Determining the Resistors in a Network

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Abstract

In this paper we show that the resistors in a rectangular network can be determined by measurements at the boundary of the voltages generated by imposed currents. We also give an algorithm for using the boundary measurements to compute the resistances.

Key words. network of resistors, inverse problem, conductivity.

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1 Introduction

We consider a rectangular network of resistors in the plane. Let $\mathbb{Z}^2$ be the lattice in $\mathbb{R}^2$ consisting of the points with integer coordinates. Two lattice points $p$ and $q$ are said to be adjacent if there is a horizontal or vertical segment of length one joining them. (The points $p$ and $q$ will also be called neighbors, and the line segment joining them will be called $pq$.) Suppose given integers $(a, b)$ with $a < b$, and $(c, d)$ with $c < d$. A rectangular network $\Omega$ is constructed as follows. The nodes of $\Omega$ are the lattice points $p = (i, j)$ for which $a \leq i \leq b$ and $c \leq j \leq d$, with the four corner points $(a, c), (b, c), (a, d)$ and $(b, d)$ excluded. The set of nodes will be denoted $\Omega_0$. For each lattice point $p$, the set of four adjacent lattice points will be called $\mathcal{N}(p)$. The interior $\text{int} \Omega_0$ of $\Omega_0$ consists of those nodes $p$ all of whose neighbors are in $\Omega_0$. The boundary $\partial \Omega_0$ is $\Omega_0 - \text{int} \Omega_0$. Every boundary node $p$ has exactly one neighbor in $\Omega_0$, which is an interior node. An edge of $\Omega$ is the horizontal and vertical line segment $\sigma = pq$ which connects a pair of adjacent nodes $p$ and $q$ in $\text{int} \Omega_0$, or which connects a boundary node $p$ to its adjacent interior node $q$. The set of edges will be denoted $\Omega_1$. Figure 1.1 shows a network with 49 edges, 20 interior nodes and 18 boundary nodes.

![Figure 1.1](image)

A network of resistors $\Gamma = (\Omega_0, \Omega_1, \gamma)$ is a network $\Omega = (\Omega_0, \Omega_1)$ together with a function $\gamma : \Omega_1 \to R^+$ where $R^+$ is the set of positive real numbers. For each edge $\sigma = pq$ in $\Omega_1$, the number $\gamma(\sigma)$ is called the conductance of $\sigma$, and
$1/\gamma(\sigma)$ is the resistance of $\sigma$. The function $\gamma$ on $\Omega_1$ is called the conductivity.

For any function $f : \Omega_0 \rightarrow R$, we define a function $L_\gamma f : \text{int } \Omega_0 \rightarrow R$ by

$$L_\gamma f(p) = \sum_{q \in \mathcal{N}(p)} \gamma(pq)(f(q) - f(p))$$

A function $f : \Omega_0 \rightarrow R$ which satisfies $L_\gamma f(p) = 0$ for all interior nodes will be called $\gamma$-harmonic. If voltage $\phi(r)$ is applied at each boundary node $r$, the network $\Omega$ will acquire a unique voltage $f(p)$ at every interior node $p$ according to Kirkhoff’s Law, which states that for each interior node $p$, $L_\gamma f(p) = 0$. (See Section 2.) The function $\phi$ defined on the boundary nodes determines a current $I_\phi(r)$ through each boundary node $r$, by $I_\phi(r) = \gamma(rq)(f(r) - f(q))$, where $q$ is the unique neighbor of $r$ in int $\Omega_0$. For each conductivity $\gamma$, a quadratic form $Q_\gamma$ is defined as follows. For two boundary functions $\psi$ and $\phi$,

$$Q_\gamma(\psi, \phi) = \sum_{r \in \partial \Omega_0} \psi(r) I_\phi(r)$$

The main result of this paper (Theorem 3.2) is the solution of the inverse conductivity problem for a network of resistors: $\gamma$ is uniquely determined by the quadratic form $Q_\gamma$. Suppose that $\Omega = (\Omega_0, \Omega_1)$ is a network with $m$ boundary nodes. Let $F$ be the space of quadratic forms on $R^m$. Let $T$ be the map from from conductivities on $\Omega_1$ to $F$ defined by $T(\gamma) = Q_\gamma(\cdot, \cdot)$. We calculate the differential $dT$ and show that $T$ is an embedding of the space of conductivities onto a submanifold of $F$.

**Remark 1** Our approach gives a direct method for calculating the conductivity of each resistor in the network.

**Remark 2** For the sake of clarity in this paper we have given the proofs for networks in the plane. Similar methods apply to all dimensions higher than two.

**Remark 3** The restriction to rectangular networks is also unnecessary. The main results (e.g. Theorems 3.2 and 5.1) are true for general finite subnetworks of the integer lattice of $Z^n$. (See the end of Section 3.)

**Remark 4** The argument for the calculation of the differential $dT$ follows the pattern originally given by A. Calderon [3].
**Remark 5** We have benefitted from discussions with G. Uhlmann and J. Sylvester, who introduced us to inverse problems associated with the conductivity equation $\text{Div}(\gamma \text{Grad}(u)) = 0$. For the conductivity equation (in dimension at least three), they proved the uniqueness in [6]. The continuity of the inverse was shown by Allessandrini in [2].

## 2 Preliminaries

In this section, we establish some facts about $\gamma$-harmonic functions.

**Lemma 2.1** Let $f$ be a $\gamma$-harmonic function on $\Omega$, and let $p$ be an interior node. Then either $f(p) = f(q)$ for all nodes $q \in \mathcal{N}(p)$ or else there is at least one node $q_1 \in \mathcal{N}(p)$ for which $f(p) > f(q_1)$ and there is at least one node $q_2 \in \mathcal{N}(p)$ for which $f(p) < f(q_2)$.

**Proof:** By 1.1,

$$\{ \sum_{q \in \mathcal{N}(p)} \gamma(pq) \} f(p) = \sum_{q \in \mathcal{N}(p)} \gamma(pq) f(q)$$

That is, $f(p)$ is the weighted average of its neighbors, with positive weights. If the value of $f$ at some neighbor is less than $f(p)$, then the value at some other neighbor is greater than $f(p)$. QED

**Corollary 2.2** Let $f$ be a $\gamma$-harmonic function on $\Omega_0$. Then the maximum and minimum values of $f$ occur on the boundary of $\Omega_0$.

**Proof:** Suppose that the maximum value occurs at $p_0 \in \text{int} \Omega_0$, and that $f(p_0) > f(q)$ for every $q \in \partial \Omega_0$. Let $\{p_0, p_1, \ldots, p_n\}$ be a sequence of nodes in $\Omega_0$ such that each $p_jp_{j+1}$ is an edge in $\Omega_0$ and $p_n \in \partial \Omega_0$. Then let $j$ be the first index for which $f(p_j) < f(p_0)$. Then $f(p_{j-1}) = f(p_0) \geq f(q)$ for all $q \in \mathcal{N}(p_{j-1})$ and $f(p_{j-1}) > f(p_j)$. This would contradict Lemma 2.1. Similarly for the minimum. QED.

**Corollary 2.3** Let $f : \Omega_0 \rightarrow R$, be a function such that $L_\gamma f(p) = 0$ for all $p \in \text{int} \Omega_0$, and $f(p) = 0$ for all $p \in \partial \Omega_0$. Then $f(p) = 0$ for all $p$.

**Proof:** Immediate from Corollary 2.2.
**Proposition 2.4** Let functions $h : \text{int } \Omega_0 \to R$, and $\phi : \partial \Omega_0 \to R$ be given. Then there is a unique function $f : \Omega_0 \to R$ such that $L_\gamma f(p) = h(p)$ for all $p \in \text{int } \Omega_0$, and $f(p) = \phi(p)$ for all $p \in \partial \Omega_0$.

**Proof:** Consider the square system of linear equations for unknowns $f(p)$:

$$L_\gamma f(p) = h(p) \quad \text{for } p \in \text{int } \Omega_0$$

$$f(p) = \phi(p) \quad \text{for } p \in \partial \Omega_0$$

This system of equations has a unique solution since

$$L_\gamma f(p) = 0 \quad \text{for } p \in \text{int } \Omega_0$$

$$f(p) = 0 \quad \text{for } p \in \partial \Omega_0$$

has zero as its unique solution, by Corollary 2.3.

We need a discrete version of Green’s Formula. Let $(p_0, p_1, \ldots, p_n)$ be the nodes along a horizontal (or vertical) line in $\Omega_1$ as in Figure 2.1.

![Figure 2.1](image)

**Lemma 2.5** Let $V_0$ be the collection of nodes $p_i$, and let $E_1$ be the collection of edges $p_ip_{i+1}$. Let $f$ and $g$ be two functions $f, g : V_0 \to R$ and let $\gamma$ be a function $\gamma : E_1 \to R$. Let $f_i = f(p_i)$, $g_i = g(p_i)$, and $\gamma_i = \gamma(p_ip_{i+1})$. Then

$$\sum_{i=0}^{n-1} \gamma_i(f_{i+1} - f_i)(g_{i+1} - g_i) = \gamma_0g_0(f_0 - f_1)$$

$$-g_1[\gamma_0(f_0 - f_1) + \gamma_1(f_2 - f_1)]$$

$$-g_2[\gamma_1(f_1 - f_2) + \gamma_2(f_3 - f_2)]$$

$$+ \cdots + \gamma_{n-1}g_n(f_n - f_{n-1})$$

**Proof:** Rearrange the terms. QED
**Notation** Henceforth, let the nodes in $\Omega_0$ be indexed $p_i$. For a function $f : \Omega_0 \to R$, let $f_i = f(p_i)$. For a function $\gamma : \Omega_1 \to R$, let $\gamma_{ij} = \gamma(\sigma_{ij})$ for each edge $\sigma_{ij} = p_ip_j \in \Omega_1$. For each boundary node $p_i$, let $e_i$ be the unique edge connecting it to its adjacent interior node denoted $nb(p_i)$.

**Proposition 2.6** (Discrete Green’s Formula) Let $\Omega = (\Omega_0, \Omega_1)$ be a network, and let $\gamma : \Omega_1 \to R$ and $f, g : \Omega_0 \to R$ be functions. Then

\[
\sum_{\sigma_{ij} \in \Omega_1} \gamma_{ij}(f_i - f_j)(g_i - g_j) = \sum_{p_i \in \partial \Omega_0} g_i\gamma(e_i)(f(p_i) - f(nb(p_i))) - \sum_{p_i \in \int \Omega_0} g_iL_\gamma f(p_i)
\]

**Proof:** Use Lemma 2.5 and compute the sum on the left by first summing horizontally and then vertically.

**Corollary 2.7** If $\Gamma = (\Omega_0, \Omega_1, \gamma)$ is a network of resistors and $f$ and $g$ are $\gamma$-harmonic functions on $\Omega_0$ with $g = \psi$ and $f = \phi$ on $\partial \Omega_0$, then

\[
Q_\gamma(\psi, \phi) = \sum_{p \in \partial \Omega_0} \psi(p)I_{\phi}(p)
= \sum_{\sigma_{ij} \in \Omega_1} \gamma_{ij}(f_i - f_j)(g_i - g_j)
= \sum_{p \in \partial \Omega_0} \phi(p)I_{\phi}(p)
= Q_\gamma(\phi, \psi)
\]

where the above notation is used.

**Proof:** Immediate from Proposition 2.6, using $L_\gamma(f) = 0$ and $L_\gamma(g) = 0$. QED

We will use a process which we call harmonic continuation. Let $\Gamma = (\Omega_0, \Omega_1, \gamma)$ be a network of resistors, and let the columns of $\Omega_0$ be $C_0, C_1, \ldots, C_n$, numbering from left to right. Let $S$ be the subset of $\Omega_0$ consisting of the nodes in columns $C_0, C_1, \ldots, C_k$. Suppose that $f$ is a function defined on $S$ which is $\gamma$-harmonic on the nodes which are interior in $S$. 

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Lemma 2.8 In this situation $f$ can be defined on $C_{k+1}$, so that $f$ is harmonic on the larger set. The definition of $f$ is uniquely determined on the interior nodes of $C_{k+1}$, and can be given arbitrary values on the endnodes of column $C_{k+1}$.

Proof: First notice that the definition of $f$ on the nodes of column $C_{k+1}$ will not affect the assumed harmonicity of $f$ at any of the nodes in columns $C_0, C_1, \ldots, C_{k-1}$. We consider a node $p$ in column $C_k$, with its four neighbors $q_i$, as in Figure 2.2.

![Figure 2.2](image)

Kirkhoff’s Law implies that we must have

$$\{ \sum_{q \in N(p)} \gamma(pq) \} f(p) = \sum_{q \in N(p)} \gamma(pq) f(q)$$

If $q$ is an interior node of $C_{k+1}$, then $f(q)$ is determined by the values to the left. In Figure 2.2, $f(q_4)$ is determined by $f(p)$, $f(q_1)$, $f(q_2)$ and $f(q_3)$. The values $f$ on the two boundary nodes of $C_{k+1}$ can be assigned arbitrarily. QED
Harmonic continuation is also valid to the left, up or down. Note that although the values of $f$ on the boundary nodes of $C_k$ are arbitrary, these values affect the next step of the continuation. Consider Figure 2.3, where the dotted line is the diagonal of slope -1 passing through the top node of $C_k$.

![Figure 2.3](image)

**Lemma 2.9** Let $\Gamma = (\Omega_0; \Omega_1; \gamma)$ be a network of resistors. Suppose a function $f$ is defined and constant on the nodes of columns $C_0, C_1, \ldots, C_k$. Then $f$ can be continued as a $\gamma$-harmonic function where $f$ is constant on or below the diagonal indicated by the dotted line. The values of $f$ at boundary nodes at the tops of columns $C_{k+1}, \ldots, C_n$ are arbitrary.

**Proof:** Immediate from Lemma 2.8. QED

This shows that it is possible for a $\gamma$-harmonic function to be locally constant, without being constant throughout $\Omega$. This is in contrast to the continuous case, where a harmonic function which is constant on an open set must be identically constant.

We need some facts (well-known) about the discrete Neumann problem. Let $f$ be a real function defined on $\Omega_0$ with $L_\gamma f = 0$. Let $\phi = f|_{\partial \Omega_0}$ and
\[ I_\phi(p) = \gamma(pq)[f(p) - f(nb(p))] \] where \( nb(p) \) is the unique neighbor of \( p \) in \( \text{int} \ \Omega_0 \). Then we have the following.

**Lemma 2.10** Suppose \( L_\gamma f(p) = 0 \) for \( p \in \text{int} \ \Omega_0 \) and \( I_\phi(r) = 0 \) for \( r \in \partial \Omega_0 \). Then there is a \( c \in R \) in such that \( f(p) = c \) for all \( p \in \Omega_0 \).

**Proof:** Since \( L_\gamma f = 0 \),
\[
Q_\gamma(\phi, \phi) = \sum_{\sigma_{ij} \in \Omega_1} \gamma_{ij}(f_i - f_j)^2 = \sum_{p \in \partial \Omega_0} \phi(p) I_\phi(p) \quad (1)
\]
Since \( I_\phi(p) = 0 \) for \( p \in \partial \Omega_0 \), \( Q_\gamma(\phi, \phi) = 0 \). Thus \( f(p_i) = f(p_j) \) for all \( p_i, p_j \in \Omega_0 \). QED

Suppose values \( \{ J_p \} \) are given; consider the system of equations (one equation for each node in \( \Omega_0 \)):
\[
L_\gamma f(p) = 0, \text{ for } p \in \text{int} \ \Omega_0 \\
\gamma(pq)[f(q) - f(p)] = -J_p, \text{ for } p \in \partial \Omega, q = nb(p) \quad (2)
\]
The unknowns are the values of \( f \).

**Lemma 2.11** This system of equations has a solution if and only if
\[
\sum_{p \in \partial \Omega_0} J_p = 0
\]
If \( f \) and \( g \) are two solutions then \( f - g \) is constant.

**Proof:** (This is an application of the Fredholm alternative to the discrete situation). Let \( B_\gamma \) be the matrix of the system of equations (3) and let \( e = [1, 1, \ldots, 1] \in R^n \), where \( n \) is the number of nodes. Then it is easy to see that
\[
e \cdot B_\gamma = 0 \quad (3)
\]
By Lemma 2.10, \( \ker(B_\gamma) = \{ c \cdot e^T : c \in R \} \). By (4) the range of \( B_\gamma \) is orthogonal to the kernel of \( B_\gamma \) (\( B_\gamma \) is a square matrix). Since \( \text{dim}(\ker(B_\gamma)) = 1 \), \( B_\gamma \) maps onto \( \{ b : e \cdot b = 0 \} \). Since \( \sum_{p \in \partial \Omega_0} J_p = 0 \), there is a solution of (3). By Lemma 2.10, it is unique up to a constant. QED
3 Global uniqueness

Let $\Omega = (\Omega_0, \Omega_1)$ be a rectangular network with $N$ edges and $m$ boundary nodes. We show that the map $T$ from $(R^+)^N$ to the space $F$ of quadratic forms on $R^m$ is $1-1$. Observe that

$$\sum_{p_j \in \partial \Omega_0} \phi(p_j) I_{\psi}(p_j) = Q_\gamma(\phi, \psi)$$

$$= \frac{1}{2} [Q_\gamma(\phi + \psi, \phi + \psi) - Q_\gamma(\phi, \phi) - Q_\gamma(\psi, \psi)] ,$$

(4)

Thus knowing $Q_\gamma(\phi, \phi)$ for all $\phi$ is equivalent to knowing the Dirichlet to Neumann map $\Lambda_\gamma$, (or its inverse the Neumann to Dirichlet map). $\Lambda_\gamma$ maps the boundary value function $\phi(p), p \in \partial \Omega_0$ to the current function $I_\phi(p), p \in \partial \Omega_0$ which is determined by the solution to the Dirichlet problem with boundary values $\phi$. (The boundary values and boundary currents are dually paired by equation (5)). In this sense we show that measurements at the boundary determine $\gamma$.

At each corner of the rectangular network there are two edges, each containing a boundary node. We first show how to determine the conductances of these edges.
Referring to Figure 3.1, we want to compute the conductances $\gamma(\sigma_0)$ and $\gamma(\sigma_1)$. Consider the following Neumann problem. The current is set equal to 0 at all boundary nodes except at the corner pair, where the current is 1 at node $q_0$ and -1 at node $q_1$. To uniquely determine the solution, the voltage is set equal to 0 at the boundary node $p_0$ at the top left. By Lemma 2.9 we know that there is a (unique) $\gamma$-harmonic function with values zero everywhere except in the upper right corner. This is also the solution of the Neumann problem we have just posed which is unique by Lemma 2.11. Thus by measuring the voltages $f(q_0)$ and $f(q_1)$, we know the conductances $\gamma(\sigma_0) = f(q_0)^{-1}$ and $\gamma(\sigma_1) = -f(q_1)^{-1}$ of these edges. Referring to Figure 3.2, consider the conductances of each edge within the strip bounded by the diagonal lines (dotted) of slope -1.
Assume inductively that we know the conductances of each edge above the diagonal from \( p_1 \) to \( q_k \). By Lemma 2.9 there is a \( \gamma \)-harmonic function \( h \) such that \( h \) is 0 on and below the lower diagonal, has current 1 at the node \( p_1 \), and has current 0 at all other exterior nodes except at nodes \( q_j \) for \( j = 1, \ldots, k \). Suppose the current at \( q_j \) is \( -\alpha_j \).

**Lemma 3.1** The numbers \( \alpha_1, \ldots, \alpha_k \) are uniquely determined by the conditions on the currents at the other exterior nodes and the condition that \( h \) have the same value at all boundary nodes on the left side.

**Proof:** This follows from Lemma 2.9. QED

For each \( j = 1, 2, \ldots, k \), let \( h_j \) be the solution of the Neumann problem:

\[
\begin{align*}
I_{h_j}(q) & = 0, \text{ if } q \neq p_1, q_j \\
I_{h_j}(p_1) & = 1 \\
I_{h_j}(q_j) & = -1 \\
h_j(p_0) & = 0
\end{align*}
\]  

Figure 3.2
Then $\sum_{j=1}^{k} \alpha_j h_j$ solves the same Neumann problem as $h$ and thus

$$\sum_{j=1}^{k} \alpha_j h_j = h$$

(6)

Now we find $\{\alpha_1, \ldots, \alpha_k\}$. We know that (6) has a unique solution. We also know that if

$$\sum_{j=1}^{k} \alpha_j = 1$$

$$\sum_{j=1}^{k} \alpha_j h_j(q) = 0$$

(7)

for all $q$ which are boundary nodes of the left column, then (7) holds by Lemma 3.1. Hence the (sometimes) overdetermined system (8) has a unique solution $\{\alpha_1, \ldots, \alpha_k\}$. Thus by using the solutions of the Neumann problems (6) and by solving (8) we can find the function $h$. We now have a $\gamma$-harmonic function $h$ with known values and currents at all boundary nodes. The values of $h$ are in fact 0 at all nodes on and to the left of the lower of the two dotted diagonals. Moreover the values of $h$ are also known at all neighbors of boundary nodes since these values are either known to be zero or can be computed from known conductances, currents, and boundary values. By using Kirchhoff’s Law and known conductances we find the values of $h$ at all remaining nodes. We will use the function $h$ and the inductively known conductances above the diagonal to compute the conductances within the diagonal strip.
In Figure (3.3) the letters $\tau_j$ stand for the conductances of resistors in the diagonal strip. The letters $p_j$ stand for the nodes in this strip. We first compute the conductance $\tau_1$ from the known current through from $p_1$ to $p_2$ (it is 1) and the voltage drop from $p_1$ to $p_2$ (it is the value of $h$ at $p_1$). We next use Kirchhoff’s law at $p_2$ to compute $\tau_2$. (Three currents are known and the voltages at $p_2$ and $p_3$ are known.) We can then compute $\tau_3$ by using Kirchhoff’s law at $p_3$ and the known voltages and conductances. We continue until we finally compute $\tau_l$ ($l = 2k$). By going to the right, left, up or down, we can inductively find the conductance of each resistor in the network. We have thus proved the following.

**Theorem 3.2** $T$ is $1 - 1$. That is, if $\gamma_1 \neq \gamma_2$ then $Q_{\gamma_1} \neq Q_{\gamma_2}$.

The proof of Theorem 3.2 describes a direct algorithm for calculating $\gamma$ from the Neumann to Dirichlet map. There is a similar direct algorithm for calculating $\gamma$ from the Dirichlet to Neumann map.

We now give a justification of Remark 3 of Section 1. Let $\Omega = (\Omega_0, \Omega_1)$ be a general network of the following form. $\Omega_1$ is a finite set of edges $pq$ where
\( p, q \) are points in the integer lattice \( Z^2 \) and \( |p - q| = 1; \) \( \Omega_0 = \{ p \in Z^2 : pq \in \Omega_1 \text{ for some } q \} \). We take the same definition of interior and boundary: \( \text{int } \Omega_0 = \{ p \in \Omega_0 : N(p) \subset \Omega_0 \}; \) \( \partial \Omega_0 = \Omega_0 - \text{int } \Omega_0 \). A node \( p \in \partial \Omega_0 \) may have one, two, or three neighbors in \( \Omega_0 \). As before, a conductivity function is a function \( \gamma \) from \( \Omega_1 \) to \( R^+ \), and a resistor network is a triple \( (\Omega_0, \Omega_1, \gamma) \). In this context if \( \phi \) is a boundary potential, and \( f \) is the harmonic function with boundary values \( \phi \), the current at a boundary node \( p \in \partial \Omega_0 \) is now be defined as

\[
I_\phi(p) = \sum_{q \in N(p) \cap \Omega_0} \gamma(pq) [f(p) - f(q)]
\]

It is straightforward to verify most of the previous results for such a network, but for the proof of Theorem 3.2 in this general setting, we need Proposition 3.3 below.

Let \( \Gamma = (\Omega_0, \Omega_1, \gamma) \) be a general resistor network, as above, with conductivity function \( \gamma \). Adjoin edges to \( \Omega \) to produce a rectangular network. Make this enlarged network into a resistor network \( (W_0, W_1, \mu) \) by setting \( \mu \) equal to \( 1 \) on the adjoined edges.

**Proposition 3.3** \( \Lambda_\gamma \) determines \( \Lambda_\mu \).

**Proof:** Let \( \phi \) be given on \( \partial W_0 \). We show how to compute \( \Lambda_\mu \phi \). Let \( \{ p_1, \ldots, p_k \} \) be the boundary nodes of \( \Omega_0 \). If \( u \) is a function defined on \( \partial \Omega_0 \) \( \Lambda_\gamma \) is regarded as a matrix acting on the vector \( [u(p_1), \ldots, u(p_k)]^T \) so that \( \Lambda_\gamma u \) is the vector of currents through \( \{ p_1, \ldots, p_k \} \). Consider the following set of equations.

\[
L_\mu f(p) = 0 \text{ for } p \in \text{int } W_0 - \Omega_0 \tag{8}
\]

\[
\Lambda_\gamma [f(p_1), \ldots, f(p_k)]^T = [J(p_1), \ldots, J(p_k)]^T \text{ for } p_j \in \partial \Omega_0 \tag{9}
\]

where \( f | \partial W_0 = \phi \). In this set of equations, when \( p \in \partial \Omega_0 \cap \text{int } W_0 \), we define

\[
J(p) = \sum_{q \in N(p) \cap \Omega_0} \mu(pq) [f(q) - f(p)]
\]

\[
= \sum_{q \in N(p) \cap \Omega_0} [f(q) - f(p)]
\]

(The second equality holds since \( \mu(pq) = 1 \) if \( pq \in W_1 - \Omega_1 \).) The unknown terms in (9) and (10) are the values \( f(p) \) for \( p \in \text{int } W_0 - \text{int } \Omega_0 \), and \( J(p) \) for
\( p \in \partial \Omega_0 \cap \partial W_0 \). The number of unknowns equals the number of equations. We now show that this system has a unique solution. We know from Proposition 2.4 that there is a unique solution of the Dirichlet problem:

\[
L_{\mu} h = 0, \ h|_{\partial W_0} = \phi \quad (10)
\]

This function \( h \) will be used to find a solution of (9) and (10). For \( p \in \partial \Omega_0 \cap \partial W_0 \) set \( J(p) = \mu(pq)[h(p) - h(nb(p))] \) where \( nb(p) \) is the unique neighbor of \( p \) in \( W_0 \), and set \( f(p) = h(p) \) for \( p \in \text{int} W_0 \). This gives a solution of (9) and (10). This is the only possible solution of (9) and (10), since a different solution of (9) and (10) would lead to a different solution of the the Dirichlet problem (11). Thus we know that the system (9) and (10) has a unique solution. Then \( \Lambda_{\mu} \) is given by

\[
\Lambda_{\mu} \phi(p) = J(p), \ i f \ p \in \partial \Omega_0 \cap \partial W_0 \quad (11)
\]

\[
\Lambda_{\mu} \phi(p) = f(p) - f(nb(p)), \ i f \ p \in \partial W_0 - \partial \Omega_0 \quad (12)
\]

where \( f \) and \( J \) are the solutions of (9) and (10). QED

**Corollary 3.4** For a general network of resistors, the conductivity \( \gamma \) is uniquely determined by \( \Lambda_{\gamma} \).

**Proof:** This follows from from Proposition 3.3 and Theorem 3.2. QED

## 4 The differential of T

For the computation of the differential we consider \( T(\gamma + \kappa) \) for a small perturbation \( \kappa \). For any function \( \alpha : \Omega_1 \to R \), the norm of \( \alpha \) is \( ||\alpha|| = \max |\alpha(\sigma)| \) for \( \sigma \in \Omega_1 \). Fix a conductivity \( \gamma \), and consider a perturbation \( \gamma + \kappa \) where \( ||\kappa|| \) is small. Consider the solutions of

\[
L_{\gamma + \kappa} u(p) = 0 \quad (13)
\]

for values \( u(p) \) for \( p \in \text{int} \ \Omega_0 \), and where the values of \( u(p) = \phi(p) \) for \( p \in \partial \Omega_0 \). Let \( u = f + g \), where \( f = \phi \) on \( \partial \Omega_0 \) and \( L_{\gamma} f = 0 \) in \( \text{int} \ \Omega_0 \). Then \( g = 0 \) on \( \partial \Omega_0 \), and (14) implies that:

\[
(L_{\gamma} + L_{\kappa}) g = -L_{\kappa} f \quad (14)
\]
in \( \text{int } \Omega_0 \). If \( h \) is given, \( L_\gamma^{-1} h \) is defined to be the solution \( v \) of \( L_\gamma v = h \) with \( v = 0 \) on \( \partial \Omega_0 \). By Proposition 2.4, this makes sense. In the present context, this implies that

\[
L_\gamma^{-1} L_\gamma g = g
\]
since \( g = 0 \) on \( \partial \Omega_0 \). Equation (15) yields

\[
(I + L_\gamma^{-1} L_\kappa)g = -L_\gamma^{-1} L_\kappa f
\]
in \( \text{int } \Omega_0 \). \( L_\kappa f \) is linear in \( \kappa \) at all nodes in \( \text{int } \Omega_0 \). Hence \( -L_\gamma^{-1} L_\kappa f \) is linear in \( \kappa \). Moreover if \( \| \kappa \| \) is small, then \( I + L_\gamma^{-1} L_\kappa \) is invertible on those \( g \) with \( g(p) = 0 \) on \( \partial \Omega_0 \). Also

\[
g = -(I + L_\gamma^{-1} L_\kappa)^{-1} L_\gamma^{-1} L_\kappa f = O(\kappa)
\]
(15)

That is, \( g \) vanishes to order 1 in \( \kappa \). We have

\[
Q_{\gamma+\kappa}(\phi, \phi) = \sum_{\sigma_{ij} \in \Omega_1} (\gamma_{ij} + \kappa_{ij})(f_i - f_j)(g_i - g_j)^2
\]
\[
= Q_\gamma(\phi, \phi) + 2 \sum_{\sigma_{ij} \in \Omega_1} \gamma_{ij}(f_i - f_j)(g_i - g_j) +
\]
\[
2 \sum_{\sigma_{ij} \in \Omega_1} \kappa_{ij}(f_i - f_j)(g_i - g_j) + \sum_{\sigma_{ij} \in \Omega_1} \gamma_{ij}(g_i - g_j)^2 +
\]
\[
\sum_{\sigma_{ij} \in \Omega_1} \kappa_{ij}(f_i - f_j)^2 + \sum_{\sigma_{ij} \in \Omega_1} \kappa_{ij}(g_i - g_j)^2
\]

By Proposition 2.6, since \( L_\gamma f = 0 \), and \( g = 0 \) on \( \partial \Omega_0 \), we have

\[
\sum_{\sigma_{ij} \in \Omega_1} \gamma_{ij}(f_i - f_j)(g_i - g_j) = 0
\]

By equation (16)

\[
2 \sum_{\sigma_{ij} \in \Omega_1} \gamma_{ij}(f_i - f_j)(g_i - g_j) + \sum_{\sigma_{ij} \in \Omega_1} \gamma_{ij}(g_i - g_j)^2 + \sum_{\sigma_{ij} \in \Omega_1} \kappa_{ij}(g_i - g_j)^2
\]

\[
= O(\kappa^2)
\]
Thus,
\[ Q_{\gamma+\kappa}(\phi, \phi) = Q_\gamma(\phi, \phi) + \sum_{i,j \in \Omega_1} \kappa_{ij}(f_i - f_j)^2 + O(\kappa^2) \] (16)

This proves the following. Recall that the space of quadratic forms on \( R^m \) is denoted \( F \).

**Theorem 4.1** The map \( T : (R^+)^N \to F \) is differentiable. The differential at \( \gamma \in (R^+)^N \) as a linear function of \( \kappa \in R^N \) is given by the quadratic form

\[ dT_\gamma \kappa(\phi, \psi) = \sum_{i,j \in \Omega_1} \kappa_{ij}(f_i - f_j)(g_i - g_j) \]

where \( f \) and \( g \) are \( \gamma \)-harmonic functions on \( \Omega_0 \), with \( f|_{\partial \Omega_0} = \phi \), and \( g|_{\partial \Omega_0} = \psi \).

**Theorem 4.2** Let \( \Gamma = (\Omega_0, \Omega_1, \gamma) \) be a network of resistors. Suppose given any function \( \kappa : \Omega_1 \to R \) and suppose that for all \( \gamma \)-harmonic \( f \) and \( g \) functions

\[ \sum_{i,j \in \Omega_1} \kappa_{ij}(f_i - f_j)(g_i - g_j) = 0 \]

Then \( \kappa(\sigma) = 0 \) for all \( \sigma \in \Omega_1 \). Hence the differential \( dT : R^N \to R^{m^2} \) is one to one.

**Proof:** We order the edges (from the outside in) as follows. First every node is assigned a *level* by \( \text{level}(p) = \min|p - r| \), where \( r \) is a boundary node. The nodes of a fixed level form the sides of a rectangle in \( \Omega_0 \). In the notation for an edge \( \sigma = pq \), it will be assumed that \( \text{level}(p) \leq \text{level}(q) \). Each edge \( \sigma = pq \) is assigned a level by \( \text{level}(pq) = \text{level}(p) + \text{level}(q) \). The edges are partially ordered by level. By ordering the edges arbitrarily within each level, we obtain a total ordering of the edges. There are two types of edges, those whose level is odd, and those whose level is even.

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Let $\sigma = pq \in \Omega_1$, be an edge of level one as indicated in Figure 4.1.

![Figure 4.1]

First we show that $\kappa(\sigma) = 0$. Let $f$ be a $\gamma$-harmonic function with $f(p) = 1$, and $f(r) = 0$ on all nodes $r$ on or below the diagonal of slope -1 which includes $q$. Similarly, let $g$ be a $\gamma$-harmonic function with $f(p) = 1$, and $f(r) = 0$ on all nodes $r$ on or below the diagonal of slope +1 which includes $q$. For $rs \neq \sigma$, $(f(r) - f(s))(g(r) - g(s)) = 0$. Furthermore, $(f(p) - f(q))(g(p) - g(q)) = 1$. Thus

$$\sum_{\sigma_{ij} \in \Omega_1} \kappa_{ij}(f_i - f_j)(g_i - g_j) = \kappa(\sigma)$$  \hfill (17)

By hypothesis, the sum is 0, so $\kappa(\sigma) = 0$.

Consider next an edge of level two like $\sigma = pq$ in Figure 4.2.
By using Lemma 3.1 as in Lemma 3.2, there is a $\gamma$-harmonic function $f$ with $f(p) = 0, f(q) = 1$ and $f(r) = 0$ for all nodes $r$ on or below the diagonal of slope 1 which includes $p$. There is also a $\gamma$-harmonic function $g$ with $g(p) = 0, g(q) = 1$ and $g(r) = 1$ for all nodes $r$ on or below the diagonal of slope +1 which includes $q$. We have already shown that $\kappa(\tau) = 0$ for each edge $\tau$ of level one. For this choice of $f$ and $g$, $(f(p) - f(q))(g(p) - g(q)) = 1$. For any other edge $rs$, either $(f(r) - f(s))(g(r) - g(s)) = 0$ or $\kappa(rs) = 0$. Thus formula (6) again holds. By hypothesis, the sum is 0, so $\kappa(\sigma) = 0$. This argument shows that $\kappa(\sigma) = 0$ for each edge $\sigma$ of level two. Considering next an edge of level three such as $\sigma$ in figure 4.3, we construct $\gamma$-harmonic functions $f, g$ which show that $\kappa(\sigma) = 0$. Continuing in this way, level by level, we find that $\kappa(\sigma) = 0$ on all edges. QED
For a given conductivity $\gamma$ on $\Omega$ and boundary potential $\phi$ the power dissipated by $\Omega$ is $Q_\gamma(\phi, \phi)$. The following proposition proves a kind of monotonicity for the power dissipated.

**Proposition 4.3** Suppose conductances $\gamma^0$ and $\gamma^1$ satisfy $\gamma^1(\sigma) \geq \gamma^0(\sigma)$ for all $\sigma \in \Omega_1$ and $\gamma^1(\tau) > \gamma^0(\tau)$ for some $\tau \in \Omega_1$. Then there is a $\phi$ such that $Q_{\gamma^1}(\phi, \phi) > Q_{\gamma^0}(\phi, \phi)$.

**Proof:** Let $\gamma^t = (1-t)\gamma^0 + t\gamma^1$, and let $\phi$ be any boundary function and define $s_\phi(t) = Q_{\gamma^t}(\phi, \phi)$. Then

$$s_\phi(1) - s_\phi(0) = \int_0^1 s'_\phi(t) dt$$

But

$$s'_\phi(t) = \sum_{\sigma_{ij} \in \Omega_1} \gamma^t_{ij}(t)(f^t_i - f^t_j)^2$$

where $L_{\gamma^t} f^t = 0$ and $f^t = \phi$ on $\partial \Omega_0$. Since $\gamma^t_{ij}(t) = \gamma^1_{ij} - \gamma^0_{ij} \geq 0$, it follows that $s_\phi(1) \geq s_\phi(0)$. If $\gamma^1_{kl} > \gamma^0_{kl}$ for some $\sigma_{kl}$, then it follows from Theorem 4.2

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that there is a $\gamma^0$-harmonic function $f^0$ such that

$$s'_{\phi}(0) = \sum_{\sigma_{ij} \in \Omega_1} (\gamma^0_{ij} - \gamma^0_{i,j})(f^0_i - f^0_j)^2 \neq 0$$

and hence $s'_{\phi}(0) > 0$, where $\phi$ is $f^0$ restricted to $\partial \Omega_0$. Thus $Q_{\gamma_1}(\phi, \phi) \geq Q_{\gamma^*}(\phi, \phi)$. QED

5 The continuity of $T^{-1}$

Let $\Omega = (\Omega_0, \Omega_1)$ be a network with $N$ edges and $m$ boundary nodes.

**Theorem 5.1** The map $T$ has a continuous inverse. The image $T((R^+)^N)$ is an embedded submanifold of $F$, and $T$ is a diffeomorphism onto its image.

**Proof:** Let $\gamma_1$ and $\gamma_2$ be two conductivity functions on $\Omega_1$. Suppose that $Q_{\gamma_1}$ is close to $Q_{\gamma_2}$. Then $\Lambda_{\gamma_1}$ is close to $\Lambda_{\gamma_2}$. This means that if $\phi$ is given then $\Lambda_{\gamma_1}(\phi)$ is close to $\Lambda_{\gamma_2}(\phi)$, and if $I$ is given with $\sum_{p \in \partial \Omega_0} I(p) = 0$ then $\phi_1 = \Lambda_{\gamma_1}^{-1}(I)$ is close to $\phi_2 = \Lambda_{\gamma_2}^{-1}(I)$. (By $\Lambda_{\gamma_1}^{-1}(I)$ we mean the function $\phi$ defined on $\partial \Omega_0$ such that $I_\phi = I$, and such that $\phi(p_0) = 0$, where $p_0$ is a fixed node on $\partial \Omega_0$ which is used to uniquely define $\Lambda_{\gamma_1}^{-1}$ and make it a linear map). Now we consider the algorithm for computing $T^{-1}$. First consider the Neumann problem:

$$I(q_0) = 1$$
$$I(q_1) = -1$$
$$I(q) = 0 \text{ if } q \neq q_0, q_1$$
$$\phi(p_0) = 0$$

where the notation refers to Figure 3.1. Let $\phi_1 = \Lambda_{\gamma_1}^{-1}(I)$ and $\phi_2 = \Lambda_{\gamma_2}^{-1}(I)$. Then $\phi_1$ is close to $\phi_2$, and so $\phi_1(q_0)$ is close to $\phi_2(q_0)$. But $\phi_1(q_0) = 1/\gamma_1(\sigma_0)$ and $\phi_2(q_0) = 1/\gamma_2(\sigma_0)$, and so $\gamma_1(\sigma_0)$ is close to $\gamma_2(\sigma_0)$. Similarly, $\gamma_1(\sigma_1)$ is close to $\gamma_2(\sigma_1)$, and we see that the corner values of $\gamma_1$ and $\gamma_2$ are close.

Next we consider the Neumann problems of equation (6), which are used in the inductive computation of the values of $\gamma$. We assume already proved
that the previously computed values of \( \gamma \) are close. The two different solutions of the Neumann problems found by using \( \Lambda_\gamma \) and \( \Lambda_{\gamma_j} \) are denoted by \( h_j^{(1)} \) and \( h_j^{(2)} \) and are close by assumption. Hence the coefficients of the equations

\[
\sum_{j=1}^{k} \alpha_j^{(1)} = 1 \\
\sum_{j=1}^{k} \alpha_j^{(1)} h_j^{(1)}(q) = 0
\]

are close to the coefficients of the equations

\[
\sum_{j=1}^{k} \alpha_j^{(2)} = 1 \\
\sum_{j=1}^{k} \alpha_j^{(2)} h_j^{(2)}(q) = 0
\]

Thus the solutions \( \{ \alpha_j^{(1)} : 1 \leq k \} \) and \( \{ \alpha_j^{(2)} : 1 \leq k \} \) are close. It then follows that the boundary values of the functions

\[
h^{(1)} = \sum_{j=1}^{k} \alpha_j^{(1)} h_j^{(1)}
\]

and

\[
h^{(2)} = \sum_{j=1}^{k} \alpha_j^{(2)} h_j^{(2)}
\]

are close. Proceeding inductively we see that the functions \( \gamma_1 \) and \( \gamma_2 \) are close.

By the results of Section 4, the map \( dT \) is injective, so \( T((R^+)^N) \) is an immersed submanifold. By what we have just proved \( T \) is a homeomorphism onto its image. QED
References


