1) (15 points) Evaluate the indicated derivatives of the following functions. **Do not simplify**.

Note: All these questions can be answered in more than one correct way, if you first rewrite the function. What I'm listing below are the most "obvious" solutions.

a) $f(x) = (x^2 + 2x + 5 + \frac{1}{x})(x^3 + e^x)$

$$f'(x) = \left(2x + 2 - x^{-2}\right)\left(x^3 + e^x\right) + \left(x^2 + 2x + 5 + \frac{1}{x}\right)(3x^2 + e^x)$$

b) $h(z) = \ln(\sqrt{z+4})$

$$h'(z) = \frac{1}{\sqrt{z+4}}\left(\frac{1}{2}(z+4)^{-1/2}\right)$$

c) $y = \frac{t^5 e^t}{-t^2 + 5}$

$$\frac{dy}{dt} = \frac{\left(5t^4e^t + t^5e^t\right)\left(-t^2 + 5\right) - t^5e^t(-2t)}{(-t^2 + 5)^2}$$

2. (15 points) Consider the function $f(x,y) = -3x^2 + 6xy - y^3 + 5$.

a) Write out the two partial derivatives, $f_x(x,y)$ and $f_y(x,y)$. You need not show work, but do work carefully!

$f_x(x,y) = \mathbf{-6x + 6y}$

$f_y(x,y) = \mathbf{6x - 3y^2}$

b) Find **all** values of (x,y) that are candidates for local maxima or local minima.

Set the two partial derivatives equal to zero, then solve the resulting system of two equations in two unknowns:

$$\begin{cases} -6x + 6y = 0 \\ 6x - 3y^2 = 0 \end{cases}$$

From the first equation we get:

$$-6x + 6y = 0$$

$$-x + y = 0$$

$$x = y$$

Replacing in the 2nd equation: $6y - 3y^2 = 0$

Now factor (or use the Quadratic Formula): $3y(2 - y) = 0$ to get**:** $y = 0$ ***or*** $y = 2$

So, there are 2 candidates **(x,y) = (0,0) or (2,2)**

Answer: (x,y)=_______ **(0,0) or (2,2)**_______(*list all*)

c) Use partial derivatives to estimate the value of: $\frac{f(1.0001,0)-f(1,0)}{0.0001}$.

This is expression is of the form $\frac{\Delta f}{\Delta x}$, = the slope of secant line to the function obtained by setting y=0, when x changes from x=1 to x=1+0.0001.

It can be approximated by slope of the tangent line at (1,0), when y is held fixed and x varies, so by the partial derivative with respect to x:

$$f_x(1,0) = -6(1) + 6(0) = -6$$

Answer: $\frac{f(1.0001,0)-f(1,0)}{0.0001} \approx \mathbf{-6}$

3. (20 points)

You sell Things.

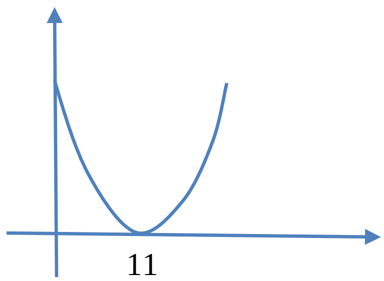
The demand curve is given by the following formula: to sell q thousand Things, you have to sell them at a **price** of $\boldsymbol{p(q) = q^2 - 22q + 121}$ dollars per Thing.

Thus, your Total Revenue is given by $\boldsymbol{TR} = pq = \boldsymbol{q^3 - 22q^2 + 121q}$ (in thousands of dollars.)

a) List all quantities q for which your demand curve make sense (i.e, the price p is decreasing and is not negative).

Our function, p(q), is a concave-up quadratic, so

- **it's decreasing before its vertex :** $\boldsymbol{q = \frac{22}{2} = 11}$**,**
- **and it's positive outside its roots. Using the Quadratic Formula (or factorization) , we get that both its roots are q=11.**



NOTE: For full credit, one needs to justify the answer either in words, as above, or at least by drawing the graph.

Answer: from q=__**0**_____ to q=____**11**_____ thousand Things.

(Recall that q is in thousands so q=1 means 1000 Things.)

b) Find all the critical numbers for your Total Revenue function.

$\boldsymbol{TR(q) = q^3 - 22q^2 + 121q}$

$\boldsymbol{TR'(q) = 3q^2 - 44q + 121}$

Set $\boldsymbol{3q^2 - 44q + 121}$ **=0, and solve for q via the** Quadratic Formula: $\boldsymbol{q \approx 3.67}$ **&** $\boldsymbol{q = 11}$**.**

[Note: since q is in thousands, if you want to be precise you should really not round off this much, but leave it as q=3.667 thousand Things, i.e. 3667 Things]

Answer: TR has critical numbers at q= _____ $\boldsymbol{3.67}$ **&** $\boldsymbol{11}$ _________.

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Recall: to sell q thousand Things, your price per Thing is $\boldsymbol{p(q) = q^2 - 22q + 121}$ dollars,

and your Total Revenue is: $\boldsymbol{TR(q) = q^3 - 22q^2 + 121q}$ thousands of dollars

c) Use the Second Derivative Test to determine whether each of the critical numbers you found in part (b) is a local maximum or a local minimum. Show all work and circle your answers.

$$TR'(q) = 3q^2 - 44q + 121$$

$$TR''(q) = 6q - 44$$

Evaluating the second derivative at the two critical points we found above:

$$TR''(3.67) = 6(3.67) - 44 < 0$$

$$TR''(11) = 6(11) - 44 > 0$$

Hence, according to the 2nd Derivative Test,

$q \approx 3.67$ is a local maximum for TR
$q = 11$ is a local minimum for TR

d) Suppose you sell quantities between 1,000 and 10,000 Things (so q is between 1 and 10). What is the price per Thing that you must charge in order to maximize your Total Revenue?

Which q between 1 and 10 maximizes TR?

Step 1: Find the critical points (CPs) of the function.

We already did this above. The only CP inside the given interval is $q \approx 3.67$

(technically, should use q=3.667 i.e. the nearest whole number of Things to our CP)

Step 2: Evaluate the function at the CP:

$$TR(3.67) = (3.67)^3 - 22(3.67)^2 + 121(3.67) \approx 197.19$$

Step 3: Evaluate the function at the endpoints:

$$TR(1) = (1)^3 - 22(1)^2 + 121(1) = 100$$

$$TR(10) = (10)^3 - 22(10)^2 + 121(10) = 10$$

Step 4: So: Max TR is at $q \approx 3.67$.

Computing the price, $p(3.67) = (3.67)^2 - 22(3.67) + 121 \approx \$53.73$

(more precisely: $p(3.667) = (3.667)^2 - 22(3.667) + 121 \approx \$53.77$)

Answer: To maximize your TR, you should charge p ≈ **$\$53.73$** dollars per Thing.