W/S 24: FUNDAMENTAL THEOREM OF CALCULUS (FTC)

FTC: $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$ for any anti-derivative $F(x)$ of $f(x)$

Example: $\int_1^3 x^2\,dx = ?$

$f(x) = x^2$

$F(x) = \frac{x^3}{3}$ is an anti-derivative of $f(x)$

FTC says: $\int_1^3 x^2\,dx = F(3) - F(1)$

$= \frac{3^3}{3} - \frac{1^3}{3}$

$= \frac{27}{3} - \frac{1}{3} = \boxed{\frac{26}{3}}$

Notation: $F(x)\Big|_a^b$ means: $F(b) - F(a)$

ex: $\frac{x^3}{3}\Big|_1^3 = \frac{3^3}{3} - \frac{1^3}{3}$.

So: $\int_1^3 x^2\,dx \underset{FTC}{=} \frac{x^3}{3}\Big|_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = \frac{27}{3} - \frac{1}{3} = \boxed{\frac{26}{3}}$

(makes it shorter to show your work)

"Fundamental" because it relates the 2 main concepts of Calculus:

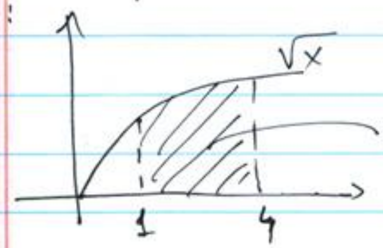
1) DERIVATIVES (anti-derivatives)

&

2) INTEGRALS (Areas under graphs)

An important application: we can now compute weird areas precisely (and without counting squares!)

Examples:

① 

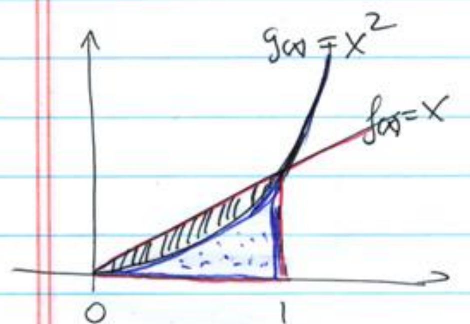
$$\text{area} = \int_1^4 \sqrt{x}\,dx$$

$$= \int_1^4 x^{1/2}\,dx$$

$$\overset{FTC}{=} \frac{x^{3/2}}{3/2}\Big|_1^4 = \frac{2}{3}x^{3/2}\Big|_1^4$$

$$= \frac{2}{3}4^{3/2} - \frac{2}{3}1^{3/2}$$

$$= \frac{16}{3} - \frac{2}{3} = \boxed{\frac{14}{3}}$$

② 

Find area between x and x^2 from $x=0$ to $x=1$

Method 1 (Longer)

$$\begin{bmatrix}\text{Area}\\\text{between}\\f \,\&\, g\end{bmatrix} = \begin{bmatrix}\text{Area}\\\text{below}\\f(x)\end{bmatrix} - \begin{bmatrix}\text{Area}\\\text{below}\\g(x)\end{bmatrix}$$

$$= \int_0^1 x\,dx - \int_0^1 x^2\,dx$$

$$= \frac{x^2}{2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1$$

$$= \left[\frac{1^2}{2} - \frac{0^2}{2}\right] - \left[\frac{1^3}{3} - \frac{0^3}{3}\right]$$

$$= \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}$$

Method II (faster)

$$\boxed{\begin{bmatrix}\text{Area}\\ \text{between}\\ \text{2 fcts}\end{bmatrix} = \int_a^b (\text{top fct}) - (\text{bottom fct})\, dx}$$

Here: area $= \int_0^1 x - x^2\, dx$

An antiderivative of $x - x^2$ is $\frac{x}{2} - \frac{x^3}{3}$, so applying FTC:

(*)
$$\int_0^1 x - x^2\, dx = \left[\frac{x}{2} - \frac{x^3}{3}\right]\Big|_0^1$$
$$= \underbrace{\left(\frac{1}{2} - \frac{1^3}{3}\right)}_{\text{plug in upper number}} - \underbrace{\left(\frac{0}{2} - \frac{0^3}{3}\right)}_{\text{plug in lower number}}$$
$$= \frac{1}{2} - \frac{1}{3} = \boxed{\frac{1}{6}}.$$

③ Compute $\int_1^e \frac{1}{x}\, dx$

An antiderivative of $\frac{1}{x}$ is $\ln x$ so:

$$\int_1^e \frac{1}{x}\, dx = \underbrace{\ln x \Big|_1^e}$$

[recall: this means: $\ln x$ evaluated at e minus $\ln x$ evaluated at 1
so $F(e) - F(1)$ for $F(x) = \ln x$]

$$\int_1^e \frac{1}{x}\, dx = \ln x \Big|_1^e = \ln e - \ln 1 = 1 - 0 = \boxed{1}$$

IMPORTANT EXAMPLE !

(4) Suppose $MR(q) = -q^2 + 32q + 190$
$MC(q) = 19q + 50$
and your fixed costs are \$500.

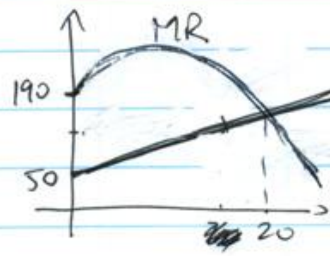
a) What is your maximal profit?

First: max profit occurs when $MR = MC$, switching from $MR > MC$ to $MR < MC$.

$$-q^2 + 32q + 190 = 19q + 50$$

$$-q^2 + 13q + 140 = 0.$$ Using Quadr. Formula:

$$q = -7 \ \& \ q = 20$$



Check: $MR > MC$ before $q = 20$
$MR < MC$ after

So: $\boxed{\text{max profit at } q = 20}$.

Second: We need to compute $P(20)$
(where $P(q)$ = profit at q)

Method 1: Find a formula for $P(q)$, then evaluate at 20.

LONGER

* TR is the antiderivative of MR which satisfies $TR(0)=0$.

$$\int MR\,dq = \int -q^2+32q+190\,dq$$
$$= -\frac{q^3}{3}+16q^2+190q+C$$

at $q=0$: $-\frac{0^3}{3}+16\cdot 0^2+190\cdot 0+\boxed{C=0}$

so: $\boxed{TR(q) = -\frac{q^3}{3}+16q^2+190q}$

* TC is the anti-derivative of MC which satisfies $TC(0)=FC$ $=500$.

$$\int MC\,dq = \int 19q+50\,dq = \frac{19}{2}q^2+50q+C$$

at $q=0$: $\frac{19}{2}(0^2)+50\cdot(0)+C = 500$

so $C=500$.

So: $\boxed{TC(q) = \frac{19}{2}q^2+50q+500}$

* $P(q) = TR(q) - TC(q)$
$$=\left(-\frac{q^3}{3}+16q^2+190q\right) - \left(\frac{19}{2}q^2+50q+500\right)$$

$$\boxed{P(q) = -\frac{q^3}{3}+6.5q^2+140q-500}$$

$$P(20) = -\frac{(20)^3}{3}+6.5(20)^2+140(20)-500 \cong 2233.33$$

So: $\boxed{\text{max profit is } \$2{,}233.33 \text{ at } q=20.}$ *

Method 2 (shorter)

Recall: $P = \overbrace{\begin{bmatrix}\text{Area between}\\ \text{MR \& MC}\\ \text{with MR > MC}\end{bmatrix} - \begin{bmatrix}\text{Area between}\\ \text{MR \& MC}\\ \text{with MR < MC}\end{bmatrix}}^{TR - VC} - FC$

So: $$\boxed{P(20) = \left(\int_0^{20} MR - MC \, dq \right) - FC}$$

$$P(20) = \left(\int_0^{20} \overbrace{(-q^2 + 32q + 190)}^{MR} - \overbrace{(19q + 50)}^{MC} \, dq \right) - 500$$

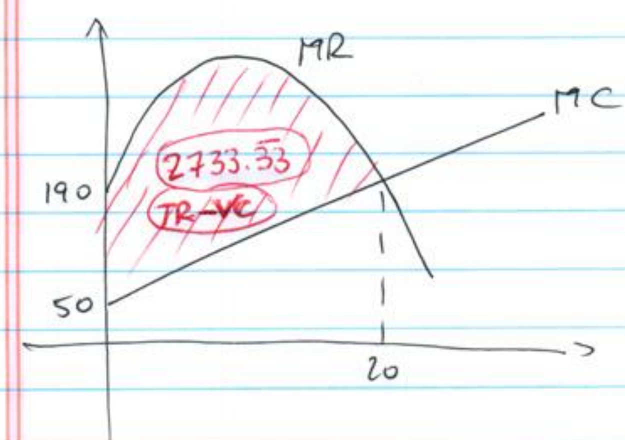
$$= \left(\int_0^{20} -q^2 + 13q + 140 \, dq \right) - 500$$

$$\stackrel{FTC}{=} -\frac{q^3}{3} + 13\frac{q^2}{2} + 140q \Big|_0^{20} - 500$$

$$= \left[-\frac{(20)^3}{3} + 13\frac{(20)^2}{2} + 140(20) \right] - \left[-\frac{0^3}{3} + 13\frac{0^2}{2} + 140(0) \right] - 500$$

$$= [2733.\overline{33}] - [0] - 500$$

$$= \boxed{\$2233.\overline{33}}$$



negative profit

Bonus Q: What is your profit (or loss) at $q = 30$?

$$P(30) = \int_0^{30} MR - MC \, dq \; - \; FC$$

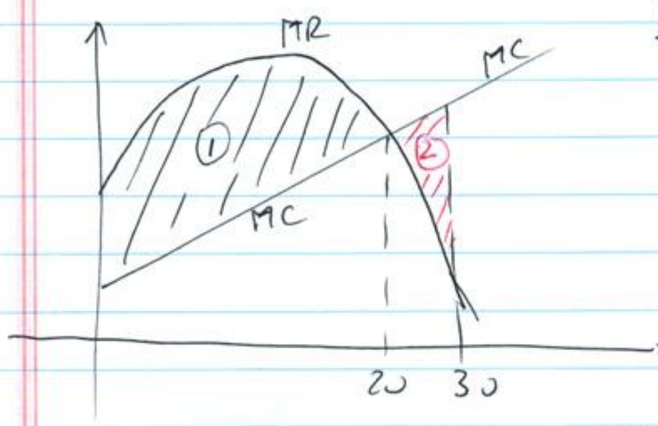
$$= \int_0^{30} -q^2 + 13q + 140 \, dq \; - \; FC$$

$$= \left(-\frac{q^3}{3} + 13\frac{q^2}{2} + 140q \right) \Big|_0^{30} \; - 500$$

$$= \underbrace{\left(-\frac{30^3}{3} + 13\frac{30^2}{2} + 140(30) \right)}_{1050} - \underbrace{\left(-\frac{0^3}{3} + 13\frac{0^2}{2} + 140(0) \right)}_{0} - 500$$

$$= 1050 \; - \; 0 \; - 500$$

$= \boxed{\$550}$ profit at $q = 30$.



Note:

$$\int_0^{30} MR - MC \, dq = 1050$$

represents area 1 (MR > MC)

minus area 2 (MR < MC)

i.e. $2733.\overline{33} - 1683.\overline{33} = 1050$

This gives TR - VC at $q = 30$.