Math 300 -- **SETS** Worksheet (Ch 6)

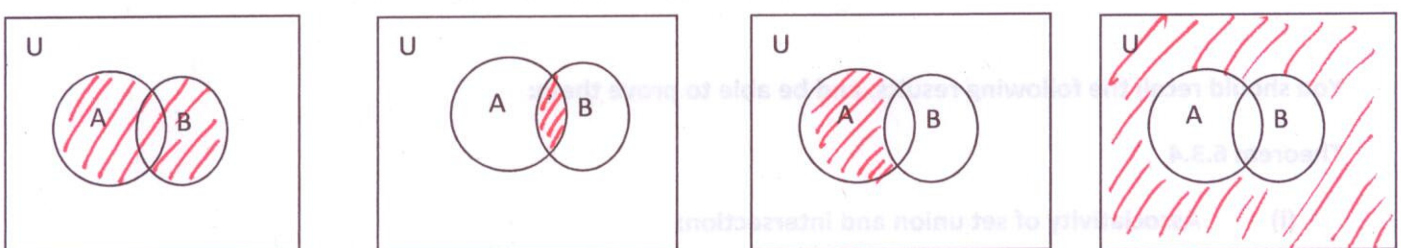
RELATIONSHIPS:

- Inclusion (subset) $A \subseteq B \iff [x \in A \Rightarrow x \in B]$
- Equality $A = B \iff [A \subseteq B \text{ and } A \supseteq B]$ (double inclusion)
  $\iff [x \in A \Leftrightarrow x \in B]$
- Proper Subset $A \subset B \iff A \subseteq B$ **and** $A \neq B$
  $\iff [(x \in A \Rightarrow x \in B)$ **and** (there exists some $a \in B - A$)]

OPERATIONS on Sets (new sets from old):

- Set Intersection: $A \cap B = \{x \mid x \in A \textbf{ and } x \in B\}$
- Set Union: $A \cup B = \{x \mid x \in A \textbf{ or } x \in B\}$
- Set Difference: $A - B = \{x \mid x \in A \textbf{ and } x \notin B\}$
- Set Complement: $A^c = U - A = \{x \in U \mid x \notin A\}$ (U is some universal set containing A)
- POWER Set: $\mathcal{P}(A) = \{X \mid X \subseteq A\}$. (the set whose elements are all the subsets of A)
  Note that by definition of $\mathcal{P}(A)$, $\boxed{X \subseteq A \iff X \in \mathcal{P}(A)}$
- Cartesian Product: $A \times B = \{(x, y) \mid x \in A \textbf{ and } y \in B\}$
  The set of all ordered pairs (x,y), with x an element of A and y an element of B.
  (this is in **Ch 7**, section **7.7**)

Venn Diagrams (useful to visualize sets, but please don't use in proofs!)



$A \cup B$ $\quad$ $A \cap B$ $\quad$ $A - B$ $\quad$ $(A \cup B)^c = A^c \cap B^c$

EMPTY SET: The empty set is the set with no elements: $\phi = \{\ \}$ (a.k.a. the null set)

Exercise: Prove that the empty set is a subset of any set.

Proof: Let A be any set.
By definition of subset, $\emptyset \subseteq A$ iff $x \in \emptyset \Rightarrow x \in A$.
Since $\emptyset$ has no elements, the statement $x \in \emptyset$ is always false.
Hence the implication $x \in \emptyset \Rightarrow x \in A$ is true. $\square$

ALTERNATIVELY: Prove the contrapositive: $x \notin A \Rightarrow x \notin \emptyset$
Let x be any element not in A. Then $x \notin \emptyset$, since $\emptyset$ has no elements. $\square$.

Examples:

1) Let $A = \{a, b, c, d, e\}$ and $B = \{x, y, a, z, c\}$.

$A \cap B = \{a, c\}$

$A \cup B = \{a, b, c, d, e, x, y, z\}$

$A - B = \{b, d, e\}$

2) $(-3, 0]^c = (-\infty, -3] \cup (0, \infty) = \{x \in \mathbb{R} \mid x \leq -3 \text{ or } x > 0\}$

3) If A={1,2,3} list all the elements of the following sets:

- $\mathcal{P}(A) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$

- $A \times A = \{ (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3) \}$

<u>It's important to distinguish between subsets and elements, and to use the correct notation!</u>

Example: Suppose A={1,2,3} . Which of the following are correct statements?

$1 = \{1\}$, $1 \in A$, $1 \subseteq A$, $\{1\} \subseteq A$, $\{1,2\} \subseteq A$, $\{1,2\} \in A$,

$\{1,2\} \in \mathcal{P}(A)$, $\{1,2\} \not\subseteq \mathcal{P}(A)$, $\{\{1,2\}\} \in \mathcal{P}(A)$, $\{\{1,2\}\} \subseteq \mathcal{P}(A)$

You should recall the following results, and be able to prove them:

Theorem 6.3.4

(i) Associativity of set union and intersection:

$\boldsymbol{A \cup (B \cup C)} = (A \cup B) \cup C$ $\qquad$ $\boldsymbol{A \cap (B \cap C)} = (A \cap B) \cap C$

(ii) Commutativity: $\boldsymbol{A \cup B} = B \cup A$ $\qquad$ $\boldsymbol{A \cap B} = B \cap A$

(iii) Distributivity: $\boldsymbol{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$, $\qquad$ $\boldsymbol{A \cap (B \cup C)} = (A \cap B) \cup (B \cap C)$

(iv) De Morgan Laws: $\boldsymbol{(A \cup B)^c} = A^c \cap B^c$ $\qquad$ $\boldsymbol{(A \cap B)^c} = A^c \cup B^c$

(v) Complementation: $\boldsymbol{A \cup A^c} = \mathcal{U}$ $\qquad$ $\boldsymbol{A \cap A^c} = \emptyset$

(vi) Double complement: $\boldsymbol{(A^c)^c} = A$

Sample Proofs:

Use logical arguments from the definitions to show: (iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Proof:

① First, we'll prove that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C)$. We need to show that $x \in (A \cup B) \cap (A \cup C)$.

Since $x \in A \cup (B \cap C)$, $x \in A$ or $x \in B \cap C$, so we have two cases:

(i) If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so indeed $x \in (A \cup B) \cap (A \cup C)$

(ii) Else, if $x \notin A$ then $x \in B \cap C$, i.e. $x \in B$ and $x \in C$. But $x \in B \Rightarrow x \in A \cup B$, and $x \in C \Rightarrow x \in A \cup C$. Hence, again, $x \in (A \cup B) \cap (A \cup C)$

② Now we'll show the other set inclusion: $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Let $x \in (A \cup B) \cap (A \cup C)$. By definition of set intersection, $x \in A \cup B$ and $x \in A \cup C$

~~Thus~~ Since $x \in A \cup B$, $x \in A$ or $x \in B$. Again, we have two cases:

(i) If $x \in A$ then $x \in A \cup (B \cap C)$.

(ii) else, $x \in B$. But we also know that $x \in A \cup C$, so, if $x \notin A$ then $x \in C$. Hence $x \notin A \Rightarrow x \in B \cap C \Rightarrow x \in A \cup (B \cap C)$.

This shows that $x \in A \cup (B \cap C)$ in all cases. QED

➔ You try: prove the counterpart statement: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Use truth tables to prove: (iv) $(A \cup B)^c = A^c \cap B^c$

$x \in A$	$x \in B$	$x \in A \cup B$	$x \in (A \cup B)^c$	$x \in A^c$	$x \in B^c$	$x \in A^c \cap B^c$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

equivalent statements.

Proposition 6.2.4: $A \cup B = (A \cap B) \cup (A - B) \cup (B - A)$

➔ Prove it by truth tables (in text) and by logical arguments (exercise)