} = 4 + c \]
\[ \Rightarrow 3 = 4 + c \iff c = -1 \]

12) a) \( f \) and \( g \) injective \( \Rightarrow f + g \) injective.

Counterexample: \( f(x) = x \), \( g(x) = -x \) (both \( : \mathbb{R} \rightarrow \mathbb{R} \))

are both injective but \( (f + g)(x) = 0 \) is a constant function so \( f + g \) not injective.
b) Same example shows that
\[ f, g \text{ surjective } \not\implies f+g \text{ surjective} \]

c) No, \( f \text{ bijective } \implies 2f \text{ bijective} \) because \( f \text{ surjective } \not\implies 2f \text{ surjective} \)

ex: \( f: \mathbb{Z} \to \mathbb{Z}, f(x) = x \) is surjective (bijective)
but \( (2f)(x) = 2x \) is not

Proof: \( f(y) \in \mathbb{Z} \), namely any odd integer, say \( y = 1 \)
such that \( y \neq 2x \) for \( x \in \mathbb{Z} \).

However, \( f \text{ injective } \implies 2f \text{ injective is TRUE} \)

Proof
\[
\begin{align*}
(2f)(x_1) &= (2f)(x_2) \\
\implies 2(f(x_1)) &= 2(f(x_2)) \\
\implies f(x_1) &= f(x_2) \\
\implies x_1 &= x_2 \\
\text{by injectivity of } f. \quad \& \text{ED}
\end{align*}
\]

left to reader \& textbook.