Your Name

Your Signature

Student ID #

Problem	Points	Possible
1		
2		
3		
4		
Total		

- No books and no notes are allowed. You may use a calculator.
- Place a box around your final answer to each question.
- If you need more room, use the back of each page and indicate to the grader how to find the logic order of your answer.
- Raise your hand if you have questions or need more paper.
- For TRUE/FALSE problems, you just need to circle the correct answer, unless you are asked to give a proof/counterexample.

Do not open the test until everyone has a copy and the start of the test is announced.

GOOD LUCK!

1. (4+5+2+2+2+2+2+2+4+5=30 points)

(a) (4 points) Give the definition of *linear independence of vectors* $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \subseteq \mathbb{R}^n$.

$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \subseteq \mathbb{R}^n$ are linearly independent, if the only solution to

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \ldots + x_m\mathbf{u}_m = \mathbf{0}$$

is the trivial solution.

(b) (5 points) Prove the following. If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, then so are $\mathbf{u}+\mathbf{v}$ and $\mathbf{v}$.

We need to determine, what the solutions for

$$x_1(\mathbf{u}+\mathbf{v}) + x_2(\mathbf{u}) = \mathbf{0}$$

are. But this equation can be rewritten as $(x_1+x_2)\mathbf{u}+x_2\mathbf{v} = \mathbf{0}$. By assumption, $\mathbf{u}, \mathbf{v}$ are linearly independent, so the previous equation can only have the trivial solution, i.e. $x_1 + x_2 = 0$ and $x_2 = 0$. From the equation, we get $x_2 = 0$ and we substitute this in the first equation to get $x_1 = 0$. Hence $x_1 = x_2 = 0$, so the linear independence follows.

(c) (2 points) TRUE or FALSE: If $\mathbf{u}_1, \mathbf{u}_2$ are linearly dependent, then there is a scalar $r \in \mathbb{R}$ such that $\mathbf{u}_1 = r\mathbf{u}_2$.

(d) (2 points) TRUE or FALSE: If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a linearly dependent set, then there is a scalar c such that either $\mathbf{u}_1 = c\mathbf{u}_2$ or $\mathbf{u}_2 = c\mathbf{u}_3$ or $\mathbf{u}_1 = c\mathbf{u}_3$.

(e) (2 points) TRUE or FALSE: If A is an (n, n)-matrix, then $A\mathbf{x} = \mathbf{0}_n$ has exactly one solution.

There could be infinitely many solutions.

(f) (2 points) TRUE or FALSE: If A is an (m, n)-matrix with $m < n$, then $A\mathbf{x} = \mathbf{0}_m$ has more than one solution.

There must be free variables because of $m < n$, and because this is a homogeneous system, we have infinitely many solutions (no solution is not possible with homogenenous systems).

(g) (2 points) TRUE or FALSE: There are three vectors in $\mathbb{R}^3$ that span $\mathbb{R}^3$ but that are not linearly independent. False by the Big Theorem

(h) (2 points) TRUE or FALSE: If $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent vectors, and $\mathbf{u}_2, \mathbf{u}_3$ are linearly independent vectors, then $\mathbf{u}_1, \mathbf{u}_3$ are also linearly independent vectors.

Consider

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

(i) (4 points) What are the possible numbers of solutions a linear system with 308 equations and 309 variables can have? Justify your answer(s).

If it is an homogeneous system, we have infinitely many solutions. If it is an inhomogeneous system, there might be none or infinitely many.

(j) (5 points) Determine, if the following set of vectors are linearly independent and justify your answer:

$$\mathcal{A} := \{ \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix} \}.$$

Perform Gauss-elimination with the corresponding matrix.

$$A := \left[\begin{array}{ccc|c} -4 & 1 & -3 & 0 \\ 2 & 3 & 5 & 0 \\ 3 & 1 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -4 & 1 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that there are free variables, hence there are infinitely many solutions. The vectors are therefore linearly dependent.

2. (2+5+1+5+10=20 points)

(a) (2 points) TRUE or FALSE: If $\mathcal{A} := \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \mathbb{R}^3$ is a linearly independent set and $\mathbf{u}_4 \in \mathbb{R}^3$ is an arbitrary vector, then $\mathbf{u}_4$ is in the span of $\mathcal{A}$. By the Big Theorem.

(b) (2 points) TRUE or FALSE: If B is a (m, n)-matrix with columns that span $\mathbb{R}^m$, then $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$. Theorem 2.3.10

(c) (1 point) TRUE or FALSE: If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ does not span $\mathbb{R}^n$, then neither does $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

(d) (5 points) Determine the values for k such that the following vectors span $\mathbb{R}^3$:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ k \end{bmatrix}.$$

Perform Gauss elimination with the associated matrix.

$$A := \left[\begin{array}{ccc|c} 1 & 2 & 0 & \star \\ 1 & 1 & -1 & \star \\ 0 & -2 & k & \star \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & \star \\ 0 & 1 & 1 & \star \\ 0 & 0 & k+2 & \star \end{array}\right]$$

In order to never have an inconsistent system, we need the last row *not* to be a zero row. Therefore, $k \neq -2$

(e) (10 points) Circle the sets of vectors, that can *never* span $\mathbb{R}^3$:

- $\{\mathbf{u}_1\}$
- $\{\mathbf{v}_1, \mathbf{v}_2, r\mathbf{v}_2\}$, where $r \in \mathbb{R}$ is a scalar.
- $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.
- $\{\mathbf{y}_1, \mathbf{y}_1 + \mathbf{y}_2, \mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3\}$
- $\{\mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3\}$

3.(5+5+3 points) Consider the function $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as follows:

$$T(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}) = \begin{bmatrix} x_1 - 2x_2 + x_3 \\ 4x_1 - x_2 + x_3 \\ 2x_1 + x_2 - 3x_3 \end{bmatrix}$$

(a) Provide a matrix A such that $A\mathbf{x} = T(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^3$.

$$\begin{bmatrix} 1 & -2 & 1 \\ 4 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix}$$

(b) Is $\mathbf{v} = \begin{bmatrix} -1 \\ 3 \\ 30 \end{bmatrix}$ in the range of T?

You could go different ways to solve this. I decided to check whether the columns of A are linearly independent and then apply the Big Theorem.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 4 & -1 & 1 & 0 \\ 2 & 1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 7 & -3 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 7 & -3 & 0 \\ 0 & 0 & -20 & 0 \end{array}\right]$$

From this matrix in echelon form we deduce, that the columns are linearly independent. Because $m = n$, we apply the Big Theorem to see that T is an isomorphism, in particular surjective. Hence, any vector of $\mathbb{R}^3$ lies in the range.

(c) Is $S : \mathbb{R}^1 \to \mathbb{R}^2, x \mapsto \begin{bmatrix} 3x \\ x^2 \end{bmatrix}$ a linear transformation? Justify your answer.

The square indicates that it is not a homomorphism. Both properties of a homomorphism are not satisfied, I'll just show the violation of scalar multiplication. $T(2 \cdot 1) = \begin{bmatrix} 6 \\ 4^2 \end{bmatrix}$ But $2T(1) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$, so the function is not linear.