Announcements

Today: Chapter 20

Recall that the next homework (from Ch 18-21) is due on Friday.

No office hours for Dr. Nichifor today.
Instead, she will hold extended hours on Wed and Thu:

- Wednesday 1:30-3:30 in Padelford C-326
- Thu: 2:30-4 in MSC
- Or by appointment

Other sources of help:

- Michael’s office hrs Tue 10-11 or Thu 2-3 in MSC
- Matt’s office hrs Wed 10:30-11:30 in Padelford C-113, or Thu 10:30-11:30 in MSC
- MSC tutors, instructors and TAs
- CLUE

Note: The homework from Ch 20 is very short. If you’re done early, it will help to start looking at the Ch 21 before Wednesday’s lecture.
Chapter 20: Exponential Modeling

Since an exponential function (in standard form) is of the form $f(x) = A_0 b^x$, in order to find an exponential model we need to compute two unknowns: $A_0$ and $b$, so we need two data points.

**Example (it’s important to understand the procedure):**

The population of the United States was 203 million in 1970, and 250 million in 1990. Assume the population grows exponentially, and let $P(t)$ denote the US population, in millions, $t$ years after 1950.

a) Find the exponential function $P(t) = P_0 b^t$ which models the given data.

b) According to this exponential model, what was the US population in 1950? Compare your answer to the actual population in 1950, which was about 150 million.

c) What population does this model predict for the current US Population?

a) Given data: $P(20)=203$ (since 1970 is $t=20$ yrs after 1950), and $P(40)=250$

Plug the two data points into the model to get two equations:

\[
\begin{align*}
203 &= P_0 b^{20} \\
250 &= P_0 b^{40}
\end{align*}
\]

To solve a system of equations in two unknowns, you need to eliminate one of the unknowns. Easiest way to do this here is to divide 2nd equation by the first, to eliminate $P_0$:

\[
\frac{250}{203} = \frac{P_0 b^{40}}{P_0 b^{20}} = b^{40-20} = b^{20}
\]

\[
b^{20} = \frac{250}{203}
\]

Solve for $b$ by taking the appropriate root:

\[
(b^{20})^{1/20} = \left(\frac{250}{203}\right)^{\frac{1}{20}}
\]

$b \approx 1.010467148$

**WARNING: Do not round off the base $b$!**
Keep as many decimals as you can, or else your answer may be badly off!

Replace $b$ in the 1st eq: $203 = P_0 (1.010467148)^{20}$ so $P_0 = \frac{203}{(1.010467148)^{20}} \approx 164.836$

So the modeling function is: $P(t)=164.836(1.010467148)^t$.

b) $P(0)=P_0=164.836$ million. Model overestimated by about 14.836 million (9.9%).

c) Year 2007 is 57 years after year 1950, so: $P(57)=164.836(1.010467148)^{57} \approx 298.41$ million.

(Actual census data: 303.4 million, so the model underestimates by about 5 million (1.65%).)
COMPOUND INTEREST:
(this is an application of exponential functions that we encounter in daily life, so it helps to understand how it works)

Example: You invest $1000 in a bank account offering an annual interest percentage of 7%, compounded annually. Assuming no further deposits, how much money will you have in that account after t years?

Let P(t) be your balance after t years. We are asked to figure out a formula for this function, P(t).

- \( P(0)=1000 \)
- \( P(1)=\text{prev balance + interest applied to previous balance}=1000+0.07\times1000=$1070 \)
- \( P(2)=P(1)+0.07P(1)=1070+0.07\times1070=1070+74.9=$1144.9 \)
- Etc.

What’s the pattern?

- \( P(1)=P(0)+0.07P(0)=1\times P(0)+0.07\times P(0)=1.07P(0) \)
- \( P(2)=P(1)+0.07P(1)=1.07P(1)=1.07(1.07P(0))=(1.07)^2P(0) \)
- \( P(3)=P(2)+0.07P(2)=1.07P(2)=1.07(1.07)^2P(0)=1.07^3P(0) \)
- Etc.

The pattern is: \( P(t)=(1.07)^tP(0) \), where \( P(0) \) is the initial deposit (principal), 1.07 comes from 1 plus the annual interest rate in decimal form, and \( t \) is the number of years.

1) Compound Amount Formula (CAF): \( P(t)=P_0(1+r)^t \) where \( P_0 \) is the principal (amount deposited), \( r \) is the interest rate in decimal form, and \( t \) is the number of compounding periods.

More generally, if a bank offers an annual rate \( r \), compounded \( n \) times a year, they actually mean: rate \( r/n \) applied \( n \) times a year.

Ex: 6% annual rate compounded monthly actually means \( (6\%)/12=0.5\% \) interest applied to your balance each month (which ends up being more that 6% a year!).

The formula to use in such cases is a slight modification of the above:

2) Discrete Compound Formula: the balance after \( t \) years is:

\[
P(t)=P_0(1+r/n)^{nt}
\]

where \( P_0 \) is the principal, \( r \) is the quoted annual rate in decimal form, \( n \) is the number of compoundings per year, and \( t \) is the number of years.

Ex: You deposit $5000 in a bank account offering 10% annual interest rate. \( P_0=5000, r=0.1 \)

How much money is in your account after 20 years if the interest is compounded:

a) Annually? \( (n=1) \) \( P(20)=5000(1.1)^{20} = $33,637.5 \)

b) Quarterly? \( (n=4) \) \( P(20)=5000(1+0.1/4)^{4\times20}=5000(1.025)^{80}=$36,047.84 \)

c) Monthly? \( (n=12) \) \( P(20)=5000(1+0.1/12)^{12\times20}=36,640.37 \)

d) Daily? \( (n=365) \) \( P(20)=5000(1+0.1/365)^{365\times20}=$36,935.16 \)
It seems to be the case that the more often the interest is compounded, the more money we get. Can we keep improving the amount forever, or is there a max case?

In fact, there is a limiting case: If we assume that the interest is compounded continuously (i.e., \( n \) is infinitely large), we don’t get an infinite amount of money. Instead, we get:

3) **Continuously Compounded Interest**: \( P(t) = P_0 e^{rt} \), where \( P_0 \) is the principal, \( r \) is the annual rate in decimal form, \( t \) is the number of years, and \( e \) is a special mathematical constant.

Type \( e^{1} \) in your calculator. What value do you get? \( \approx 2.718281828 \ldots \)

The constant \( e \) is an irrational number, very much like \( \pi \) (that is, cannot be written as a fraction and has infinitely many decimal digits.)

The number \( \pi \) is special because of its relationship to circles (ratio of circumference to diameter).

The number \( e \) is special for many different reasons, some of which you’ll see in Math 124. It comes up in continuously compounded interest because it’s the limit (horiz. asymptote) of the function \((1+1/x)^x\) as \( x \) gets larger and larger. More details in textbook, if you’re curious.

Basically: if we take larger and larger \( n \), the “piece” \((1+r/n)^n\) from the discretely compounded interest formula “approaches” \( e^r \), so the entire formula: \( P_0(1+r/n)^n \) behaves, asymptotically, like the formula \( P_0e^{rt} \)

**Example**: In the previous example, how much money do you have after 20 years if the interest is compounded continuously?

\[
P(20) = 5000e^{0.1*20} = 36,945.28
\]

**Extra Q**: What is the real (effective) annual interest rate in a bank acct offering 10% per year, continuously compounded?

\[
P(1) = P_0 e^{0.1*1} = P_0(1.10517\ldots).
\]

Comparing to the discrete formula with \( n=1 \): \( P(1) = P_0(1+r)^1 \), we get that \( 1+r=1.10517\ldots \), so:

\[
r = 0.10517\ldots \text{ i.e. about 10.517%}
\]

This means that 10% per year compounded continuously is equivalent to 10.517% per year, compounded annually.

Alternative method:

the % change in 1 year is \[
\frac{P(1)-P(0)}{P(0)} = \frac{P_0 e^{0.1}-P_0}{P_0} = \frac{P_0(e^{0.1}-1)}{P_0} = \frac{e^{0.1}-1}{1} \approx 0.10517, \text{ so about 10.517%}.
\]