Domain Restrictions and Constraints for Optimization

Master identifying domain restrictions from both mathematical functions and real-world contexts. These skills are essential for setting up optimization problems correctly!

Key Domain Restrictions to Remember

Mathematical Restrictions:

- Division by zero: $\frac{1}{x-3}$ requires $x \neq 3$
- Square roots of negatives: $\sqrt{x-5}$ requires $x \geq 5$
- Logarithms: $\ln(x-2)$ requires $x > 2$ (argument must be positive)
- Even roots of negatives: $\sqrt[4]{x+1}$ requires $x \geq -1$

Real-World Restrictions:

- Physical quantities: lengths, areas, volumes must be positive
- Time can be negative (before a reference point) or positive
- Quantities like "number of items" must be non-negative integers

Strategy: Always consider BOTH mathematical and contextual restrictions!

Section 1: Mathematical Domain Restrictions

Identify domain restrictions based purely on mathematical function requirements.

Example

Find the domain of $f(x) = \frac{\sqrt{x+3}}{x-7}$
Step 1: Square root requires $x + 3 \geq 0$, so $x \geq -3$
Step 2: Denominator cannot be zero: $x - 7 \neq 0$, so $x \neq 7$
Step 3: Combine restrictions: $x \geq -3$ AND $x \neq 7$
Domain: $[-3, 7) \cup (7, \infty)$ or $x \geq -3, x \neq 7$

1. $g(x) = \sqrt{2x-6}$

   Domain restriction: ________________________________

2. $h(x) = \frac{5}{x+4}$

   Domain restriction: ________________________________

3. $f(x) = \ln(3x - 9)$

   Domain restriction: ________________________________

4. $k(x) = \frac{\sqrt{x-1}}{x^2-9}$

   Domain restrictions: ________________________________

5. $m(x) = \sqrt[4]{2 - x} + \frac{1}{x-5}$

   Domain restrictions: ________________________________

6. $p(x) = \ln(x^2 - 4)$

   Domain restriction: ________________________________

Section 2: Real-World Context Domain Restrictions

Identify domain restrictions based on the physical or practical meaning of variables.

Example

A square is cut from each corner of a 10-inch by 8-inch rectangle to form a box. If x is the side length of the cut-out squares, what are the domain restrictions?
Physical meaning: x = side length of cut-out squares
Restrictions: 1. $x > 0$ (must cut something out) 2. $x < 4$ (can't cut more than half the width: $2x < 8$) 3. $x < 5$ (can't cut more than half the length: $2x < 10$)
Most restrictive: $0 < x < 4$

7. The height of a projectile is $h(t) = -16t^2 + 64t + 80$ where t is time in seconds. What are reasonable domain restrictions for this context?

   Domain restriction: ______________________________

   Explanation: ____

8. A rectangular garden has length $(x+5)$ feet and width $(10-x)$ feet. What domain restrictions make sense?

   Domain restriction: ______________________________

   Explanation: ____

9. A company's profit function is $P(x) = -2x^2 + 400x - 5000$ where x is the number of items produced. What are reasonable domain restrictions?

   Domain restriction: ______________________________

   Explanation: ____

10. A cylindrical can has radius r and height $(20 - 2r)$. What domain restrictions make physical sense?

    Domain restriction: ______________________________

    Explanation: ____

11. The speed of a car t hours after starting a trip is $v(t) = 60 - 5t$ mph. For how long is this model reasonable?

    Domain restriction: ______________________________

    Explanation: ____

Section 3: Combined Mathematical and Contextual Domain Restrictions

Consider both mathematical function requirements AND real-world meaning to find the complete domain.

Example

The area of a rectangle is $A(x) = x\sqrt{100 - x^2}$ where x is the width in feet.
Mathematical restrictions: $100 - x^2 \geq 0$, so $x^2 \leq 100$, which gives $-10 \leq x \leq 10$
Contextual restrictions: Width must be positive, so $x > 0$
Combined domain: $0 < x \leq 10$
Note: We use $x \leq 10$ (not < 10) because when $x = 10$, the area is zero, which is mathematically valid even if not practically useful.

12. The volume of a box is $V(x) = x(12 - 2x)(8 - 2x)$ where x is the height of the box formed by cutting squares from corners.

    Mathematical restrictions: ______________________________

    Contextual restrictions: ______________________________

    Combined domain: ______________________

13. The distance between two moving objects is $d(t) = \sqrt{(3t)^2 + (4t - 20)^2}$ where t is time in seconds.

    Mathematical restrictions: ______________________________

    Contextual restrictions: ______________________________

    Combined domain: ______________________

14. A Norman window has area $A(x) = 2x\sqrt{16 - x^2} + \frac{\pi x^2}{4}$ where x is the width.

    Mathematical restrictions: ______________________________

    Contextual restrictions: ______________________________

    Combined domain: ______________________

15. The concentration of a drug is $C(t) = \frac{5t}{t^2+4}$ where t is hours after injection.

    Mathematical restrictions: ______________________________

    Contextual restrictions: ______________________________

    Combined domain: ______________________

16. The strength of a beam is $S(x) = \frac{x\sqrt{144-x^2}}{12}$ where x is the width in inches.

Mathematical restrictions: ____

Contextual restrictions: ____

Combined domain: ______________________________

Section 4: Complete Optimization Setup with Domain Analysis

Practice setting up optimization problems by identifying the objective function, constraints, and complete domain.

Example

A farmer has 120 feet of fence to create a rectangular pen against a barn (one side needs no fence). Express the area as a function of width and find the domain.
Let w = width, l = length
Constraint: $2w + l = 120$, so $l = 120 - 2w$
Objective function: $A(w) = w \cdot l = w(120 - 2w) = 120w - 2w^2$
Domain analysis: - Mathematical: No restrictions from the function itself - Contextual: $w > 0$ (positive width) and $l > 0$ (positive length) - From $l = 120 - 2w > 0$: $w < 60$
Combined domain: $0 < w < 60$

17. A cylindrical can must hold 500 cubic inches. Express the surface area as a function of radius and find the domain.

    Constraint equation: ______________________________

    Surface area function: $S(r) =$ ______________________________

    Domain: ____________________

18. A box is made by cutting squares of side x from a 20-inch by 30-inch piece of cardboard. Express the volume as a function of x and find the domain.

    Volume function: $V(x) =$ ______________________________

    Mathematical restrictions: ______________________________

    Contextual restrictions: ______________________________

    Combined domain: ____________________

19. A window consists of a rectangle topped by a semicircle. If the perimeter is 24 feet, express the area as a function of the width w and find the domain.

    Constraint equation: ______________________________

    Area function: $A(w) =$ ______________________________

    Domain: ____________________

Remember: Always check that your domain makes sense both mathematically and in the context of the problem. The domain tells you the valid range for optimization!