Math 310 -- **SETS** Worksheet (Ch 6)

<u>RELATIONSHIPS:</u>

- Inclusion (subset) $\boldsymbol{A \subseteq B} \quad \Leftrightarrow \quad [\boldsymbol{x \in A} \Rightarrow \boldsymbol{x} \in \boldsymbol{B}]$
- Equality $\boldsymbol{A = B} \quad \Leftrightarrow \quad [\boldsymbol{A \subseteq B} \text{ and } \boldsymbol{A \supseteq B}]$ (double inclusion)

  $\Leftrightarrow \quad [\boldsymbol{x \in A} \Leftrightarrow \boldsymbol{x} \in \boldsymbol{B}]$
- Proper Subset $\boldsymbol{A \subset B} \quad \Leftrightarrow \quad \boldsymbol{A \subseteq B} \text{ and } \boldsymbol{A \neq B}$

  $\Leftrightarrow \quad [(\boldsymbol{x \in A} => \boldsymbol{x} \in \boldsymbol{B}) \quad \textbf{and} \quad (\text{there exists some } \boldsymbol{a} \in \boldsymbol{B} - \boldsymbol{A})]$

<u>OPERATIONS on Sets (new sets from old):</u>

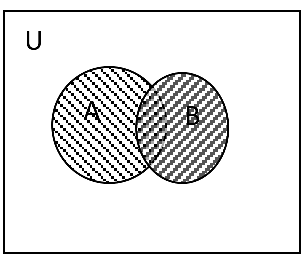
- Set Intersection: $\boldsymbol{A \cap B} = \{\boldsymbol{x} \mid \boldsymbol{x \in A} \textbf{ and } \boldsymbol{x} \in \boldsymbol{B}\}$
- Set Union: $\boldsymbol{A \cup B} = \{\boldsymbol{x} \mid \boldsymbol{x \in A} \textbf{ or } \boldsymbol{x} \in \boldsymbol{B}\}$
- Set Difference: $\boldsymbol{A - B} = \{\boldsymbol{x} \mid \boldsymbol{x \in A} \textbf{ and } \boldsymbol{x} \notin \boldsymbol{B}\}$
- Set Complement: $\boldsymbol{A^c} = \boldsymbol{U} - \boldsymbol{A} = \{\boldsymbol{x \in U} \mid \boldsymbol{x} \notin \boldsymbol{A}\}$ (U is some universal set containing A)
- POWER Set: $\boldsymbol{\mathcal{P}(A)} = \{\boldsymbol{X} \mid \boldsymbol{X \subseteq A}\}$. (the set whose elements are all the subsets of A)

  Note that by definition of $\mathcal{P}(A)$, $\boxed{X \subseteq A <=> X \in \mathcal{P}(A)}$
- Cartesian Product: $\boldsymbol{A \times B} = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x \in A} \textbf{ and } \boldsymbol{y \in B}\}$

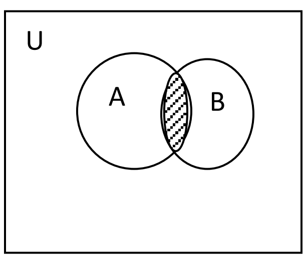
  The set of all ordered pairs (x,y), with x an element of A and y an element of B.

  (this is in **Ch 7**, section **7.7**)

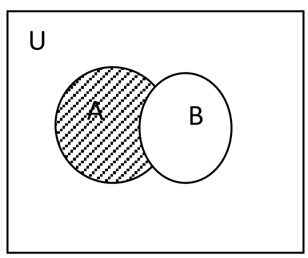
Venn Diagrams (useful to visualize sets, but please don't use in proofs!)



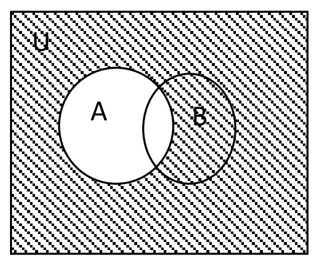
AUB



$A \cap B$



A-B



A^c

<u>EMPTY SET</u>: The empty set is the set with no elements: $\boldsymbol{\phi} = \{\ \}$ (a.k.a. the null set)

Exercise: Prove that the empty set is a subset of any set.

Proof:

By definition of subset, $\{\ \} \subseteq A$ if and only if $x \in \{\} \Longrightarrow x \in A$. Since the empty set has no elements, the hypothesis of the implication is always false, so the implication is true ("vacuously true", i.e. true because nothing satisfies the hypothesis).

Examples:

1) Let $A = \{a, b, c, d, e\}$ and $B = \{x, y, a, z, c\}$.

   $A \cap B = \{a, c\}$

   $A \cup B = \{a, b, c, d, e, x, y, z\}$

   $A - B = \{b, d, e\}$

2) $(-3, 0]^c = (-\infty, -3] \cup (0, \infty)$

3) If A={1,2,3} list all the elements of the following sets:

   - $\mathcal{P}(A) = \{\ \{\}, \{\mathbf{1}\}, \{\mathbf{2}\}, \{\mathbf{3}\}, \{\mathbf{1}, \mathbf{2}\}, \{\mathbf{1}, \mathbf{3}\}, \{\mathbf{2}, \mathbf{3}\}, \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}\ \}$

   - $A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

<u>It's important to distinguish between subsets and elements, and to use the correct notation!</u>

Example: Suppose A={1,2,3} . Which of the following are correct statements? The boxed ones are correct.

$1 = \{1\}$, $\boxed{1 \in A}$, $1 \subseteq A$, $\boxed{\{1\} \subseteq A}$, $\boxed{\{1,2\} \subseteq A}$, $\{1,2\} \in A$,

$\boxed{\{1,2\} \in \mathcal{P}(A)}$, $\boxed{\{1,2\} \nsubseteq \mathcal{P}(A)}$, $\{\{1,2\}\} \in \mathcal{P}(A)$, $\boxed{\{\{1,2\}\} \subseteq \mathcal{P}(A)}$

You should recall the following results, and be able to prove them:

Theorem 6.3.4 (this theorem is in the textbook)

(i) **Associativity of set union and intersection:**

$\boldsymbol{A} \cup (\boldsymbol{B} \cup \boldsymbol{C}) =$ $\boldsymbol{A} \cap (\boldsymbol{B} \cap \boldsymbol{C}) =$

(ii) **Commutativity:** $\boldsymbol{A} \cup \boldsymbol{B} =$ $\boldsymbol{A} \cap \boldsymbol{B} =$

(iii) **Distributivity:** $\boldsymbol{A} \cup (\boldsymbol{B} \cap \boldsymbol{C}) = (\boldsymbol{A} \cup \boldsymbol{B}) \cap (\boldsymbol{A} \cup \boldsymbol{C})$ **,** $\boldsymbol{A} \cap (\boldsymbol{B} \cup \boldsymbol{C}) =$

(iv) **De Morgan Laws:** $(\boldsymbol{A} \cup \boldsymbol{B})^{\boldsymbol{c}} =$ $(\boldsymbol{A} \cap \boldsymbol{B})^{\boldsymbol{c}} =$

(v) **Complementation:** $\boldsymbol{A} \cup \boldsymbol{A}^{\boldsymbol{c}} =$ $\boldsymbol{A} \cap \boldsymbol{A}^{\boldsymbol{c}} =$

(vi) **Double complement:** $(\boldsymbol{A}^{\boldsymbol{c}})^{\boldsymbol{c}} =$

Sample Proofs:

(iii) Use logical arguments from the definitions to show: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Remark: Since we need to show that two sets are equal, we need to show double inclusion, generally we need to do two separate proofs (unless the argument can be written clearly & correctly using " $\Leftrightarrow$ " at every step).

Proof: We'll show the set equality by proving both set inclusions:

1) "$\subseteq$ ": Show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$, i.e. that $x \in A \cup (B \cap C) \Longrightarrow x \in (A \cup B) \cap (A \cup C)$.

   $x \in A \cup (B \cap C) \Leftrightarrow \boldsymbol{x} \in A$ **or** $\boldsymbol{x} \in B \cap C$ (by definition of union)

   Case 1: If $\boldsymbol{x} \in A$, then certainly $x \in A \cup B$ and $x \in A \cup C$ (by definition of union applied twice), so $x \in (A \cup B) \cap (A \cup C)$ (by definition of intersection).

   Case 2: If $\boldsymbol{x} \in B \cap C$, then $\boldsymbol{x} \in B$ and $\boldsymbol{x} \in C$ (by definition of intersection), so $x \in A \cup B$ and $x \in A \cup C$ (by definition of union, applied twice). Hence $x \in (A \cup B) \cap (A \cup C)$, by definition of intersection.

   In all possible cases, $x \in (A \cup B) \cap (A \cup C)$, which proves the desired set inclusion.

2) "$\supseteq$ ": Show that $A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C)$, i.e. that $x \in (A \cup B) \cap (A \cup C) \Longrightarrow x \in A \cup (B \cap C)$.

   $x \in (A \cup B) \cap (A \cup C) \Leftrightarrow x \in (A \cup B)$ *and* $(A \cup C)$ (by definition of intersection)

   $\Leftrightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$ (by definition of union)

   Case 1: If $x \in A$, then $x \in A \cup (B \cap C)$ by definition of union.

   Case 2: If $x \notin A$, then, since it is true that $[(x \in A \text{ or } x \in B)$ and $(x \in A \text{ or } x \in C)]$, it must be that $x \in B$ and $x \in C$ (by definition of "or" & "and"), so $x \in B \cap C$ (by definition of intersection). Hence, in this case as well, $x \in A \cup (B \cap C)$.

   In all possible cases, $x \in (A \cup B) \cap (A \cup C) \Longrightarrow x \in A \cup (B \cap C)$, which proves the desired reverse set inclusion.

QED

➔ You try: prove the counterpart statement: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(iv) Use truth tables to prove $(A \cup B)^c = A^c \cap B^c$

$\boldsymbol{x \in A}$	$\boldsymbol{x \in B}$	$\boldsymbol{x \in A \cup B}$	$\boldsymbol{x \in (A \cup B)^c}$	$\boldsymbol{x \in A^c}$	$\boldsymbol{x \in B^c}$	$\boldsymbol{x \in A^c \cap B^c}$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Proposition 6.2.4: $\boldsymbol{A \cup B = (A \cap B) \cup (A - B) \cup (B - A)}$

➔ Prove it by truth tables (in text) and by logical arguments (exercise)