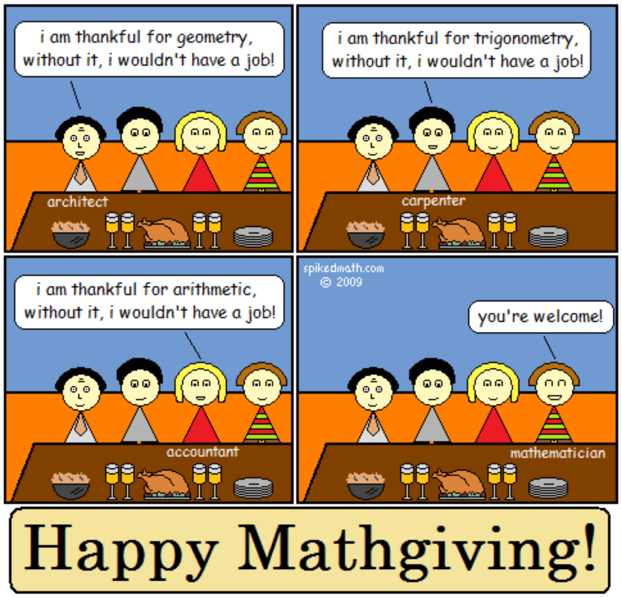
Monday, November 19, 2012

Today: 6.2 Geometric Sequences & Compound Interest

Office hours: Today: 3-4 in PDL C-326 & CDH 109 1:30-2 (if there are questions)
Tuesday 10-11 PDL C-326 & **2:30-330 in CMU B-006**.

To do: **Section 6.1 is due Tuesday night.**

Print & bring Activity 08 to quiz section on TUES



Recall from last time:

A sequence is called ARITHMETIC (additive) if the next term can be gotten from the previous one by **always adding the same amount $\boldsymbol{d}$**, called "the common difference" or the increment.

Then the n-th term is: $\boldsymbol{a_n = a_1 + (n-1)d}$
where n-1 is the number of times the common difference is added.

For instance:
If $\$P$ are invested at a rate of $r \times 100\%$ in simple interest, then the interest is always $\$rP$
The balances then form an arithmetic sequence with
$a_1 = P$, and common difference $d = rP$
and the balance after the interest is applied t times is:

$$S = P + (Pr)t$$

Ex: Suppose you invest $800 at an annual simple interest rate of 7%.

Then each year you earn $d = rP = 0.07 \times \$800 = \$56$. This is the common difference.

Your balance in year n (after n-1 years) is the principal $800,
plus the interest $56 added n-1 times: $S_n = P + (rP)(n-1)$

year 1: $S_1 = \$800$
year 2: $S_2 = \$800 + \$56 = \$856$
year 3: $S_3 = \$800 + (\$56) * 2 = \$912$
...etc...

$S_3 - S_2 = \$56 \leftarrow d$
$S_{100} - S_{99} = \$56 \leftarrow d$

6.2: Geometric Sequences

A sequence is called GEOMETRIC (multiplicative) if the next term can be gotten from the previous one by **always MULTIPLIED by the same amount m**, called "the common ratio" (or the multiplier)

Ex: 5, 10, 20, 40, ... $\leftarrow m=2$ $\quad \frac{10}{5}=2, \frac{20}{10}=2, \ldots$

Then the n-th term is: $\boxed{a_n = m^{n-1} a_1}$

$a_1 = m^0 a_1 = 5$
$a_2 = m a_1 = 2(5)$
$a_3 = m^2 a_1 = 2^2(5)$
$a_4 = m^3 a_1 = 2^3(5)$
...
$a_n = m^{n-1} a_1$

where n-1 is the number of times the common ratio is multiplied (number of steps).

ex: In the sequence 5, 10, 20, 40, ..., what is $a_{20} = (2^{19})5 = 2{,}621{,}440$

Application:
If $\$P$ are invested at a rate of $r \times 100\%$ in **COMPOUND** interest, then the interest is applied to the entire balance.
The balances then form an **geometric** sequence with common ratio $m = 1 + r$
and the balance after the interest is compounded n times is:

$$\boxed{S = (1+r)^n P}$$

Ex: **Suppose you invest \$800 at an interest rate of 7%, compounded annually.** $\quad r = 0.07 = \frac{7}{100}$

Then the common ratio is: $m = 1 + r = 1.07$

Your balance in year n (after n-1 years) is:

year 1: $S_1 = \$800$
year 2: $S_2 = \$800 + 0.07 * \$800 = (1.07) * \$800 = \$856$
year 3: $S_3 = \$856 + 0.07 * (\$856) = (1.07) * \$856 = (1.07)(1.07) * 800 = \$915.92$
...etc...

$S_4 = \$915.92 + 0.07\,\$915.92 \cong \$980.03$
$= (1.07)^3\, 800 = \$980.03$

2 types:

1. <u>Compounded m times a year.</u>

If you invest \$P at $r \times 100\%$ (<u>nominal</u>) annual rate compounded m times a year

then the bank is really giving you: $\frac{r}{m} \times 100\%$ applied m times each year.

So t years after the initial deposit, your balance is:

$$\boxed{S = \left(1+\frac{r}{m}\right)^{mt} P}$$

<u>Ex:</u> \$50,000 at 10% annual rate, <u>compounded quarterly</u> $m=4$

You actually get $\frac{10\%}{4} = 2.5\%$ <u>every 3 months</u>.

(so your actual <u>annual percentage yield</u> is higher than 10%!)

~~Afte~~ In year 10 ($\underbrace{\text{9 years}}_{t=9}$ since initial deposit)

$$S = \left(1+\frac{0.1}{4}\right)^{4\times 9} \$50{,}000 = (1.025)^{36}\,\$50{,}000$$

$$\simeq \boxed{121{,}626.77\,\$}$$

2. <u>CONTINUOUSLY COMPOUNDED</u>

If you deposit \$P at $r \times 100\%$ (<u>nominal</u>) annual rate, <u>compounded continuously</u> ($m \longrightarrow \infty$)

Then your balance t years after your

in decimal form

Then your balance t years after your initial deposit is:

rate, in decimal form

# of years since deposit

$$\boxed{S = Pe^{rt}}$$

$e = 2.71828\ldots$

future v

principal (present value)

Examples

1) Find the future value in the 10th year if $50,000 is invested at 5%

a) Compounded annually

P $\quad r=0.05$

9 compoundings:

$$S = (1.05)^9\, 50{,}000 = \boxed{\$77{,}566.41}$$

b) Compounded monthly, $m=12$

$$S = \left(1+\frac{0.05}{12}\right)^{12\times 9} 50{,}000 = \underbrace{(1.00416666\ldots)}_{\text{DO NOT ROUND!}}^{108} 50{,}000$$

$$\simeq \boxed{\$78{,}342.34}$$

c) Compounded continuously $S = 50{,}000\, e^{(0.05\times 9)}$ $\quad (S = Pe^{rt})$

$$50{,}000\ e\ \boxed{\wedge}\ (0.05\times 9) = 50{,}000\, e^{0.45} \simeq \boxed{78{,}415.61}$$

$$\times\ 50{,}000\ e\ \boxed{\wedge}\ 0.05\ \boxed{\times}\ 9 = 50000\,(e^{0.05})\,9$$

2) **For each investment situation in Problems 5–8, identify (a) the annual interest rate, (b) the length of the investment in years, (c) the periodic interest rate, and (d) the number of periods of the investment.**

a) 8% compounded quarterly for 7 years

b) 12% compounded monthly for 3 years

a) 8% compounded quarterly for 7 years

$r = 0.08 \left(= \frac{8}{100}\right)$

$t = 7$ years

2% every 3 months $\rightarrow 0.02$

\# of periods $= 4\times 7 = 28$

$$S = (1+0.02)^{28}\, \underline{P}$$

b) 12% compounded monthly for 3yrs

$r = 0.12$

$t = 3$

periodic interest rate: $\frac{12\%}{12} = 1\%$ ie 0.01

$$S = (1+0.01)^{36}\, \underline{P}$$

# of periods: $3 \times 12 = 36$

3) What lump sum should be deposited in an account that will earn 9%, compounded quarterly, to grow to \$300,000 for retirement ~~in~~ after 25 years?

0.09

$t = 25$

$$S = \left(1 + \frac{r}{m}\right)^{mt} P$$

300,000

4

$$300{,}000 = \underbrace{\left(1 + \frac{0.09}{4}\right)^{4 \times 25}}\, P$$

Solve for P

$$300{,}000 = (1.0225)^{100}\, P$$

$$P = \frac{300{,}000}{(1.0225)^{100}} = \boxed{32{,}418.25}$$