I. Review of formal DEFINITION of limits:

Given a sequence of real numbers \( a_n \), and a real number \( L \), \( \lim_{n \to \infty} a_n = L \) iff

\[
\forall \epsilon > 0, \exists N \in \mathbb{Z}^+, \text{ such that } \forall n \geq N, |a_n - L| < \epsilon.
\]

Notes:

a) The part "\( \exists N \in \mathbb{Z}^+, \text{ such that } \forall n \geq N \)" can be thought of less formally as: "for all \( n \) large enough"

b) Also less formally: \( \epsilon \) is often used mathematically to denote a quantity which is positive but can be taken as small as we wish.

II. A few sample solutions:

1. Show that \( \lim_{n \to \infty} \frac{1}{n + 1} = 0 \).

PROOF:

By definition, \( \lim_{n \to \infty} \frac{1}{n + 1} = 0 \) iff

\[
\forall \epsilon > 0, \exists N \in \mathbb{Z}^+, \text{ such that } \forall n \geq N, \left| \frac{1}{n + 1} \right| < \epsilon.
\]

Note that

\[
\left| \frac{1}{n + 1} \right| = \frac{1}{n + 1} < \epsilon
\]

(cross-multiplying and noting that \( n > 0 \) and \( \epsilon > 0 \))

\[
\Leftrightarrow \frac{1}{\epsilon} < n + 1
\]

(subtracting 1 from both sides)

\[
\Leftrightarrow \frac{1}{\epsilon} - 1 < n.
\]

Pick any \( \epsilon > 0 \).

Then, by the above discussion, there exists \( N \in \mathbb{Z}^+ \), namely any \( N > \frac{1}{\epsilon} - 1 \), such that \( \forall n \geq N, \left| \frac{1}{n + 1} \right| < \epsilon \). QED

2. Show that \( \lim_{n \to \infty} (n - 1) \neq 0 \).

PROOF:

By the negation of the definition of limit of a sequence, \( \lim_{n \to \infty} (n - 1) \neq 0 \) iff

\[
\exists \epsilon > 0, \forall N \in \mathbb{Z}^+, \text{ such that } \exists n \geq N, |n - 1| \geq \epsilon.
\]
Note that for \( n \geq 1, |n - 1| = n - 1 \), so

\[ |n - 1| = n - 1 \geq \epsilon \iff [n \geq \epsilon + 1]. \]

Let \( \epsilon = 1 \).

Then, \( \forall N \in \mathbb{Z}^+ \), we can take \( n = N + 1 \), such that \( n \geq N \) (clearly, by the way we chose it) and, since \( N \geq 1, n = N + 1 \geq 2 = \epsilon + 1 \). By the above discussion, this implies \( |n - 1| \geq \epsilon \).

In other words,

\[ \exists \epsilon = 1 > 0, \forall N \in \mathbb{Z}^+, \text{ such that } \exists n = N + 1 \geq N, |n - 1| \geq \epsilon. \]

QED
III. Sample solutions for the functions on sets problems:

Problems II: Problem 20 (iii) Prove that \( \vec{f}(A_1 \cup A_2) = \vec{f}(A_1) \cup \vec{f}(A_2) \).

Comments:
Set equality usually means having to show two set inclusions. I will only show the first inclusion, namely: \( \vec{f}(A_1 \cup A_2) \subseteq \vec{f}(A_1) \cup \vec{f}(A_2) \).

The usual way to show set inclusion is by picking an arbitrary element in the first set and showing it must be an element of the second set. The intermediate steps use the definitions of the starting and ending sets, and whatever other hypotheses you have, if any. Let’s do it:

PROOF:
Let \( y \in \vec{f}(A_1 \cup A_2) \). By definition of the set \( \vec{f}(A_1 \cup A_2) \), this means that \( y = f(x) \) for some \( x \in A_1 \cup A_2 \). That is, \( y = f(x) \) for some element \( x \) such that \( x \in A_1 \) or \( x \in A_2 \). If \( x \in A_1 \), then \( y = f(x) \in \vec{f}(A_1) \), and if \( x \in A_2 \), then \( y = f(x) \in \vec{f}(A_2) \). Either way, \( y = f(x) \in \vec{f}(A_1) \cup \vec{f}(A_2) \).
Since we’ve shown that \( y \in \vec{f}(A_1 \cup A_2) \Rightarrow y \in \vec{f}(A_1) \cup \vec{f}(A_2) \), the set inclusion follows.

[now do the other direction]

Problems II: Problem 21, part of part (ii) Let \( f : X \rightarrow Y \) be a function. Prove that \( f \) is surjective \( \iff \vec{f} \) is surjective.

Comments: Again, there are two directions to prove, this time because it’s a double implication. Let’s do ”\( \Rightarrow \)”. We need to show that \( \vec{f} : P(X) \rightarrow P(Y) \) is surjective, under the hypothesis that \( f : X \rightarrow Y \) is surjective. But what does \( \vec{f} \) surjective mean, by definition? It means
\[ \forall B \in P(Y), \exists A \in P(X), B = \vec{f}(A) \]

By definition of the power sets and of \( \vec{f} \) this is equivalent to
\[ \forall B \subseteq Y, \exists A \subseteq X, B = \{ f(x) \mid x \in A \} \]

So this is what we have to show. For a given \( B \) we must produce the set \( A \), probably by using the surjectivity of \( f \).
Let’s do it:

PROOF: Let \( B \) be an arbitrary subset of \( Y \). Since \( f \) is surjective, for every \( b \in B \subseteq Y \) there is some \( a \in X \) such that \( b = f(a) \). Let \( A \) be the set of all those \( a \). Then \( B = \{ f(a) \mid a \in A \} \), so, by definition of \( \vec{f} \), \( B = \vec{f}(A) \). This shows that given any \( B \in P(Y) \), the surjectivity of \( f \) provides a set \( A \in P(X) \) such that \( B = \vec{f}(A) \). Hence \( \vec{f} \) is surjective.