Winter 2008 (Nichifor), Lectures A&B, version 1

1) (12 points) Evaluate the indicated derivatives of the following functions. **Do not simplify**.

a) $g(t) = \frac{t^2+2}{5t+7}$

$$g'(t) = \frac{(2t)(5t+7) - (t^2+2)(5)}{(5t+7)^2}$$

(Quotient Rule)

b) $y = e^z \ln z \sqrt{z+1}$

Group any two parts together and apply the Product Rule. I will group the first two functions together:

$$\frac{dy}{dz} = (e^z \ln z)' \sqrt{z+1} + e^z \ln z \; (\sqrt{z+1})'$$

$$= \left(e^z \ln z + e^z \frac{1}{z}\right)\sqrt{z+1} + e^z \ln z \left(\frac{1}{2}(z+1)^{-1/2}\right)$$

c) $f(x) = [1 + (\ln x)^3]^5$

This is a double Chain Rule.

$$f'(x) = 5[1 + (\ln x)^3]^4 [1 + (\ln x)^3]' =$$

$$= 5[1 + (\ln x)^3]^4 [0 + 3(\ln x)^2 (\ln x)']$$

$$= 5[1 + (\ln x)^3]^4 \left[3(\ln x)^2 \frac{1}{x}\right]$$

2. (14 points) Consider the function $z = f(x, y) = -x^3 + 12xy - 4y^2 + 3y + 15$.

a) (4pts) Write out the two partial derivatives, $f_x(x, y)$ and $f_y(x, y)$. You need not show work.

$$\boldsymbol{f_x(x, y) = -3x^2 + 12y}$$

$$\boldsymbol{f_y(x, y) = 12x - 8y + 3}$$

b) (5 pts) Find the largest value of the function $f(x, 1)$ over the interval from $x = 0$ to $x = 3$. Show all steps.

First, compute $h(x) = f(x, 1) = -x^3 + 12x + 14$. Then:

1. **Find the Critical Points:**
   set $h'(x) = f_x(x, 1) = -3x^2 + 12 = 0$ and solve for x to get $x = \pm 2$.
2. **Evaluate $f(x, 1)$ at all CPs within the given interval, i.e. at $x = 2$**
   $f(2, 1) = -(2)^3 + 12(2) + 14 = 30.$
3. **Evaluate f(x,1) at the endpoints of the interval: $x = 0$ & $x = 3$**
   $f(0, 1) = 14$
   $f(3, 1) = -3^3 + 12(3) + 14 = 23$

The largest value is 30, at x=2.

Answer: ______**30**________________

d) (5 pts) Which graph is steeper:

i. the graph of the function $f(3, y)$ at $y = 4$ OR

ii. the graph of the function $f(x, 1)$ at $x = 2$?

Show all work.

The first is a function of y only, and the second is a function of x only, so to compare their steepness (slopes), we can compare the corresponding partial derivatives at the given points:

$$\boldsymbol{i.\ f_y(3, 4) = 12(3) - 8(4) + 3 = 7}$$

$$\boldsymbol{ii.\ f_x(2, 1) = -3(2)^2 + 12(1) = 0}$$

Answer: ________**(i)**______________ is steeper

3. (12 points)

Suppose that in order to achieve monthly sales of q thousand Items you have to sell your Items at a **price**

$$\boldsymbol{p(q) = q^2 - 25q + 150} \text{ (dollars per Item)}$$

a) Determine all quantities for which your demand curve $p(q)$ is decreasing and not negative. Justify your answer.

The price is a quadratic function, whose graph is concave-up. There are two things to check:

1) **Decreasing: A concave-up parabola decreases before its vertex, so, in this case, from 0 to 12.5.**
2) **Not-negative: A concave-up parabola is negative between its roots, and positive outside of its roots. Using the Quadratic Formula or factorization to solve $q^2 - 25q + 150 = 0$, we get that the roots are q=10 and q=15.**

Answer: from q =__0____ to q =___10____ thousand Items.

b) Find all the critical points for your **Total Revenue** function. Round your answers to 2 decimal digits.

$$TR(q) = q(q^2 - 25q + 150) = q^3 - 25q^2 + 150q$$

The critical points (numbers) are the zeroes of the derivative: $TR'(q) = 3q^2 - 50q + 150$

Use the Quadratic Formula to solve $3q^2 - 50q + 150 = 0$.

Answer: TR has critical points at q = _**3.92 & 12.74**_____thousand Items

c) Use the Second Derivative Test to determine whether each of the critical points you found in part (b) is a local maximum or a local minimum for the total revenue. Show all work and circle your answers.

$$TR''(q) = 6q - 50$$

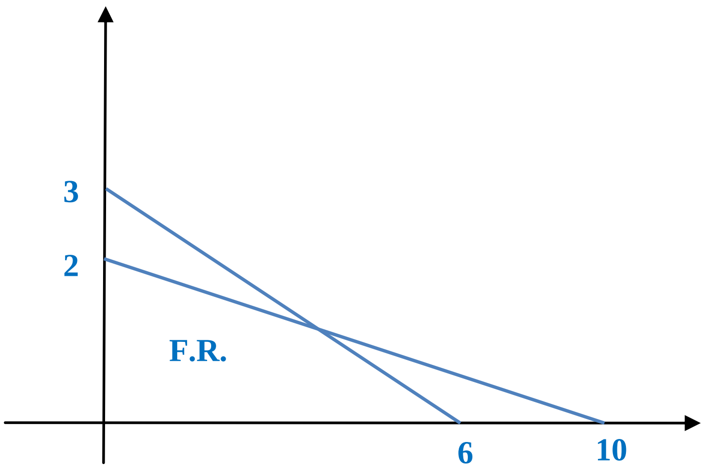
$$TR''(3.92) = 6(3.92) - 50 < 0$$

$$TR''(12.74) = 6(12.74) - 50 > 0$$

Hence, TR has a local max at q=3.92 and a local min at q=12.74.

4. (12 pts) The constraints for a linear programming problem are: $2x + 4y \leq 12$ and $x + 5y \leq 10$.

a) Sketch the two constraints and label the feasible region.



$2x + 4y = 12$
y-intercept: x=0, y=3
x-intercept: y=0, x=12/2=6

x+5y=10
y-intercept: x=0, y=2
x-intercept: y=0, x=10

b) Find the coordinates (x, y) of all the vertices of the feasible region.

To get the x and y coordinates of the 4th vertex, we need to solve the system: $\begin{cases} 2x + 4y = 12 \\ x + 5y = 10 \end{cases}$

From the 2nd equation: $x = -5y + 10$. Replacing in the first:

$$2(-5y + 10) + 4y = 12$$
$$-10y + 20 + 4y = 12$$
$$-6y = -8$$
$$y = \frac{4}{3} \cong 1.33$$
$$x = -5\left(\frac{4}{3}\right) + 10 = \frac{10}{3} \cong 3.33$$

Answer: $(x, y) =$ **(0.0), (0,2), (6,0), (3.33, 1.33)** *(list all vertices)*

(c) Subject to the given constraints, find the maximum value of the following objective function:

$$f(x, y) = 6x - 3y$$

Evaluate the objective function at all vertices of the feasible region.

$f(0,0) = 0$

$f(0,2) = -6$

$f(6,0) = 36$

$f\left(\frac{10}{3}, \frac{4}{3}\right) = 6\frac{10}{3} - 3\frac{4}{3} = 16$

Answer: max $f(x, y) =$____**36**_____