Solving Equations Equal to Zero

Master the fundamental techniques for solving equations set equal to zero. These skills are essential for finding critical points, roots, and solving optimization problems!

Important Tips to Remember

Key Strategy: The main idea behind solving equations is to keep "undoing" what is happening around the variable x. Work backwards from the operations applied to x.
Remember: Always check your solutions in the original equation, especially when dealing with radicals, logarithms, or fractions that might create undefined expressions!
For Products: If $A \cdot B = 0$, then $A = 0$ OR $B = 0$ (or both).
For Fractions: If $\frac{A}{B} = 0$, then $A = 0$ AND $B \neq 0$.
When Combining Fractions: Find a common denominator first, then treat as a single fraction equal to zero.

Section 1: Products Equal to Zero

When a product equals zero, at least one factor must equal zero.

Example

Solve: $(x-3)(2x+1) = 0$
Since the product equals zero, either: $x - 3 = 0 \quad \text{OR} \quad 2x + 1 = 0$
From $x - 3 = 0$: $x = 3$ From $2x + 1 = 0$: $x = -\frac{1}{2}$
Solutions: $x = 3$ and $x = -\frac{1}{2}$

1. $(x-2)(x+5) = 0$

2. $(3x-6)(x+1) = 0$

3. $x(2x-7) = 0$

4. $(x-4)^2(x+3) = 0$

5. $2x(x-1)(x+2)=0$

Section 2: Fractions Equal to Zero

A fraction equals zero when the numerator equals zero AND the denominator is non-zero.

Example

Solve: $\frac{x^2-4}{x+1} = 0$
Step 1: Set numerator equal to zero: $x^2 - 4 = 0$ $(x-2)(x+2) = 0$, so $x = 2$ or $x = -2$
Step 2: Check that denominator $\neq 0$: When $x = 2$: $2 + 1 = 3 \neq 0$ When $x = -2$: $-2 + 1 = -1 \neq 0$
Solutions: $x = 2$ and $x = -2$

6. $\frac{x-5}{x+3} = 0$

7. $\frac{x^2-9}{2x-1} = 0$

8. $\frac{x(x-6)}{x^2+1} = 0$

9. $\frac{(x-1)(x+4)}{x-7} = 0$

10. $\frac{x^2-2x}{x^2-4} = 0$

Section 3: Using Basic Trigonometric Identities

Use fundamental trig identities like $\sin^2 x + \cos^2 x = 1$ *to simplify and solve equations.*

Example

Solve: $\sin^2 x - \cos^2 x = 0$
Step 1: Use the identity $\cos^2 x = 1 - \sin^2 x$: $\sin^2 x - (1 - \sin^2 x) = 0$
Step 2: Simplify: $\sin^2 x - 1 + \sin^2 x = 0$ $2\sin^2 x - 1 = 0$
Step 3: Solve for $\sin x$: $\sin^2 x = \frac{1}{2}$, so $\sin x = \pm\frac{\sqrt{2}}{2}$
Solutions: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ (in $[0, 2\pi)$)

11. $\cos^2 x - \sin^2 x = 0$

12. $2\sin^2 x + \cos x - 1 = 0$ (Hint: Use $\sin^2 x = 1 - \cos^2 x$)

13. $\tan^2 x - 1 = 0$

14. $\sin x \cos x = 0$

15. $1 - 2\cos^2 x = 0$

Section 4: Combining Fractions, Then Using Product/Quotient Methods

First combine fractions over a common denominator, then apply previous techniques.

Example

Solve: $\frac{x}{x-1} - \frac{2}{x+1} = 0$

Step 1: Find common denominator and combine: $\frac{x(x+1)-2(x-1)}{(x-1)(x+1)} = 0$

Step 2: Simplify numerator: $\frac{x^2+x-2x+2}{(x-1)(x+1)} = 0$ $\frac{x^2-x+2}{(x-1)(x+1)} = 0$

Step 3: Set numerator equal to zero: $x^2 - x + 2 = 0$

Using quadratic formula: $x = \frac{1 \pm \sqrt{1-8}}{2} = \frac{1 \pm \sqrt{-7}}{2}$

No real solutions (discriminant is negative).

16. $\frac{1}{x} + \frac{1}{x-2} = 0$

17. $\frac{x}{x+1} - \frac{3}{x-1} = 0$

18. $\frac{2x}{x-3} + \frac{1}{x+2} = 0$

19. $\frac{x-1}{x} - \frac{x}{x+1} = 0$

20. $\frac{1}{x-1} - \frac{2}{(x-1)^2} = 0$

Section 5: Advanced Algebraic Manipulations

Undo cube roots, handle higher-order terms, and work with more complex expressions.

Example

Solve: $\sqrt[3]{x+2} - 3 = 0$
Step 1: Isolate the cube root: $\sqrt[3]{x+2} = 3$
Step 2: Cube both sides: $x + 2 = 27$
Step 3: Solve for x: $x = 25$
Check: $\sqrt[3]{25+2} = \sqrt[3]{27} = 3$

21. $\sqrt{2x+1} - 5 = 0$

22. $(x-3)^3 + 8 = 0$

23. $x^{2/3} - 4 = 0$

24. $e^{2x} - 9 = 0$

25. $\ln(x+1) - 2 = 0$

26. $x^4 - 16 = 0$ (Hint: Factor as difference of squares twice)

27. $\frac{1}{\sqrt{x-1}} - 2 = 0$

28. $(x^2 - 1)^{3/2} = 8$

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