Math 308 **Final Exam** **Winter 2015, 3-18-2015**

Your Name

Your Signature

Student ID #

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	Form	Bonus	$\sum$
Points														
of	50	13	12	17	8	3	7	6	3	4	6	6	(10)	135

- No books are allowed. But you are allowed one sheet (10 x 8) of handwritten notes (back and front). You may use a calculator.
- Place a box around [your final answer] to each question.
- If you need more room, use the back of each page and indicate to the grader how to find the logic order of your answer.
- Raise your hand if you have questions or need more paper.
- For TRUE/FALSE problems, you just need to cross the right box. For each correct answer, you will get 1 point, for each incorrect answer, -1 point is added. For no answer you will get zero points. In each subsection of the TRUE/FALSE part, you can never get less than zero points.
- In order to get points for formal correctness, underline vectors, use {}-brackets for sets, declare parameters, mark equivalent matrices properly and keep a reasonable order and neatness.

Do not open the test until everyone has a copy and the start of the test is announced.

GOOD LUCK!

1.) For each correct answer in the TRUE/FALSE part, you will get 1 point, for each incorrect answer, there will be one point subtracted, i.e. you get -1 point. For no answer, you get 0 points. You can not get less than 0 points out of one subproblem (which are the problems, (a)-(h))

(a)	Cross the right box for the statements about linear systems.		
	A homogeneous system can either have no solution, a unique solution or infinitely many solutions.	☐ TRUE	☐ FALSE
	It is possible that an inhomogeneous system does not have a solution.	☐ TRUE	☐ FALSE
	A homogeneous system with 5 variables and 5 equations has exactly one solution.	☐ TRUE	☐ FALSE
	If $\mathbf{s} \neq \mathbf{0}$ is a solution to a linear system of the form $A\mathbf{x} = \mathbf{0}$ for a matrix A, then this system has infinitely many solutions.	☐ TRUE	☐ FALSE
	A homogeneous system with at least one free variable has infinitely many solutions.	☐ TRUE	☐ FALSE
	Let A be a $(5, 7)$-matrix. Then any solution to $A\mathbf{x} = \mathbf{0}$ is a vector in $\mathbb{R}^7$.	☐ TRUE	☐ FALSE
	Let A be a $(3, 3)$-matrix with linearly independent rows. Then $A\mathbf{x} = \mathbf{b}$ has exactly one solution for any $\mathbf{b}$ in $\mathbb{R}^3$.	☐ TRUE	☐ FALSE
(b)	Cross the right box for the statements about span.		
	Let $S := \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} \subseteq \mathbb{R}^n$. Then $\mathbf{u}_1 - 2\mathbf{u}_1 + 5\mathbf{u}_2 - 0\mathbf{u}_4$ is an element of $\operatorname{span}(S)$.	☐ TRUE	☐ FALSE
	The span of $\{\mathbf{u}\}$ has infinitely many elements for any choice of $\mathbf{u} \in \mathbb{R}^n$.	☐ TRUE	☐ FALSE
	$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} \subseteq \mathbb{R}^3$ spans $\mathbb{R}^3$ for any choice of vectors $\mathbf{u}_i$.	☐ TRUE	☐ FALSE
	Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \subseteq \mathbb{R}^n$. Then $\operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \subsetneq \{\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$.	☐ TRUE	☐ FALSE
	Let A be an (n, m)-matrix and let $\mathbf{u} \in \operatorname{col}(A)$. Then $A\mathbf{x} = \mathbf{u}$ has a solution.	☐ TRUE	☐ FALSE
	Let $\mathbf{0} \neq \mathbf{u}$ be a vector in $\mathbb{R}^n$. Then $\mathbf{0} \notin \operatorname{span}\{\mathbf{u}\}$.	☐ TRUE	☐ FALSE
	Let $\mathbf{u}_0 \in \operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} \subseteq \mathbb{R}^n$. Then $\{\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ is linearly dependent.	☐ TRUE	☐ FALSE
	Let A be an (m, n)-matrix. Then $\{A\mathbf{u} \mid \mathbf{u} \in \mathbb{R}^n\} = \operatorname{col}(A)$.	☐ TRUE	☐ FALSE
(c)	Cross the right box for the statements about linear independence, span, bases and dimensions.		
	For any vector $\mathbf{u}$ and $a \in \mathbb{R}$, the set $\{\mathbf{u}, a\mathbf{u}\}$ is linearly dependent.	☐ TRUE	☐ FALSE
	Let $S = \operatorname{span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Then $\dim(S) = 2$.	☐ TRUE	☐ FALSE
	Let $S \subseteq \mathbb{R}^4$ be a subspace of dimension 3. Then S has a uniquely determined basis with 4 elements.	☐ TRUE	☐ FALSE
	Let $S_1 \subseteq S_2$ be subspaces of $\mathbb{R}^n$. Then $\dim(S_1) < \dim(S_2)$.	☐ TRUE	☐ FALSE
	Let S be a subspace of $\mathbb{R}^n$ with $\dim(S) = m$ and let $U := \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subseteq S$. If $k > m$ then U is linearly dependent.	☐ TRUE	☐ FALSE
	$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \mathbb{R}^2$ is a linearly dependent set.	☐ TRUE	☐ FALSE
	Let A be an (n, m)-matrix. Then $\operatorname{null}(A)$ is a subspace of $\mathbb{R}^m$.	☐ TRUE	☐ FALSE
	Let A be a matrix. If $\operatorname{nullity}(A) = 3$, then $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.	☐ TRUE	☐ FALSE

(d)	Cross the right box for the statements about matrices and homomorphisms.		
	Let T be a homomorphism with corresponding matrix A_T. If $\text{nullity}(A_T) \geq 1$ then T is not injective.	☐ TRUE	☐ FALSE
	Let $T : \mathbb{R}^4 \to \mathbb{R}^8$ be a homomorphism. Then T can be surjective, but not injective.	☐ TRUE	☐ FALSE
	Let T be a homomorphism with corresponding matrix A_T. Then $\ker(T) = \text{null}(A_T)$.	☐ TRUE	☐ FALSE
	Let $T : \mathbb{R}^{12} \to \mathbb{R}^4$. Then $\dim(\ker(T))$ must be 8 or greater.	☐ TRUE	☐ FALSE
	Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a homomorphism. Then $\text{range}(T) \subseteq \mathbb{R}^n$.	☐ TRUE	☐ FALSE
	The function $T : \mathbb{R}^m \to \mathbb{R}^m, \mathbf{u} \mapsto \mathbf{u}$ is a linear homomorphism.	☐ TRUE	☐ FALSE
	If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a surjective homomorphism with corresponding matrix A_T, then $\mathbf{b} \in \text{col}(A_T)$ for any $\mathbf{b} \in \mathbb{R}^n$.	☐ TRUE	☐ FALSE
	If T is an isomorphism with domain $\mathbb{R}^n$ and corresponding matrix A_T, then A_T is an (n, n)-matrix.	☐ TRUE	☐ FALSE
(e)	Cross the right box for the statements about matrices.		
	Let A, B be (n, n)-matrices . Then $A^2 - B^2 = (A+B)(A-B)$	☐ TRUE	☐ FALSE
	Let A be an (n, m)-matrix. Then A^2 is defined.	☐ TRUE	☐ FALSE
	Let A be a square matrix. Then $\det(2A) = 2\det(A)$.	☐ TRUE	☐ FALSE
	A square matrix A is singular, if and only $\det(A) = 0$.	☐ TRUE	☐ FALSE
	If A is invertible and $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{s}$, then $\mathbf{b}$ is a solution to $A^{-1}\mathbf{x} = \mathbf{s}$.	☐ TRUE	☐ FALSE
	Let A, B be equivalent matrices. Then $\det(A) = \det(B)$.	☐ TRUE	☐ FALSE
(f)	Cross the right box for the statements about column- and rowspace of a matrix A.		
	Let A, B be equivalent matrices. Then $\text{row}(A) = \text{row}(B)$.	☐ TRUE	☐ FALSE
	The rank of a matrix A is equal to the dimension of the row space of A.	☐ TRUE	☐ FALSE
	Let A be a matrix. Then $\text{row}(A) = \text{col}(A)$.	☐ TRUE	☐ FALSE
	Let A be an (m, n)-matrix. Then $\text{rank}(A)$ is less than or equal to m.	☐ TRUE	☐ FALSE
(g)	Cross the right box for the statements about eigenvalues and eigenspaces of an (n, n)-matrix A.		
	Any vector $\mathbf{0} \neq \mathbf{u} \in \mathbb{R}^n$ that satisfies $A\mathbf{u} = \lambda\mathbf{u}$ for some $\lambda \in \mathbb{R}$ is an eigenvector of A.	☐ TRUE	☐ FALSE
	Let λ be an eigenvalue for A. Then the set of eigenvectors of A with eigenvalue λ forms the eigenspace E_λ of A.	☐ TRUE	☐ FALSE
	The zero vector is always an eigenvector for any eigenvalue for A because it satisfies the defining property $A\mathbf{0} = \lambda\mathbf{0}$.	☐ TRUE	☐ FALSE
	If $\text{rank}(A)$ is less than the number of columns of A, then 0 is an eigenvalue of A.	☐ TRUE	☐ FALSE
	The matrix A may not have an eigenvalue.	☐ TRUE	☐ FALSE
(h)	Cross the right box for the statements about orthogonality of vectors.		
	Let $\mathbf{u}_1, \mathbf{u}_2$ be orthogonal to $\mathbf{u}$, then $\mathbf{u}_1 + \mathbf{u}_2$ is also orthogonal to $\mathbf{u}$.	☐ TRUE	☐ FALSE
	Two vectors in $\mathbb{R}^1$ can only be orthogonal if at least one of them is the zero vector.	☐ TRUE	☐ FALSE
	There is no nonzero vector $\mathbf{u}$ that is orthogonal to $\mathbf{u}$.	☐ TRUE	☐ FALSE
	Let A be a matrix and $\mathbf{u} \in \text{null}(A)$. Then $\mathbf{u} \in \text{row}(A)^\perp$.	☐ TRUE	☐ FALSE

2.(2+4+3+4 points) Consider the following linear homomorphism:

$$T : \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \rightarrow \begin{bmatrix} u_1 + 3u_2 + u_3 \\ u_2 + u_3 \\ 2u_1 - 3u_3 \end{bmatrix}.$$

(a) Find the corresponding matrix A, such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

(b) Find the kernel of T.

(c) Calculate the determinant of A. Is T invertible? Justify your answer.

(d) Find the inverse T^{-1} of T using A.

3.(4+2+4+2 points) Consider the following matrix:

$$A = \begin{bmatrix} 1 & -2 & 3 & -2 & 2 \\ 3 & -4 & -1 & -2 & 0 \\ 2 & -3 & 1 & -2 & 1 \end{bmatrix}.$$

(a) Find the null space of A.

(b) Verify that the vector $v = [0,\ -1,\ 0,\ 2,\ 1]^t$ is an element of null(A).

(c) Find a basis of null(A), that contains the vector $v = [0,\ -1,\ 0,\ 2,\ 1]^t$.

(d) What is the nullity of A? What is the rank of A?

4.(5+4+4+2+2 points) Consider the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 5 & 1 \\ 0 & 0 & -3 & -1 \\ 0 & 2 & 14 & 4 \end{bmatrix}$$

(a) Determine the characteristic polynomial χ_A of A. Show all your work! (Key, so that you can continue: The characteristic polynomial is $\chi_A = \lambda^4 - 4\lambda^3 + 5\lambda^2 - 2\lambda$.)

(b) Determine the eigenvalues for A.

(c) Compute the eigenspace for the eigenvalue $\lambda = 1$. What is the dimension of this eigenspace?

(e) What is the only possible dimension of the eigenspace with eigenvalue $\lambda = 0$? Answer this question with the help of χ_A and justify your answer.

(f) Based on the knowledge about the eigenvalues of this matrix, what can be said about the determinant of A?

5.(4+2+2 points) Let $S = \text{span}\{\mathbf{s}_1 = \begin{bmatrix} -1 \\ 4 \\ 2 \\ 0 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 4 \end{bmatrix}, \mathbf{s}_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}\}$.

(a) Find a basis for $S^{\perp}$.

(b) Compute the norm $||\mathbf{s}_1||$ of $\mathbf{s}_1$.

(c) What is the norm of

$$\frac{1}{||\mathbf{s}_1||}\mathbf{s}_1?$$

6. (3 points) Find a matrix that has $\chi = \lambda^2 + 2$ as its characteristic polynomial.

7.(2+1+1+3 points) Suppose that A is a $(5, 16)$-matrix.
(a) What is the maximum possible value for $\text{rank}(A)$?

(b) What is the minimum possible value for $\text{nullity}(A)$?

(c) Suppose that $\dim(\text{col}(A)) = 5$. What is $\text{nullity}(A)$?

(d) Consider the homomorphism $T : \mathbb{R}^{16} \to \mathbb{R}^5, \mathbf{x} \mapsto A\mathbf{x}$.
(i) What does the nullity of A represent in terms of T?

(ii) What is the dimension of the range of T if the nullity of A is at its minimum value? Is T then surjective?

8.(4+1+1 points) Let $S \subseteq \mathbb{R}^5$ be a subspace of dimension 4.
(a) What are the possible dimensions of subspaces S_i, that are subsets of S, i.e. $S_i \subseteq S$?

(b) How many elements does the subspace of S of dimension 0 have?

(c) How many elements does a subspace of S of dimension 1 have?

9.(3 points) Calculate

$$\left(\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 1 & 1 \end{bmatrix} \right)^2.$$

10.(4 points) Let $\mathcal{B}_1 = \{\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \end{bmatrix}\}$ be a basis for $\mathbb{R}^2$ and let $\mathbf{u} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$ be a vector represented with respect to the standard matrix. What is the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}_1$?

11.(3+3 points) Let

$$A = \begin{bmatrix} 1 & 17 & -3 & 23 & 3 & -3 & 2 & 6 \\ 10 & 170 & -30 & 230 & 30 & -30 & 20 & 60 \\ 0 & 3 & 0 & -2 & -51 & 12 & -27 & 9 \\ 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 2 & -5 & 10 & -1 & 3 & -1 & 1 & 1 \\ 4 & -10 & 20 & -2 & 6 & -2 & 2 & 2 \\ 7 & 3 & -1 & 0 & 8 & 7 & 1 & 0 \\ 0 & 0 & 0 & -2 & 12 & 9 & 11 & -2 \end{bmatrix}.$$

(a) Have a close look at A and find its determinant *without* actually computing it.

(b) Is $\lambda = 0$ an eigenvalue of A? Justify your answer.