The Dirichlet to Neumann Map For A Cubic Resistor Network

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Abstract

Relations that characterize the Dirichlet to Neumann map for a cubic network of resistors are given. Using these relations, a parameterization of a subset of Dirichlet to Neumann maps is given (hopefully).

1 Introduction.

We consider the particular cubic network in figure 1.

Such a network Ω is a restriction upon the cubic networks examined in [7]. Each point of intersection between two or more line segments, or end of a line segment, is called a *node*. The set of nodes is denoted Ω_0. The *interior* of Ω_0, called int Ω_0, consists of those nodes which are the vertices of the cube. The *boundary* of Ω_0, called $\partial\Omega_0$, is $\Omega_0 - \text{int } \Omega_0$. Each interior node p has five neighboring nodes, all of which are in Ω_0. The set of five neighbor nodes is called $\mathcal{N}(p)$. Each boundary node p has exactly one neighboring node which is an interior node. An *edge pq* is a line segment which connects a pair of neighboring nodes p and q in int Ω_0, or which connects a boundary node p to its neighboring interior node q. This set of edges is denoted Ω_1. An edge pq, where p is a boundary node and q is its neighboring interior node will be called a boundary edge or boundary spike.

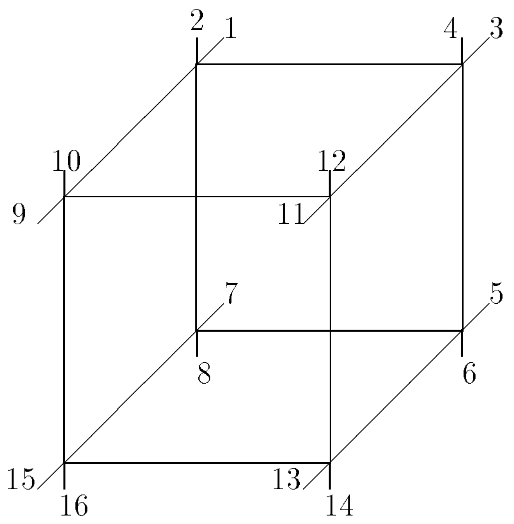

Figure 1: A unit cube resistor network, with nodes labelled.

A *cubic network of resistors* of this type is a network $\Omega = (\Omega_0, \Omega_1)$ together with a positive real-valued function γ on Ω_1. The function γ is called the *conductivity.* For each edge pq in Ω_1, the number $\gamma(pq)$ is the *conductance* of pq, and $1/\gamma(pq)$ is the *resistance* of pq. If u is a function on Ω_0, Ohm's Law gives a *current* along each conductor pq where $I(pq) = \gamma(pq)(u(p) - u(q))$ is the current from p to q. the function u is called γ*-harmonic* if for each interior node p,

$$\sum_{q \in \mathcal{N}(p)} \gamma(pq)(u(p) - u(q)) = 0.$$

This property of a γ-harmonic function, which asserts that the sum of the currents flowing out of each interior node is zero, is *Kirchhoff's Law.* If a function ϕ is defined at the boundary nodes, there will be a unique γ-harmonic function u, with $u(p) = \phi(p)$ for each boundary node p. The function u is called the *potential* due to ϕ. The potential drop across the conductor pq is $\Delta u(pq) = u(p) - u(q)$. The function u determines a current $I_\phi(p)$ through each boundary node p, by $I_\phi(p) = \gamma(pq)(u(p) - u(q))$, where q is the interior neighbor of p.

For each conductivity γ on Ω_1, the linear map Λ_γ from boundary functions to boundary functions is defined by $\Lambda_\gamma(\phi) = I_\phi$. The boundary function ϕ is called Dirichlet data, and the boundary current I_ϕ is called Neumann data. The map Λ_γ which takes potentials at the boundary of Ω to currents through the boundary nodes of Ω is called the Dirichlet-to-Neumann map.

The inverse problem is to recover the conductivity γ from the map Λ_γ. This leads to four problems.

1. Uniqueness: If $\Lambda_\gamma = \Lambda_\mu$, does it necessarily follow that $\gamma = \mu$?
2. Continuity: If Λ_γ is near to Λ_μ, does it necessarily follow that γ is near to μ?
3. Reconstruction: Give an algorithm for using Λ_γ to compute γ.
4. Characterization: For each integer n, which n by n matrices are of the form Λ_γ for some γ?

In [7], we showed that the Neumann-to-Dirichlet map uniquely determines the conductivity γ, and we gave an algorithm for computing γ from the Neumann-to-Dirichlet map. We will (attempt) to show here that for a unit cube, 28 entries of the Λ matrix give a parameterization.

2 Definitions and Motivations

Let us make analagous definitions to those we find in [6]. A *spherical graph* is a graph with a boundary which is embedded in a sphere $\mathcal{B}$ so that the boundary nodes lie on the surface C which bounds $\mathcal{B}$, and the rest of Ω is in the interior of $\mathcal{B}$. A pair of sequences of boundary nodes $(P; Q) = (p_1, \dots, p_k; q_1, \dots, q_k)$ such that the sequence $(p_1, \dots, p_k; q_1, \dots, q_k)$ can be split by an object homeomorphic to a circle on the sphere such that all $p \in P$ are on the inside of the circle and all $q \in Q$ are in its complement on the sphere, will be called a *spherical pair.*

A spherical pair $(P; Q) = (p_1, \dots, p_k; q_1, \dots, q_k)$ of boundary nodes is said to be *connected through* Ω if there are k disjoint paths $\alpha_1, \dots, \alpha_k$ in Ω, such that α_i starts at p_i, ends at q_i, and passes through no other boundary nodes. We say that α is a *connection* from P to Q.

We say that removing an edge *breaks the connection* from P to Q if there is a connection from P to Q through Ω, but there is not a connection from P to Q after the edge is removed. A network Ω is called *critical* if the removal of any edge breaks some connection.

The original definition of cubic networks from [7] had three boundary spikes for each of the eight vertices of the cube. We would like to reducce the number of parameters and determinantal conditions to be dealt with while maintaining the recovery properties of the network.

Suppose we reduce the number of boundary spikes to one per vertex. Consider the analagous planar case. Removing a boundary spike breaks no connections. Therefore the network is not critical. By theorem 2 of [6], we cannot recover γ from Λ_γ for this network. This leads us to believe that a similar situation exists in the cubic case, that having only one boundary spike per vertex is not sufficient to maintain recoverability.

Suppose we instead consider the unit cube network with two boundary spikes per vertex. Again looking at the analagous planar case, removal of a boundary spike does not break a connection. Therefore the network is critical, and we may recover γ from Λ_γ for this network. Again, this leads us to believe that a similar situation exists for the cubic case. We will show in §3.2 that there is a recovery algorithm for the unit cube with two boundary spikes per vertex.

Conjecture: Suppose Ω is a critical cubic network with 28 conductors. The removal of any edge breaks some connection, therefore some determinantal condition is zero. If all γ are recoverable by determinantal conditions from Λ_γ, then criticality implies recoverability in the cubic case. However, from numerical experiments and a statistical analysis of the distribution of known determinantal conditions, it would appear that there are non-determinantal conditions necessary to recover all γ in Ω, using only 28 parameters, therefore criticality does not imply recoverability.

3 Relations in Λ.

Λ has the block structure:

$$\Lambda = [\begin{array}{cc} A & B \\ B^T & C \end{array}]$$

By theorem 3.2 of [6], the Λ matrix is symmetric, thus $\lambda_{i,j} = \lambda_{j,i}$.

From corollary 2.2 of [2], if the boundary potential ϕ has value 1 at all bounday nodes, then the potential u has value 1 at all interior nodes, and hence the current $I_\phi = 0$. From this follow the sum relations: for each $i = 1, 2, \ldots, 16$,

$$\sum_{i=1}^{16} \lambda_{i,j} = 0.$$

3.1 Derivation of 2×2 determinantal conditions

Consider how we can recover the conductance of a boundary spike i. We wish to restrict the flow of current to the boundary spikes i, j at some vertex of the cube. To do this, we impose the conditions of 0 voltage and 0 current to the selected boundary spikes. (figure 3.1). Now place a voltage of 1 on i, α on j. α must be chosen such that the 0 current conditions we have placed on some nodes k hold true. This leads to the equation:

$$\lambda_{i,k} + \alpha\lambda_{j,k} = 0 \tag{1}$$

where k is any boundary node with the 0 current condition.

We may solve this equation for α:

$$\alpha = -\frac{\lambda_{i,k}}{\lambda_{j,k}} \tag{2}$$

Now that we know α, we can calculate the currents through i, j:

$$i : \lambda_{i,i} + \alpha\lambda_{j,i}$$
$$j : \lambda_{i,j} + \alpha\lambda_{j,j}$$

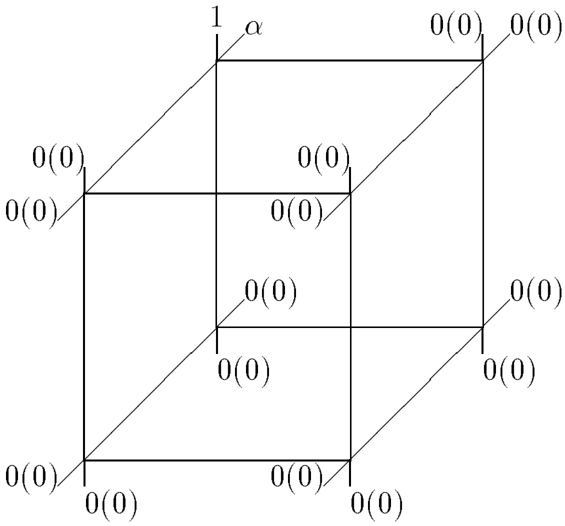

Figure 2: Recovering γ for a boundary spike

But we know the voltage and the current across i and j, so we may solve for the conductivity by $G = I/V$ (using our assumption that Ohm's Law holds), yielding:

$$i : \frac{\lambda_{i,i} + \alpha\lambda_{j,i}}{1-0} = \lambda_{i,i} + \alpha\lambda_{j,i}$$
$$j : \frac{\lambda_{i,j} + \alpha\lambda_{j,j}}{\alpha - 0} = \frac{\lambda_{i,j} + \alpha\lambda_{j,j}}{\alpha}$$

Since we may use this formula to solve for all boundary spikes i, j at any vertex of the cube, we can extend equation 1 to hold for all $l \neq i, j$. Now let us look at substituting in α from equation 2 into equation 1. We obtain:

$$\lambda_{i,l} - \frac{\lambda_{i,k}}{\lambda_{j,k}}\lambda_{j,l} = 0 \tag{3}$$
$$\lambda_{i,l}\lambda_{j,k} - \lambda_{i,k}\lambda_{j,l} = 0 \tag{4}$$

which is a 2×2 determinantal relation which holds $\forall l \neq i, j$.

In particular, this relationship shows us that the values of the Λ matrix across row i is a multiple of the corresponding values in row j, except in the 2×2 block delimeted by $\Lambda_{i,i}$ and $\Lambda_{j,j}$. It can easily be shown that this ratio of entries $\frac{\lambda_{i,m}}{\lambda_{j,m}}$ is exactly the ratio of the conductivities of $i, j \forall m \neq i, j$.

Having shown that there are certain 2×2 determinantal conditions, we ask if there are 3×3, 4×4, ... conditions on the entries of the Λ matrix. It can be easily shown that any 3×3 determinants which must always equal 0 contain 2×2 determinants which force this condition.

There are, however, 4×4 determinantal conditions which are independent of any 2×2 sub-determinantal conditions. These conditions can be shown to arise from a recovery algorithm for the interior edges of the network (§3.2). Further, by the structure of the unit cube network, any 5×5 or larger determinantal conditions would be implied by 2×2 and 4×4 sub-determinantal conditions.

3.2 Derivation of 4×4 determinantal conditions

Let us now consider how to recover the conductance of an interior edge pq. We may view the determinantal conditions as arising from spherical pairs or from the recovery process itself. Figure 3.2 shows a spherical pair (a,b,c,d);(α,β,γ,δ) for which there is no connection. If we make the limiting assumption that the Λ we consider contain no 3×3 sub-matrices M such that $\det(M) = 0$, then this is a unique relation with $\det(($a,b,c,d$);(\alpha,\beta,\gamma,\delta)) = 0$. Many such patterns are possible, all of which hinge on some element of (a, b, c, d) being isolated by the other three. The conditions resulting from spherical pairs formed in this manner are summarized in §5.

4 Parameters for Λ.

Let $\Omega = (\Omega_0, \Omega_1)$ be a unit cube with two boundary spikes per vertex with conductivity γ. The Dirichlet-to-Neumann map is represented by a 16×16 matrix $\Lambda = \{\lambda_{i,j}\}$, as in §3. The parameters of Λ we take are the following:

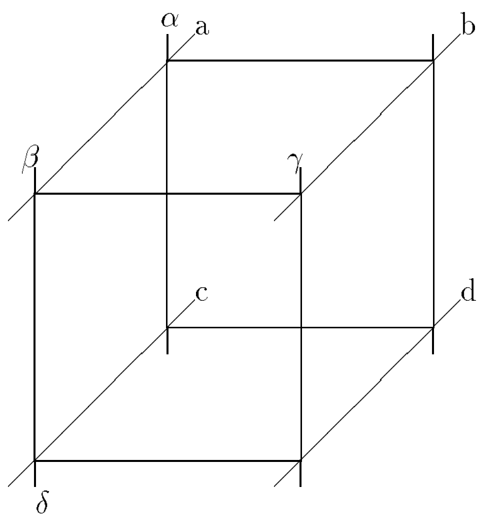

Figure 3: Recovering γ for an interior edge

(so far, best to date is 30 parameters for the 28 conductors. Work continues....)

(1,9), (1,10), (2,9), (3,15), (3,16), (4,15), (5,13), (5,14), (6,13), (7,11), (7,12), (8,11)

(1,3), (1,5), (1,7), (1,11), (1,13), (1,15)

(3,5), (3,7), (3,9), (3,11), (3,13)

(5,7), (5,9), (5,11), (5,15)

(7,8), (7,13), (13,14)

4.1 Numerical experiments

In an effort to locate parameterizations, a set of computer programs were written to perform various tasks. One of these programs performed a frequency distribution for the number of times an entry $\lambda_{i,j}$ was referenced in the

Λ matrix by the various known determinantal conditions. It was hoped that analysis of the relative frequencies of "hits" would lead to a way of choosing the 28 parameters for Λ. This has not yielded the hoped-for results.

A second program allows the seeding of any number of locations in the Λ matrix, and then allowing the program to randomly select a user-specified number of parameters. It then uses all known determinantal conditions and the symmetry property to determine if the starting parameterization yields all other entries in Λ.

To date, four weeks and billions of iterations on 28 random parameter choices have yielded no such successful parameterization. A similar situation exists for 29 random parameters. However, allowing for 30 parameters has yielded solutions in a few hours to a few days. Further, seeding the Λ matrix with a minimal set of 12 parameters which describe the relationships between all pairs of boundary spikes, and allowing 18 random choices, has produced successful results in a few hours of run time.

The results from this experiment would seem to lend support to the conjecture that there are some non-determinantal conditions necessary to recover all γ in the cubic case, and that criticality does not imply recoverability.

5 Appendix–A Listing of All Known 4×4 Determinantal Conditions

For the following eight relations, the set Q is restricted by the listed condition, and the further condition that no i, j exist such that $i + 1 = j$ and i odd (i.e. no two in the set share the same vertex/no 2×2 sub-determinants).

(1,3,5,11) ;($\neq$ 4)
(1,3,7,9) ;($\neq$ 2)
(1,5,7,15) ;($\neq$ 8)
(1,9,11,15) ;($\neq$ 10)
(3,5,7,13) ;($\neq$ 6)
(3,9,11,13) ;($\neq$ 12)
(5,11,13,15) ;($\neq$ 14)
(7,9,13,15) ;($\neq$ 16)

For the following conditions, it is equally true that (for example) $\det((1,3,5,7);(2,9,11,16)) = 0$. But since we know all adjacent rows and columns of the form $2i-1, 2i$ are multiples, it is unnecessary to explicitly list the additional 7 conditions per listed spherical pair.

(1,3,5,7) ;(2,9,11,15)
(1,3,5,7) ;(4,9,11,13)
(1,3,5,7) ;(6,11,13,15)
(1,3,5,7) ;(8,9,13,15)

(1,3,5,9) ;(7,10,13,15) (1,3,5,9) ;(6,11,13,15)

(1,3,5,13) ;(2,9,11,15) (1,3,5,13) ;(7,9,14,15)

(1,3,5,15) ;(4,9,11,13) (1,3,5,15) ;(7,9,13,16)

(1,3,7,11) ;(5,12,13,15) (1,3,7,11) ;(8,9,13,15)

(1,3,7,13) ;(2,9,11,15) (1,3,7,13) ;(5,11,14,15)

(1,3,7,15) ;(4,9,11,13) (1,3,7,15) ;(5,11,13,16)

(1,3,9,11) ;(2,5,7,15)
(1,3,9,11) ;(4,5,7,13)
(1,3,9,11) ;(5,12,13,15)
(1,3,9,11) ;(7,10,13,15)

(1,3,9,13) ;(2,5,7,15) (1,3,9,13) ;(5,11,14,15)

(1,3,9,15) ;(4,5,7,13) (1,3,9,15) ;(5,11,13,16)

(1,3,11,13) ;(2,5,7,15) (1,3,11,13) ;(7,9,14,15)

(1,3,11,15) ;(4,5,7,13) (1,3,11,15) ;(7,9,13,16)

(1,5,7,9) ;(3,10,11,13) (1,5,7,9) ;(6,11,13,15)

(1,5,7,11) ;(3,9,12,13) (1,5,7,11) ;(8,9,13,15)

(1,5,7,13) ;(2,9,11,15) (1,5,7,13) ;(3,9,11,14)

(1,5,9,11) ;(7,10,13,15) (1,5,9,11) ;(3,6,7,13)

(1,5,9,15) ;(3,10,11,13) (1,5,9,15) ;(3,6,7,13)

(1,5,11,13) ;(2,3,7,9) (1,5,11,13) ;(7,9,14,15)

(1,5,11,15) ;(2,3,7,9)
(1,5,11,15) ;(3,9,12,13)
(1,5,11,15) ;(7,9,13,16)
(1,5,11,15) ;(3,6,7,13)

(1,5,13,15) ;(2,3,7,9) (1,5,13,15) ;(3,9,11,14)

(1,7,9,11) ;(5,12,13,15) (1,7,9,11) ;(3,5,8,13)

(1,7,9,13) ;(2,3,5,11) (1,7,9,13) ;(5,11,14,15)

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(1,7,9,15) ;(3,5,8,13)

(1,7,11,15) ;(3,9,12,13) (1,7,11,15) ;(3,5,8,13)

(1,7,13,15) ;(2,3,5,11) (1,7,13,15) ;(3,9,11,14)

(1,9,11,13) ;(2,5,7,15) (1,9,11,13) ;(3,5,7,14)

(1,9,13,15) ;(2,3,5,11) (1,9,13,15) ;(3,5,7,14)

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(9,11,13,15) ;(3,5,7,14)

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