Example 3: Prove that \((x_1y_1 + x_2y_2)^2 \leq (x_1^2 + x_2^2)(y_1^2 + y_2^2)\) for all real numbers \(x_1, x_2, y_1, y_2\).

Note: this can be worded as an implication:
“if \(x_1, x_2, y_1, y_2\) are real numbers, then \((x_1y_1 + x_2y_2)^2 \leq (x_1^2 + x_2^2)(y_1^2 + y_2^2)\)”

“Proof” attempt:

\[(x_1y_1 + x_2y_2)^2 \leq (x_1^2 + x_2^2)(y_1^2 + y_2^2)\]

By the properties of multiplication, \((x_1y_1 + x_2y_2)^2 = x_1^2y_1^2 + 2x_1x_2y_1y_2 + x_2^2y_2^2\)
and \((x_1^2 + x_2^2)(y_1^2 + y_2^2) = x_1^2y_1^2 + x_1^2y_2^2 + x_2^2y_1^2 + x_2^2y_2^2\), so we get that:

\[x_1^2y_1^2 + 2x_1x_2y_1y_2 + x_2^2y_2^2 \leq x_1^2y_1^2 + x_1^2y_2^2 + x_2^2y_1^2 + x_2^2y_2^2\]

Using the additive law we can subtract equal amounts from both sides of the inequality:

\[2x_1x_2y_1y_2 \leq x_1^2y_2^2 + x_2^2y_1^2\]

Using the additive law again to subtract \(2x_1x_2y_1y_2\) from both sides, we get:

\[0 \leq x_1^2y_2^2 - 2x_1x_2y_1y_2 + x_2^2y_1^2\]

By the properties of multiplication again, \(x_1^2y_2^2 - 2x_1x_2y_1y_2 + x_2^2y_1^2 = (x_1y_2 - x_2y_1)^2\), so the last statement is just:

\[0 \leq (x_1y_2 - x_2y_1)^2\]

This last statement is always true because \(x_1y_2 - x_2y_1\) is a real number and by Example 2 (or prop 3.1.4), the square of any real number is non-negative.

QED

What is wrong with this proof? Each step is correct, but it does not prove the inequality is true!
What we did is to show that if the inequality holds, then we get something true. This says nothing about the truth value of the inequality itself. After all, if Q is true then any implication P=>Q is true, regardless whether P is true or not! So, just because our inequality implies something true, it does not make it true.

What we need is a chain of true statements and correct implications that ends with the inequality, not starts with it – this particular “proof” is written backwards. The process of working backwards from the desired conclusion is a good way to search for a proof when you’re thinking about your solution, but to have a valid proof we need to reverse the steps and see if it still works.

Note: In fact, in our argument above each statement is equivalent to the next, so we could just modify the above proof to indicate “if and only if” or “is equivalent to” in each step. But let’s just reverse the argument to see how it would go.
Good proof:

Since $x_1, x_2, y_1, y_2$ are given to be real numbers, and addition or multiplication of real numbers results in a real number (we say that “the real numbers are closed under addition, subtraction and multiplication”), $x_1 y_2 - x_2 y_1$ is also a real number.

By Example 2 in the handout for Chapter 3 (or by the textbook Proposition 3.1.4), the square of any real number is non-negative. Hence,

$$(x_1 y_2 - x_2 y_1)^2 \geq 0.$$ 

Expanding the left hand side:

$$x_1^2 y_2^2 - 2x_1 x_2 y_1 y_2 + x_2^2 y_1^2 \geq 0$$

By the Additive Law, adding $2x_1 x_2 y_1 y_2$ to both sides we get:

$$x_1^2 y_2^2 + x_2^2 y_1^2 \geq 2x_1 x_2 y_1 y_2$$

By the Additive Law again, adding $x_1^2 y_1^2 + x_2^2 y_2^2$ to both sides we get:

$$x_1^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_1^2 + x_2^2 y_2^2 \geq x_1^2 y_1^2 + 2x_1 x_2 y_1 y_2 + x_2^2 y_2^2$$

Factoring both the left-hand side:

$$x_1^2 y_1^2 + x_1^2 y_2^2 + x_2^2 y_1^2 + x_2^2 y_2^2 = (x_1^2 + x_2^2)(y_1^2 + y_2^2)$$

and the right-hand side:

$$x_1^2 y_1^2 + 2x_1 x_2 y_1 y_2 + x_2^2 y_2^2 = (x_1 y_1 + x_2 y_2)^2$$

we get the desired inequality:

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) \geq (x_1 y_1 + x_2 y_2)^2$$

QED