Winter 2008 (Nichifor), Lectures A&B, version 2

1) (12 points) Evaluate the indicated derivatives of the following functions. **Do not simplify**.

a) $g(t) = \frac{-t^2+7}{3t+1}$

$$g'(t) = \frac{(-2t)(3t+1) - (-t^2+7)(3)}{(3t+1)^2}$$

(Quotient Rule)

b) $z = e^y \ln y \sqrt{y+1}$

Group any two parts together and apply the Product Rule. I will group the first two functions together:

$$\frac{dz}{dy} = (e^y \ln y)' \sqrt{y+1} + e^y \ln y \left(\sqrt{y+1}\right)' =$$

$$= \left(e^y \ln y + e^y \frac{1}{y}\right)\sqrt{y+1} + e^y \ln y \left(\frac{1}{2}(y+1)^{-1/2}\right)$$

c) $f(x) = [3 + (\ln x)^2]^7$

This is a double Chain Rule.

$$f'(x) = 7\left[3 + (\ln x)^2\right]^6 \left[3 + (\ln x)^2\right]'$$

$$= 7\left[3 + (\ln x)^2\right]^6 \left[0 + 2(\ln x)(\ln x)'\right]$$

$$= 7[3 + (\ln x)^2]^6 \left[2(\ln x)\left(\frac{1}{x}\right)\right]$$

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2. (14 points) Consider the function $z = f(x, y) = -x^3 + 27xy - 10y^2 + 2y + 15$.

a) (4pts) Write out the two partial derivatives, $f_x(x, y)$ and $f_y(x, y)$. You need not show work.

$$\boldsymbol{f_x(x, y) = -3x^2 + 27y}$$

$$\boldsymbol{f_y(x, y) = 27x - 20y + 2}$$

b) (5 pts) Find the largest value of the function $f(x, 1)$ over the interval from $x = 0$ to $x = 4$.

Show all steps.

First, compute $h(x) = f(x, 1) = -x^3 + 27x + 7$. Then:

1. **Find the Critical Points:**

   set $h'(x) = f_x(x, 1) = -3x^2 + 27 = 0$ and solve for x to get $x = \pm 3$.
2. **Evaluate $h(x) = f(x, 1)$ at all CPs within the given interval, i.e. at $x = 3$**

   $h(3) = f(3, 1) = -(3)^3 + 27(3) + 7 = 61$.
3. **Evaluate $h(x) = f(x, 1)$ at the endpoints of the interval, $x = 0$ & $x = 4$**

   $f(0, 1) = 7$

   $f(4, 1) = 51$

The largest value is 61, at x=3.

Answer: _________**61**______________

d) (5 pts) Which graph is steeper:

i. the graph of the function $f(x, 1)$ at $x = 3$ OR

ii. the graph of the function $f(2, y)$ at $y = 2$?

Show all work.

The first is a function of x only, and the second is a function of y only, so to compare their steepness (slopes), we can compare the corresponding partial derivatives at the given points:

$$\boldsymbol{i.\ f_x(3, 1) = -3(3)^2 + 27(1) = 0}$$

$$\boldsymbol{ii.\ f_y(2, 1) = 27(2) - 20(2) + 2 = 16}$$

Answer: __________**(ii)**__________ is steeper

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3. (12 points)

Suppose that in order to achieve monthly sales of q thousand Items you have to sell your Items at a **price**

$$\boldsymbol{p(q) = q^2 - 45q + 500} \text{ (dollars per Item)}$$

a) Determine all quantities for which your demand curve $p(q)$ is decreasing and not negative.

Justify your answer.

The price is a quadratic function, whose graph is concave-up. There are two things to check:

1) **Decreasing: A concave-up parabola decreases before its vertex, so, in this case, from 0 to 22.5.**
2) **Not-negative: A concave-up parabola is negative between its roots, and positive outside of its roots. Using the Quadratic Formula or factorization to solve $q^2 - 45q + 500 = 0$, we get that the roots are q=20 and q=25.**

Answer: from q =__**0**__ to q =__**20**___ thousand Items.

b) Find all the critical points for your **Total Revenue** function. Round your answers to 2 decimal digits.

$$TR(q) = q(q^2 - 45q + 500) = q^3 - 45q^2 + 500q$$

The critical points (numbers) are the zeroes of the derivative: $TR'(q) = 3q^2 - 90q + 500$

Use the Quadratic Formula to solve $3q^2 - 90q + 500 = 0$ **.**

Answer: TR has critical points at q = ______**7.36 and 22.64**_____thousand Items

c) Use the Second Derivative Test to determine whether each of the critical points you found in part (b) is a local maximum or a local minimum for the total revenue. Show all work and circle your answers.

$$TR''(q) = 6q - 90$$

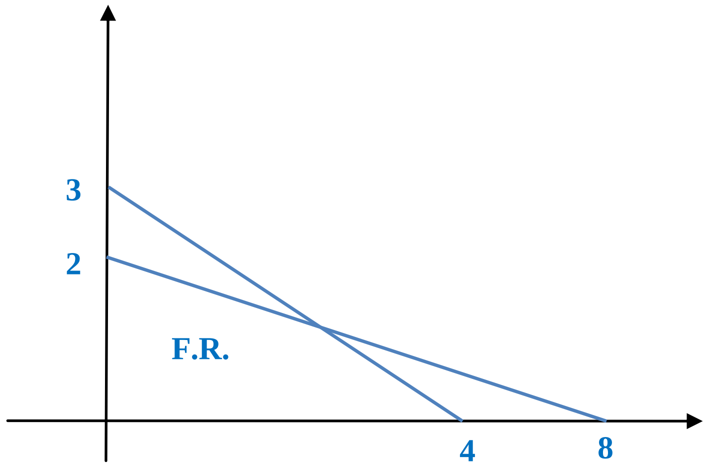
$$TR''(7.36) = 6(7.36) - 90 < 0$$

$$TR''(22.64) = 6(22.64) - 90 > 0$$

Hence, TR has a local max at q=7.36 and a local min at q=22.64.

4. (12 pts) The constraints for a linear programming problem are: $3x + 4y \leq 12$ and $x + 4y \leq 8$.

a) Sketch the two constraints and label the feasible region.



$3x + 4y = 12$
y-intercept: x=0, y=3
x-intercept: y=0, x=12/3=4
(0,3) & (4,0)

x+4y=8
y-intercept: x=0, y=2
x-intercept: y=0, x=8
(0,2) & (8,0)

b) Find the coordinates (x, y) of all the vertices of the feasible region.

To get the x and y coordinates of the 4th vertex, we need to solve the system: $\begin{cases} 3x + 4y = 12 \\ x + 4y = 8 \end{cases}$

From the 2nd equation: $x = -4y + 8$. Substituting in the first:

$$3(-4y + 8) + 4y = 12$$
$$-12y + 24 + 4y = 12$$
$$-8y = -12$$
$$y = \frac{3}{2} = 1.5$$
$$x = -4\left(\frac{3}{2}\right) + 8 = 2$$

Answer: $(x, y) =$ **(0.0), (0,2), (4,0), (2,1.5)** *(list all vertices)*

(c) Subject to the given constraints, find the maximum value of the following objective function:

$$f(x, y) = 6x - 3y$$

Evaluate the objective function at all vertices of the feasible region.

$f(0,0) = 0$

$f(0,2) = -6$

$f(4,0) = 24$

$f(2,1.5) = 6(2) - 3(1.5) = 7.5$

Answer: max $f(x, y) =$____**24**______