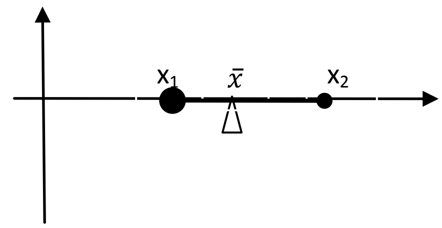
8.3 Center of Mass (Centroid)

The **center of mass** of a system of weights (or of a region in the plane) is the point $(\bar{x}, \bar{y})$ where the system balances.

Basic Case: The center of mass of a system of two weights connected by a thin rod along the x-axis, with mass m_1 at coordinate x_1 and m_2 at coordinate x_2, has x-coordinate:

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$



(we deduce this formula from Archimedes' Law of Lever, a.k.a. "the See-saw Law": $m_1 d_1 = m_2 d_2$)

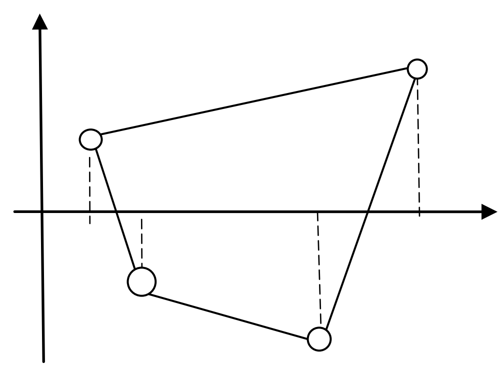
Generalized to multiple weights in the xy-plane: The center of mass of a system of n weights:
mass m_1 at point $P_1 = (x_1, y_1)$, … , m_n at point $P_n = (x_n, y_n)$ is the point with coordinates:

$$\begin{cases} \bar{\boldsymbol{x}} = \dfrac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} = \dfrac{\sum_{i=1}^{n} m_i x_i}{m} = \dfrac{\boldsymbol{M_y}}{\boldsymbol{m}} \\ \bar{\boldsymbol{y}} = \dfrac{m_1 y_1 + m_2 y + \cdots + m_n y_n}{m_1 + m_2 + \cdots + m_n} = \dfrac{\sum_{i=1}^{n} m_i y_i}{m} = \dfrac{\boldsymbol{M_x}}{\boldsymbol{m}} \end{cases}$$

The numerators $\boldsymbol{M_y} = \sum_{i=1}^{n} m_i x_i$ and $\boldsymbol{M_x} = \sum_{i=1}^{n} m_i y_i$ are called the **moments of the system about the y-axis** and **x-axis**, respectively, and measure the tendency of the system to rotate about each axis (i.e. how unbalanced the system is with respect to each axis).

Example: Compute the moments and the center of mass for the system:

$m_1 = 2$ at point $P_1 = (1,1)$
$m_2 = 6$ at point $P_2 = (2,-1)$,
$m_3 = 3$ at point $P_3 = (5,-2)$
$m_4 = 1$ at point $P_4 = (7,2)$



$$\boldsymbol{M_x} = \sum_{i=1}^{n} m_i y_i =$$

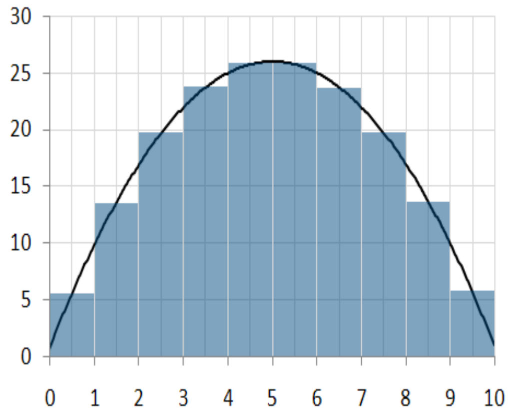
$$\boldsymbol{M_y} = \sum_{i=1}^{n} m_i x_i =$$

Total mass is $m = 2 + 6 + 3 + 1 = 12$.

Center of mass is at: $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ =**??**

Calculus Problem: compute the centroid of a lamina (flat plate), of <u>uniform</u> density ρ, occupying a region $\mathcal{R}$ in the plane.

a) **When $\mathcal{R}$ is the area under $y = f(x)$:**



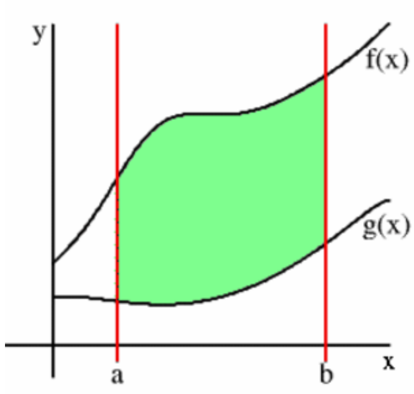
$$Area:\ A = \int_a^b f(x)\ dx, \qquad Total\ mass:\ m = \rho A$$

$$M_y = \rho \int_a^b x f(x)\, dx, \qquad M_x = \rho \int_a^b \frac{1}{2} f^2(x)\, dx$$

The density ρ cancels out, so the centroid is: $\overline{\boldsymbol{x}} = \frac{\boldsymbol{M_y}}{\boldsymbol{m}} = ??$ $\qquad \overline{\boldsymbol{y}} = \frac{\boldsymbol{M_x}}{\boldsymbol{m}} = ??$

Formulas:

b) **When $\mathcal{R}$ is the area bounded above by $y = f(x)$ and below by $y = g(x)$:**



Note: If $\mathcal{R}$ has a line of symmetry, the centroid lies along that line (so a center of symmetry is a center of mass too!)

Ex1: Find the centroid of the region bounded by the curves: $y = x + 2$ and $y = x^2$. (answer: $(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{8}{5}\right)$)

Ex 2: Find the center of mass of the triangle with vertices at points (-1, 0), (0,2) and (1,0). (answer: $(\bar{x}, \bar{y}) = \left(0, \frac{2}{3}\right)$)