Combining the limit laws, we see that for many functions (for instance, polynomials, powers, etc) we can compute the limit by just substituting. In fact, as long as the function is defined and "continuous" (ch 2.5), the limit can be computed by direct substitution. For example, compute:

\[
\lim_{x \to 2} \frac{x^2 + 3x - 7}{\sqrt{x} + 2} = \lim_{x \to 2} (x^2 + 3x - 7) = \lim_{x \to 2} (x^2) + \lim_{x \to 2} (3x) - \lim_{x \to 2} (7) = \ldots
\]

\[
= \frac{2^2 + 3(2) - 7}{2 + 2} = \frac{3}{2}.
\]

What to do when direct substitution fails?

First, recall that if the limit is of the form \(\frac{1}{0}\), then the limit does not exist and the one-sided limits are \(\pm\infty\). Else:

- Use algebra to simplify, factor, or rationalize the function until you can determine the limit
- Use the Squeeze Theorem (below) and compare your function to simpler ones
- (Later, in Ch 4.4 we’ll learn another method, called L’Hospital’s Rule)

Examples:

1) \(\lim_{x \to 5^-} \frac{-x + 2}{(x - 3)(x - 5)} = +\infty\)

This limit is of the type "non-zero numerator, zero denominator" so it will be either \(+\infty\) or \(-\infty\), depending on the sign of the numerator and whether we "divide" by \(0^+\) or \(0^-\).

When \(x \to 5^-\), the numerator \((-x + 2) \to -3\), while the denominator \((x - 3)(x - 5) \to 0^-\) (since \((x - 3) \to 2\) and \((x - 5) \to 0^-\)). So this limit is of the type "\((-3) \times \frac{1}{0^-} = (-3) \times (-\infty) = +\infty\)".

2) \(\lim_{x \to 2^-} \frac{x^2 - 4}{x - 2}\)

This limit is of the type \(\frac{0}{0}\), which is indeterminate. However, we can rewrite algebraically, which will allow us to compute its value. Recall that \(a^2 - b^2 = (a - b)(a + b)\), so \(x^2 - 4 = (x - 2)(x + 2)\).

\[
\lim_{x \to 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^-} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2^-} (x + 2) = 4.
\]

Note: It is not true that \(\frac{(x-2)(x+2)}{x^2-2} = x + 2\), so please don’t write it as a function equality (the function on the left is undefined at 2, while the one on the right is defined everywhere). However, as long as \(x - 2 \neq 0\) we can cancel the \(x - 2\) factor in the fraction. Since the limit as \(x \to 2\) only considers values of \(x\) arbitrarily close to 2, but not at 2 itself, it is OK to cancel \(x - 2\) in the fraction inside the limit, but not if you are writing it as a function, without the limit preceding it.

Hint on factorizations: note that \((x + a)(x + b) = x^2 + (a + b)x + ab\), so, conversely, any quadratic of the form \(x^2 + (a + b)x + ab\) factors as \((x + a)(x + b)\). For instance:

\[
x^2 - 2x - 8 = x^2 + (-4 + 2)x + (-4)(2) = (x - 4)(x + 2)
\]

(Alternatively, you can factor a quadratic by using the quadratic formula to find its roots.)
3) [2.3.21] \( \lim_{t \to 9} \frac{9 - t}{3 - \sqrt{t}} \)

This limit is also of the indeterminate type \( \frac{0}{0} \). One way to try simplifying it is to rationalize it (get rid of the square root in its denominator) by multiplying the denominator (and the numerator, of course) by \( 3 + \sqrt{t} \). This eliminates the square root in the denominator because \((a-b)(a+b) = a^2 - b^2\); so \((3 - \sqrt{t})(3 + \sqrt{t}) = 3^2 - (\sqrt{t})^2 = 9 - t \).

\[
\lim_{t \to 9} \frac{9 - t}{3 - \sqrt{t}} = \lim_{t \to 9} \left( \frac{9 - t}{3 - \sqrt{t}} \right) \left( \frac{3 + \sqrt{t}}{3 + \sqrt{t}} \right) = \lim_{t \to 9} \frac{(9 - t)(3 + \sqrt{t})}{(3 - \sqrt{t})(3 + \sqrt{t})} = \lim_{t \to 9} \frac{(9 - t)(3 + \sqrt{t})}{9 - t} = \lim_{t \to 9} (3 + \sqrt{t}) = 3 + \sqrt{9} = 6
\]

Warning: when writing your solution, do not drop the lim part until you actually computed the limit! Else you’re writing something incorrect.

The Squeeze Theorem

If your function can be "squeezed" or "sandwiched" between two functions that both approach the same limit \( L \) at point \( a \), then your function must also have limit \( L \) at \( a \).

Ex 1: Suppose you know that \( 3x \leq f(x) \leq x^3 + 2 \) near \( x = 1 \). What is \( \lim_{x \to 1} f(x) \)?

ANSWER: Since \( \lim_{x \to 1} (3x) = 3 = \lim_{x \to 1} (x^3 + 2) \), by the squeeze theorem \( f(x) \) must have the same limit. So \( \lim_{x \to 1} f(x) = 3 \).

This example was easy because the inequality was provided for you. In a more difficult question, you first need to figure out the inequality to use, like in the following example:

Ex 2: Compute \( \lim_{x \to 0} \left( |x| \sin \frac{1}{x} \right) \). (typical application for the Squeeze Thm)

First, think about what is going on:

1) what happens to \( |x| \) as \( x \to 0 \)? (Ans: \( |x| \to 0 \).)
2) What happens to \( \sin \frac{1}{x} \)? (Ans: It keeps cycling from -1 to 1.)
3) Can you bound ("squeeze") \( \sin \frac{1}{x} \) between two simple functions? (Ans: \( -1 \leq \sin \frac{1}{x} \leq 1 \).)

So, at all non-zero values of \( x \):

\[ -1 \leq \sin \frac{1}{x} \leq 1 \]

Multiplying this inequality by \( |x| \) and noting that \( |x| \) is positive for all nonzero \( x \), so it preserves the inequality direction, we get:

\[ -|x| \leq |x| \sin \frac{1}{x} \leq |x| \]

Since both \(-|x|\) and \(|x|\) go to 0 as \( x \to 0 \), so does our middle function. That is:

\[
\lim_{x \to 0} \left( |x| \sin \frac{1}{x} \right) = 0
\]