RECALL: An infinite set \( X \) is **countable** iff there exists a bijection \( f : X \rightarrow \mathbb{N} \).

**CANTOR HOTEL Handout:**

Think of the hotel **rooms** as the set of natural numbers \( \mathbb{N} = \{1, 2, 3, \ldots \} \).

Think of the **guests** at the hotel as a set which is either countable or uncountable depending on whether there is a possible one-to-one room assignment of one guest per each room or not.

What the handout makes us prove:

1) \( X \) countable \( \Rightarrow \) \( X \cup \{a\} \) is countable
2) \( X \) countable \( \Rightarrow \) \( X \cup \{a_1, a_2, \ldots, a_n\} \) is countable
3) \( X, Y \) countable \( \Rightarrow \) \( X \cup Y \) is countable
4) \( X, Y \) countable \( \Rightarrow \) \( X \times Y \) is countable
   (alternatively: a union of countably many sets each of which is countable is countable)
5) The rationals are countable
6) The real numbers are not countable.
   In fact, not even \([0,1]\) is countable!

You should know how to write a formal proof of any of these results.

The other results from Chapter 14, you should understand and be able to use them, but you don’t need to worry about their proofs
(for the more ambitious, the proof of 14.3.3 is tough but beautiful!)

Some of the proofs:

2) \( \text{X countable} \Rightarrow \text{X} \cup \{a_1, a_2, \ldots, a_n\} \) is countable (union of disjoint sets)

Handout idea:

Ask everyone already in the hotel to move over by \( n \) rooms, (therefore freeing up the first \( n \) rooms.)

Give the new \( n \) guests the first \( n \) hotel rooms.

Formal Proof:

The set \( X \) is countable \( \text{if and only if} \) there exists a bijection \( f : X \rightarrow \mathbb{N} \).

Construct a bijection \( g : X \cup \{a_1, a_2, \ldots, a_n\} \rightarrow \mathbb{N} \) as follows:

\[
g(a_i) = i \quad \text{for} \ 1 \leq i \leq n
\]

\[
g(x) = f(x) + n \quad \text{for all} \ x \in X
\]

Check this function maps \( X \cup \{a_1, a_2, \ldots, a_n\} \) to \( \mathbb{N} \) bijectively. Hence \( X \cup \{a_1, a_2, \ldots, a_n\} \) is countable.
4) $X, Y$ countable $\Rightarrow X \times Y$ is countable

Handout idea:
Denote the $n^{th}$ player on the $m^{th}$ team by $(m, n)$.
Then all the players can be listed in an infinite matrix of the form:

\[
\begin{pmatrix}
(1,1) & (1,2) & (1,3) & \ldots \\
(2,1) & (2,2) & (2,3) & \ldots \\
(3,1) & (3,2) & (3,3) & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

We can count the players across diagonals.
First note that:
- The $i^{th}$ diagonal has $i$ elements on it
- The $i^{th}$ diagonal consists of all elements $(x,y)$ with $x+y=i+1$

Hence:
- An element $(m,n)$ is on the \underline{\underline{_______________}} diagonal
- There are $1 + 2 + \cdots + (m + n - 2) = \frac{(m+n-2)(m+n-1)}{2}$ elements on the previous $m + n - 2$ diagonals and $(m, n)$ is the $n^{th}$ element in its own diagonal
- So the $(m,n)$ player gets the $\left(\frac{(m+n-2)(m+n-1)}{2} + n\right)^{th}$ key!

Formal proof: see text /do yourself
Infinite Cardinalities:

Recall:

- $|X| = |Y|$ def there exists a bijection $f: X \rightarrow Y$

- $\aleph_0 \cong |\mathbb{N}|$ ("aleph null")
  
  Hence any infinite countable set has cardinality $\aleph_0$ (*why?*)

Any infinite set must have cardinality at least $\aleph_0$. (*why?*)

Are there infinite sets of larger cardinality? Yes. We saw that $\mathbb{R}$ is not countable. So, $|\mathbb{R}| > \aleph_0$.

- Theorem 14.3.3: For any set $X$ (finite or infinite)
  
  $|X| < |\mathcal{P}(X)|$

  This proves that no largest cardinality exist! (*why?*)

- Cantor showed: $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})| > \aleph_0$

- Define $\aleph_1$ to be the next smallest infinite cardinality larger than $\aleph_0$

  *Continuum hypothesis*: $\aleph_1 = |\mathbb{R}|$

  This is an undecidable statement.

  For simplicity, you may assume this as an axiom in your homework