Ch 9: Injectivity, Surjectivity, Inverses & Functions on Sets

DEFINITIONS:

1. The **identity function** on a set X is the function $I_X: X \to X, I_X(x) = x$ for all $x \in X$.

Suppose $f: X \to Y$ is a function. Then:

- The ***image of f*** is defined to be: $Im(f) = \{f(x) | x \in X\} = \{y \in Y | y = f(x) \text{ for some } x \in X\}$

- The ***graph of f*** can be thought of as the set $G_f \subseteq X \times Y$, $G_f = \{(x, f(x)) | x \in X\}$.

We say that $f: X \to Y$ is:

- ***f is injective*** iff: $\forall x_1, x_2 \in X, [x_1 \neq x_2] \Longrightarrow [f(x_1) \neq f(x_2)]$

  More useful in proofs is the contrapositive: $\forall x_1, x_2 \in X, [f(x_1) = f(x_2)] \Longrightarrow [x_1 = x_2]$

- ***f is surjective*** iff: $\forall y \in Y, \exists x \in X, y = f(x)$.
  Note that this is equivalent to saying that $Y = Im(f)$

- ***f is bijective*** iff it's both injective and surjective.

- ***f invertible*** (has an inverse) iff $\exists g: Y \to X, g \circ f = I_X \text{ and } f \circ g = I_Y$.
  This function g is called the inverse of f, and is often denoted by f^{-1}.
  Theorem 9.2.3: A function $f: X \to Y$ is invertible if and only if it is a bijection. Further, if it is invertible, its inverse is unique. (proof is in textbook)

Induced Functions on Sets: Given a function $f: X \to Y$, it naturally induces two functions on power sets:

- the forward function $\overrightarrow{f}: \mathcal{P}(X) \to \mathcal{P}(Y)$, defined by $\overrightarrow{f}(A) = \{f(x) | x \in A\}$ for any set $A \epsilon \mathcal{P}(X)$.
  Note that $\overrightarrow{f}(A)$ is simply the image through *f* of the subset A.
- the pre-image function $\overleftarrow{f}: \mathcal{P}(Y) \to \mathcal{P}(X)$, defined by $\overleftarrow{f}(B) = \{x \in X | f(x) \in B\}$ for any set $B \in \mathcal{P}(Y)$.
  Note that $\overleftarrow{f}(B)$ is simply the set of all the elements that *f* maps to elements in the subset B of the codomain.

EXAMPLES & PROBLEMS:

1. Give an example of a function with domain $\mathbb{Z}$, whose image is $\mathbb{Z}^+$.

   $f: \mathbb{Z} \to \mathbb{Z}, f(x) = |x| + 1$

2. Write the graph of the identity function on $\mathbb{Z}$, $I_{\mathbb{Z}}: \mathbb{Z} \to \mathbb{Z}, I_{\mathbb{Z}}(x) = x$, as a subset of $\mathbb{Z} \times \mathbb{Z}$.

$G = \{(x, x) | x \in \mathbb{Z}\}$

3. Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 5x - 1$.

   a. Is this function injective? Yes/No. Proof:

   Suppose that there exist two values $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. Then

   $f(x_1) = f(x_2) \Longrightarrow 5x_1 - 1 = 5x_2 - 1 \Longrightarrow 5x_1 = 5x_2 \Longrightarrow x_1 = x_2.$

   Since for any $x_1, x_2 \in \mathbb{R}$, $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$, the function f is injective. QED

   b. Is this function surjective? Yes/No. Proof:

   For any $y \in \mathbb{R}$ there exists some $x \in \mathbb{R}$, namely $x = \frac{y+1}{5}$, such that

   $$f(x) = f\left(\frac{y+1}{5}\right) = 5\left(\frac{y+1}{5}\right) - 1 = y.$$

   This proves that the function is surjective.QED

   c. Is it bijective? Yes/No. If yes, find its inverse.

   $$f^{-1}\colon \mathbb{R} \to \mathbb{R}, f^{-1}(x) = \frac{x+1}{5}$$

   d. Compute $f \circ f$.

   $f \circ f\colon \mathbb{R} \to \mathbb{R}, f \circ f(x) = f(f(x)) = f(5x - 1) = 5(5x - 1) - 1 = 25x - 6$

4. Let : $\mathbb{R} \to \mathbb{R}$, $g(x) = x^2$.

   a. Is this function injective? Yes/No Proof:

   There exist two real values of x, for instance $x_1 = -2$ and $x_2 = 2$, such that
   $f(x_1) = f(x_2) = 4$, but $x_1 \neq x_2$.

   b. Is this function surjective? Yes/No Proof:

   There exist some $y \in \mathbb{R}$, for instance $y = -1$, such that for all x$\in \mathbb{R}$

   $$f(x) = x^2 > -1.$$

   This shows that -1 is in the codomain but not in the image of *f*, so *f* is not surjective. QED

   c. Is it bijective? Yes/No. If yes, find its inverse.

5. Let $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$. Determine the following sets:

$\vec{g}(\mathbb{R}) = [0, \infty)$	$\overleftarrow{g}(\mathbb{R}) = \mathbb{R}$
$\vec{g}([0,1]) = [0, 1]$	$\overleftarrow{g}([0,1]) = [-1, 1]$
$\vec{g}([-2,7)) = [0, 49)$	$\overleftarrow{g}((1,2)) = (-\sqrt{2}, -1) \cup (1, \sqrt{2})$

6. Given $f\colon X \to Y$, and two subsets $A, B \subseteq X$, prove that $\overrightarrow{f}(A \cap B) \subseteq \overrightarrow{f}(A) \cap \overrightarrow{f}(B)$.

   Proof:

   Let $y \in \overrightarrow{f}(A \cap B)$. By definition of $\overrightarrow{f}$, this means that there exists an element $x \epsilon A \cap B$ such that $y = f(x)$.
   Since $x \epsilon A$ and $y = f(x)$, we get that $y \in \overrightarrow{f}(A)$. Since $x \epsilon B$ and $y = f(x)$ we must have $y \in \overrightarrow{f}(B)$ as well.
   Hence, by definition of set intersection, $y \in \overrightarrow{f}(A) \cap \overrightarrow{f}(B)$. This proves that $\overrightarrow{f}(A \cap B) \subseteq \overrightarrow{f}(A) \cap \overrightarrow{f}(B)$. QED