The Midterm will take place Feb 11th, during lecture time. It will cover the material of Chapter 1, Chapter 2 and Chapter 3, Section 3.1. You do not need to bring paper to write on, because there will be enough room on the test paper.

Be reminded that anything you write on the midterm test paper must be your own work. If there is evidence that you are claiming credit for work that is not your own work during the test period, I need to give you a zero on the test and turn the evidence over to the Dean’s Committee on Academic Conduct.

Study suggestions: To be successful on the test I highly recommend to catch up on the homework. After that repeat solving problems. Find out with which kind of problems you are still struggling. Study those over and over again, so that you develop a routine. If there are questions that you just cannot find an answer for, do not hesitate to visit me during office hours. I am more than happy to explain whatever is unclear.

Exam advise: Start with whatever problem seems easy to you. Having solved something right away calms down and gives a secure feeling. Do not waste too much time on a specific problem if you get stuck. Rather start with another problem and come back later, if there is time left. I appreciate very much, if your work is laid down in a logic order and in readable writing. If you need more space than is available on the paper, give clear instructions about where to find the rest of your work. Place a box around your final answer.

On the following pages you find review problems. You need to turn your answers in at latest by Monday, 2/9, in order to get points towards the Midterm. Just for turning in this practice test, you will get approximately 10% of the possible points of the exam - independently of how well you do in the practice test.
1.) For each correct answer in the TRUE/FALSE part, you will get 1 point, for each incorrect answer, there will be one point subtracted, i.e. you get -1 point. For no answer, you get 0 points. You can not get less than 0 points out of one subproblem.

<table>
<thead>
<tr>
<th>(a)</th>
<th>Cross the right box for the statements about linear systems.</th>
</tr>
</thead>
<tbody>
<tr>
<td>There are homogeneous systems with no solution.</td>
<td>☐ TRUE ☒ FALSE</td>
</tr>
<tr>
<td>Every linear system with more variables than equations has at least one solution.</td>
<td>☐ TRUE ☒ FALSE</td>
</tr>
<tr>
<td>If a linear system of the form (Ax = b) with (A) an ((n, m))-matrix and (b) a vector in (\mathbb{R}^n) has a solution, then this solution can be written as a vector in (\mathbb{R}^m).</td>
<td>☒ TRUE ☐ FALSE</td>
</tr>
<tr>
<td>Let (u_i) be vectors in (\mathbb{R}^n). Then the linear system ([u_1, u_2, \ldots, u_n]u_2) has ([0, 1, 0, \ldots, 0]^T) as a solution.</td>
<td>☒ TRUE ☐ FALSE</td>
</tr>
<tr>
<td>Any homogeneous system with 5 variables and 3 equations has infinitely many solutions.</td>
<td>☒ TRUE ☐ FALSE</td>
</tr>
<tr>
<td>If (Ax = b) has a solution and (Ax = 0) has infinitely many solutions, then (Ax = b) has also infinitely many solutions.</td>
<td>☒ TRUE ☐ FALSE</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>(b)</th>
<th>Cross the right box for the statements about linear independence and span.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Including the zero vector in (\mathbb{R}^n) always gives a linearly dependent set of vectors in (\mathbb{R}^n).</td>
<td>☒ TRUE ☐ FALSE</td>
</tr>
<tr>
<td>({u_1, u_2, u_3, u_4, u_5} \subseteq \mathbb{R}^4) is always linearly dependent.</td>
<td>☒ TRUE ☐ FALSE</td>
</tr>
<tr>
<td>({u_1, u_2, u_3, u_4, u_5} \subseteq \mathbb{R}^4) always spans (\mathbb{R}^4).</td>
<td>☐ TRUE ☒ FALSE</td>
</tr>
<tr>
<td>If (u_1) and (u_2) are linearly dependent, then there exists a scalar (c \in \mathbb{R}) such that (u_1 = cu_2).</td>
<td>☐ TRUE ☒ FALSE</td>
</tr>
<tr>
<td>If ({u_1, u_2, u_3}) spans (\mathbb{R}^3), then so does ({u_1, u_2, u_3, u_4}), for a vector (u_4 \in \mathbb{R}^3).</td>
<td>☒ TRUE ☐ FALSE</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(c)</th>
<th>Cross the right box for the statements about linear systems and linear independence.</th>
</tr>
</thead>
<tbody>
<tr>
<td>If (A) is an ((n, m))-matrix and (b) a vector in (\mathbb{R}^n) and the columns of (A) are linearly independent, then the corresponding linear system cannot have free variables.</td>
<td>☒ TRUE ☐ FALSE</td>
</tr>
<tr>
<td>A set with two different vectors is linearly independent.</td>
<td>☐ TRUE ☒ FALSE</td>
</tr>
<tr>
<td>If (u_1), (u_2), and (u_3) are pairwise linearly independent, then ({u_1, u_2, u_3}) is also linearly independent.</td>
<td>☐ TRUE ☒ FALSE</td>
</tr>
<tr>
<td>A homogeneous system with 3 variables and 3 equations has exactly one solution.</td>
<td>☐ TRUE ☒ FALSE</td>
</tr>
<tr>
<td>The trivial solution is always a solution to a linear system.</td>
<td>☐ TRUE ☒ FALSE</td>
</tr>
</tbody>
</table>
2. Consider the following system:

\[
\begin{bmatrix}
2x_1 + 3x_3 - 5x_4 &= 4 \\
x_1 + x_2 - x_3 + x_4 &= 8 \\
3x_1 + x_2 + 2x_3 - 3x_4 + 10 &= 0
\end{bmatrix}
\]

(a) Write down the corresponding augmented matrix.

\[
\begin{bmatrix}
2 & 0 & 3 & -5 & 4 \\
1 & 1 & -1 & 1 & 8 \\
3 & 1 & 2 & -3 & 10
\end{bmatrix}
\]

(b) Perform Gauss-Jordan algorithm.

\[
\begin{bmatrix}
2 & 0 & 3 & -5 & 4 \\
1 & 1 & -1 & 1 & 8 \\
3 & 1 & 2 & -3 & 10
\end{bmatrix} \rightarrow \begin{bmatrix}
2 & 0 & 3 & -5 & 4 \\
0 & -2 & 5 & -7 & -12 \\
0 & 0 & 0 & 2 & -4
\end{bmatrix} \rightarrow \begin{bmatrix}
2 & 0 & 3 & -5 & 4 \\
0 & -2 & 5 & -7 & -12 \\
0 & 0 & 0 & 1 & -2
\end{bmatrix}
\]

(c) After Gauss-Jordan, what is the form of the matrix called? It is called ‘reduced form’.

(d) Identify the leading variables. From the reduced form, we deduce that the leading variables are \(x_1, x_2, x_4\).

(e) Identify the free variables. As the system turns out to be consistent, we can go ahead and identify free variables. From (d) we deduce that \(x_3\) is a free variable.

(f) Determine the solutions of the variables. Write the solution in vector form (i.e. \(x = v + sw\), where you need to specify \(v\) and \(w\)). Backward substitution gives

\[
S = \{ -3 \\ 13 \\ -2 \} + t \begin{bmatrix}
-1.5 \\
2.5 \\
1
\end{bmatrix} \mid t \in \mathbb{R}\}.
\]

3.

(a) Determine \(h\), so that the following vectors span \(\mathbb{R}^3\).

\[
\begin{bmatrix}
2 \\
-2 \\
0
\end{bmatrix}, \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}, \begin{bmatrix}
0 \\
0 \\
h
\end{bmatrix}
\]
The system needs to be consistent for any choice of \( \ast \). This means, me must exclude the zero row in the coefficient part of the augmented matrix. Hence,
\[
h \neq -6.
\]

(b) If you choose some \( h \) so that the three vectors do span \( \mathbb{R}^3 \), apply the Big Theorem and conclude the right statement about linear independence of these vectors. If the vectors span \( \mathbb{R}^3 \), we conclude that the vectors are linearly independent by the Big Theorem.

4. (a) Write down the precise definition of ‘Linear Independence’ of the vectors \( u_1, u_2, \ldots, u_m \).

\( u_1, u_2, \ldots, u_m \) are linearly independent if the only solution to the vector equation
\[
c_1u_1 + c_2u_2 + \ldots + c + mu_m = 0
\]
is the trivial solution.

(b) Let \( u, v \) be vectors in \( \mathbb{R}^n \). Then \( u - v \) is which kind of object? Addition (hence subtraction) is an operation on vectors of one Euclidean space. This means, that \( u = v \) is a vector of \( \mathbb{R}^n \).

(c) Which equation do you have to consider, in order to determine the linear independence of \( (u - v) \) and \( u + v \). Let us give those vectors a name:
\[
w_1 := u - v, \quad w_2 := u + v.
\]

Then \( w_1 \) and \( w_2 \) are linearly independent, if
\[
a_1w_1 + a_2w_2 = 0
\]
has only the trivial equation. Let us plug in the definition of \( w_1 \) and \( w_2 \). We then get the following equation we need to consider:
\[
a_1(u_v) + a_2(u + v).
\]

(d) Assume that \( u \) and \( v \) are linearly independent. Use this and the approach of (c) in order to show that \( (u - v) \) and \( u + v \) are linearly independent, too. Let us start with the equation from (c). We need to show that \( a_1 = a_2 = 0 \).

\[
0 = a_1(u - v) + a_2(u + v)
\]
\[
= (a_1 + a_2)u - (a_1 - a_2)v
\]
Have a look at the last equation. We linear combine $u$ and $v$, so that the result is the zero vector. But the assumption is that they are linearly independent. So the only possible linear combination of the zero vector is the trivial linear combination, i.e. $a_1 + a_2 = 0$ and $a_1 - a_2 = 0$. Which $a_i$'s satisfy those to equations? The second equation gives $a_1 = a_2$. Plug that in the first equation to get $2a_1 = 0$, hence $a_1 = 0$ and $a_2 = 0$.

5. Consider the following linear homomorphism:

$$T : \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \rightarrow \begin{bmatrix} u_1 + 2u_2 - u_3 \\ -4u_1 - 7u_2 + 7u_3 \\ -u_1 - u_2 + 5u_3 \end{bmatrix}.$$  

(a) Find the corresponding matrix $A$, such that $T(x) = Ax$ for all $x \in \mathbb{R}^3$.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 7 \\ -1 & -1 & 5 \end{bmatrix}$$

(b) Find the kernel of $T$. We need to find the solution to the homogeneous equation $Ax = 0$.

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ -4 & -7 & 7 & 0 \\ -1 & -1 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

As this system only has the trivial solution, we see that $\ker(T) = \{0\}$.

(c) Find the image of

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

under $T$. As $T(x) = Ax$, we have

$$T(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 7 \\ -1 & -1 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ 6 \end{bmatrix}$$