FINDING THE CONDUCTORS IN CIRCULAR NETWORKS FROM BOUNDARY MEASUREMENTS

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Abstract. We give an algorithm for computing the values of the conductors in a circular network from voltages and currents measured at the boundary. We characterize the collections of boundary measurements which can come from such networks. We also give some results of numerical reconstruction of the values of the conductors from boundary measurements.

Key words. network of resistors, inverse problem, conductivity

AMS(MOS) subject classifications. 31A25, 31B25, 35R30, 90B10

1. Introduction. We consider circular networks as in the following figure.

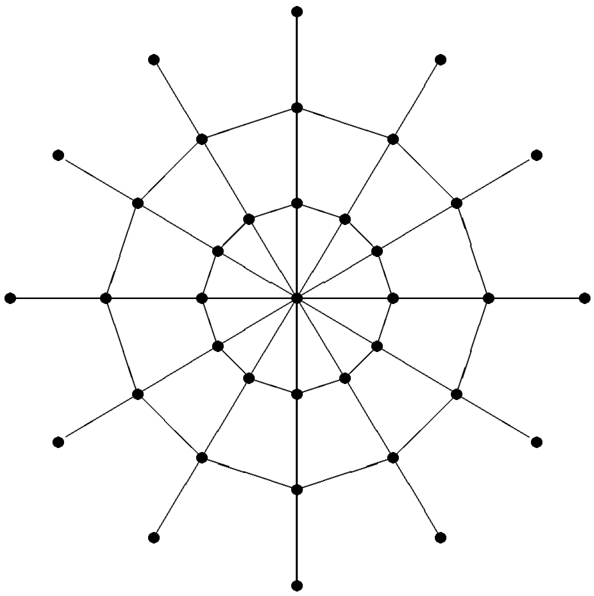

FIG 1.1

Such a network Ω with m circles and n rays will be called a circular network of type $C(m,n)$. FIG 1.1 shows a circular network of type $C(2,12)$. Other circular networks will be considered in Section 9. The *nodes* of Ω are the points in the plane consisting of the center node $p(0,0)$ and points $p(i,j)$, for $1 \le i \le m+1$ and $1 \le j \le n$. The node $p(i,j)$ is described in polar coordinates by $p(i,j) = (i, 2\pi j/n)$. We consider the nodes labelled cyclically; that is, $p(i,j+n) = p(i,j)$ for all integers j. The set of nodes is denoted Ω_0. The *interior* of Ω_0, called int Ω, consists of the nodes $p(i,j)$

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for $0 \leq i \leq m$ and $1 \leq j \leq n$ The *boundary* of Ω, called $\partial\Omega$, consists of the nodes $p(m+1, j)$ for $1 \leq j \leq n$. The boundary nodes are labelled $p_j = p(m+1, j)$ for $1 \leq j \leq n$. Each interior node except the center node, has four neighboring nodes; the center node $p(0,0)$ has n neighbors. The set of nodes which are neighbors of p is called $\mathcal{N}(p)$. Each boundary node has exactly one neighboring node which is an interior node. A circular network of type $C(m,n)$ has $1 + mn$ interior nodes and n boundary nodes. An *edge* pq of Ω is a radial line segment $p(i,j)p(i+1,j)$ for $0 \leq i \leq m$ and $1 \leq j \leq n$, or a circular arc $p(i,j)p(i,j+1)$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. The set of edges is denoted Ω_1. There are $n(2m+1)$ edges.

A *circular network of resistors* of type $C(m,n)$ is a network $\Omega = (\Omega_0, \Omega_1)$ together with a positive real-valued function γ on Ω_1. The function γ is called called the *conductivity*. For each edge pq in Ω_1, the number $\gamma(pq)$ is the *conductance* of pq, and $1/\gamma(pq)$ is the *resistance* of pq. If u is a function on Ω_0, *Ohm's Law* gives a *current* along each conductor $pq: I(pq) = \gamma(pq)(u(p) - u(q))$ is the current from p to q. The function u is called a γ-*harmonic* function on Ω if for each interior node p,

$$\sum_{q \in \mathcal{N}(p)} \gamma(pq)(u(p) - u(q)) = 0$$

This property of a γ-harmonic function, which asserts that the sum of the currents flowing out of each interior node is zero, is *Kirchhoff's Law.* If a function ϕ is defined at the boundary nodes, there will be a unique γ-harmonic function u, defined on all the nodes with $u(p) = \phi(p)$ for each boundary node p (see Lemma 2.5). The function u is called the *potential* due to ϕ. The potential drop across the conductor pq is $\Delta u(pq) = u(p) - u(q)$. The potential u determines a current $I_u(p)$ into each boundary node p by $I_u(p) = \gamma(pq)(u(p) - u(q))$, where q is the interior neighbor of p.

For each conductivity γ on Ω_1, the linear map Λ_γ from boundary functions to boundary functions is defined by $\Lambda_\gamma(\phi) = I_u$. The boundary function ϕ is called Dirichlet data, and the boundary current I_u is called Neumann data. The map Λ_γ which takes potentials at the boundary of Ω to currents through the boundary nodes of Ω is called the Dirichlet-to-Neumann map.

The inverse problem is to recover the conductivity γ from the map Λ_γ. In our situation, this leads to four problems.

(1) Uniqueness: If $\Lambda_\gamma = \Lambda_\mu$, does it necessarily follow that $\gamma = \mu$?

(2) Continuity: If Λ_γ is near to Λ_μ, does it necessarily follow that γ is near to μ?

(3) Reconstruction: Give an algorithm for using Λ_γ to compute γ.

(4) Characterization: For each integer n, which n by n matrices are of the form Λ_γ for some γ?

In Section 5, we give an algorithm for computing the conductivity γ from the Dirichlet-to-Neumann map Λ_γ for circular networks of type $C(m,n)$, where $n \geq 4m+3$. For these networks, we show that the Dirichlet-to-Neumann map uniquely determines the conductivity (see Theorem 5.2). The algebraic formulas of the algorithm show the continuity of the inverse. For circular networks of type $C(m,n)$, where $n = 4m+3$, the set of Dirichlet-to-Neumann maps forms a manifold of dimension $n(n-1)/2$ in the space of n by n matrices. In Section 6, we show that the $n(n-1)/2$ entries of Λ above the diagonal parametrize this manifold, and we describe explicitly the domain over which these parameters may vary. Theorem 6.2 gives a characterization of the Dirichlet-to-Neumann maps for such circular resistor networks. By considerations of duality, there is a similar characterization of Neumann-to-Dirichlet maps. Some numerical results based on the reconstruction algorithm of Section 5 are given in Section 13 and in [2]. Similar results may be obtained for other types of circular networks (e.g., where the outer conductors are not present), which are discussed in Section 9. In [4] and [3] we solved the four problems above for square resistor networks. The methods presented here are simplifications of those of [4] and [3]. For related work on the inverse conductivity problem see [1], [2], [5] and [6].

2. Functions on Networks. We collect some facts about γ-harmonic functions on circular networks, some of which were proved for rectangular networks in [3]. Throughout this Section, let $\Omega = (\Omega_0, \Omega_1)$ be a circular network of type $C(m,n)$, with a conductivity function γ on Ω_1.

LEMMA 2.1. *Let u be a γ-harmonic function on Ω, and let p be an interior node. Then either $u(p) = u(q)$ for all nodes $q \in \mathcal{N}(p)$ or there is at least one node $q \in \mathcal{N}(p)$ for which $u(p) > u(q)$ and there is at least one node $r \in \mathcal{N}(p)$ for which $u(p) < u(r)$.*

Proof. Kirchhoff's Law may be rewritten as

$$\left(\sum_{q\in\mathcal{N}(p)} \gamma(pq)\right) u(p) = \sum_{q\in\mathcal{N}(p)} \gamma(pq)u(q)$$

This says that the value of u at each interior node is the weighted average of the values at the neighboring nodes. □

COROLLARY 2.2. (MAXIMUM PRINCIPLE FOR FUNCTIONS) *Let u be a γ-harmonic function on Ω. Then the maximum and minimum values of u occur on the boundary of Ω.*

Proof. If the maximum value of u were to occur at an interior node, then by Lemma 2.1, the value of u at all the neighbors would be the same. Thus either u is a constant or the maximum and minimum values do not occur at an interior node and so must occur at boundary nodes. □

COROLLARY 2.3. *Let u be a γ-harmonic function on Ω such that $u(p) = 0$ for all $p \in \partial\Omega$. Then $u(p) = 0$ for all $p \in \Omega$.*

LEMMA 2.4. (MAXIMUM PRINCIPLE FOR CURRENTS) *Let u be a γ-harmonic function on Ω. The current across any conductor pq is less than or equal to the sum of the positive currents into the boundary nodes.*

Proof. Assume that $u(p) > u(q)$. Let $I_u(pq)$ be the current through pq in the direction of p to q. Construct a subnetwork Γ of Ω as follows. Let $\Gamma^{(1)}$ consist of all edges $rp \in \Gamma$ such that $u(r) > u(p)$, and r is a neighbor of p. Inductively, having defined $\Gamma^{(j)}$, let $\Gamma^{(j+1)}$ consist of all edges in $\Gamma^{(j)}$ and all edges st in Ω such that $t \in \Gamma^{(j)}$ and $u(s) > u(t)$. (Each edge includes its endpoints.) This gives an increasing sequences of subnetworks

$$\Gamma^{(1)} \subseteq \Gamma^{(2)} \subseteq \Gamma^{(3)} \subseteq \dots$$

Eventually no new edges are added and the process ends. Let Γ be the union of the $\Gamma^{(j)}$. For each boundary node r, let $I_u(r)$ be the current into Ω through r. The boundary of Γ consists of nodes of two types:

(i) nodes which are in $\partial\Omega$

(ii) nodes which are not in $\partial\Omega$.

At those nodes of $\partial\Gamma$ which are also in $\partial\Omega$, the current into $\partial\Gamma$ is positive (except possibly at node q itself). At all other nodes of $\partial\Gamma$ the current into $\partial\Gamma$ is ≤ 0. The (algebraic) sum of the currents into $\partial\Gamma$ is 0. Hence

$$I_u(pq) \leq \sum_{r \in \partial\Gamma \cap \partial\Omega} I_u(r) \leq \sum_{r \in \partial\Omega, I_u(r) > 0} I_u(r)$$

□

The following lemma shows that there is a unique γ-harmonic function with prescribed boundary values.

LEMMA 2.5. *Let $\Omega = (\Omega_0, \Omega_1)$ be a circular network of type $C(m, n)$, with a conductivity γ. Suppose given the boundary values $\phi(p_j)$ for all boundary nodes p_j. Then there is a unique γ-harmonic function u with $u(p_j) = \phi(p_j)$ for each boundary node p_j.*

Proof. For each interior node, Kirchhoff's Law becomes a linear equation for the values of u. We then have a $(1 + mn)$ by $(1 + mn)$ matrix equation

$$Ku = b$$

Here u is the vector of values $u(p)$ as p varies over the interior nodes; b is obtained by moving the terms in Kirchhoff's Law which involve boundary values of u to the right hand side. If the boundary values of u are all 0, Corollary 2.3 shows that u must be zero at all interior nodes also. Thus the matrix K is non-singular. □

As a result, Λ_γ is a well-defined linear map from boundary functions to boundary functions. Lemma 2.2 shows that the kernel of Λ_γ consists of the constant functions.

LEMMA 2.6. *Let u be a γ-harmonic function on Ω. Let p be an interior node and q a neighbor of p. The value of $u(q)$ is determined by the values of $\gamma(pr)$ for all neighbors r of p, the value of $u(p)$, and the values of $u(r)$ for all neighbors r of p other than q.*

Proof. In Kirchhoff's Law at node p, all the terms except $\gamma(pq)u(q)$ are given. The value of $u(q)$ is then determined because of the assumption that $\gamma(pq) \neq 0$. □

Let α be any real-valued function defined on the set of edges Ω_1. For any function f on Ω_0, let $L_\alpha f$ be the function defined on on Ω_0 by

$$L_\alpha f(p) = \sum_{q \in \mathcal{N}(p)} \alpha(pq)(f(p) - f(q))$$

L_α is a linear operator on the set of functions defined on int Ω_0. In the case where γ is a conductivity function on Ω_1, a function f which satisfies $L_\gamma f(p) = 0$ for all nodes $p \in$ int Ω is γ-harmonic. For any boundary node p, $L_\gamma f(p)$ is the current through p due to f, which is called $I_f(p)$.

LEMMA 2.7. *Let $\Omega = (\Omega_0, \Omega_1)$ be a circular network with conductivity γ, and let f and g be functions on Ω_0. Then*

$$\sum_{pq \in \Omega_1} \gamma(pq)(f(p) - f(q))(g(p) - g(q)) = \sum_{p \in \Omega_0} g(p) L_\gamma f(p)$$

Proof. For each edge $pq \in \Omega_1$,

$$\gamma(pq)(f(p) - f(q))(g(p) - g(q)) = g(p)\gamma(pq)(f(p) - f(q)) + g(q)\gamma(pq)(f(q) - f(p))$$

Summing over all edges in Ω_1 gives the result. □

COROLLARY 2.8. *Let g be a function on Ω_0 and let f be a γ- harmonic function on Ω_0. Then*

$$\sum_{pq \in \Omega_1} \gamma(pq)(f(p) - f(q))(g(p) - g(q)) = \sum_{p \in \partial\Omega_0} g(p) I_f(p)$$

The following is a discrete form of one of Green's identities.

LEMMA 2.9. *Let f and g be γ-harmonic functions on Ω. Then*

$$\sum_{p \in \partial\Omega_0} g(p) I_f(p) = \sum_{p \in \partial\Omega_0} f(p) I_g(p)$$

Proof. By Lemma 2.8, both sides are equal to

$$\sum_{pq \in \Omega_1} \gamma(pq)(f(p) - f(q))(g(p) - g(q))$$

□

The following lemma provides a way to construct γ-harmonic functions with prescribed data, some of which are boundary values, and some of which are boundary currents. This will be used extensively in the reconstruction algorithm of Section 5.

LEMMA 2.10. *Let $\Omega = (\Omega_0, \Omega_1)$ be a circular network of type $C(m, 4m+3)$, with a conductivity γ. Suppose given the boundary values $u(p_j)$ for $0 \le j \le 2m+1$, and* *suppose given the values of the current $I_u(p_j)$ for $1 \le j \le 2m+1$. Then there is a unique γ-harmonic function u with this boundary data.*

Proof. Using Ohm's law, we find the value of u at the nodes $p(m,j)$ for $1 \le j \le 2m+1$. Using Lemma 2.6, we find the values of u at the nodes $p(i,j)$ for $i = m-1, m-2, \ldots, 2, 1$, and $j = m+1-i, \ldots, m+1+i$, and then at the center node $p(0,0)$. Working outward from $p(0,0)$, using Lemma 2.6, and the value of u at p_0, the values of u are obtained at all the remaining nodes. □

LEMMA 2.11. *Let $\Omega = (\Omega_0, \Omega_1)$ be a circular network of type $C(m, 4m+3)$ with a conductivity γ. Let u be a γ-harmonic function on Ω. Suppose that $u(p_j) = 0$ for $1 \le j \le 2m+1$, and also that the boundary current $I_u(p_j) = 0$ for $1 \le j \le 2m+1$. Then either u is identically 0 or the values of $u(p_j)$ for $2m+2 \le j \le 4m+3$ are all non-zero and alternate in sign.*

Proof. If $u(p_{4m+3}) = 0$, Lemma 2.10 applies to show that $u(p) = 0$ for all nodes p. If $u(p_{4m+3}) \ne 0$, the values of u at all nodes are found by Lemma 2.6 just as in the proof of Lemma 2.10. The following diagram shows this situation for a network of type $C(2, 11)$ where $u(p_{4m+3})$ is assumed to be positive. At each node p where $u(p) \ne 0$, the sign of $u(p)$ is indicated by $+$ or $-$.

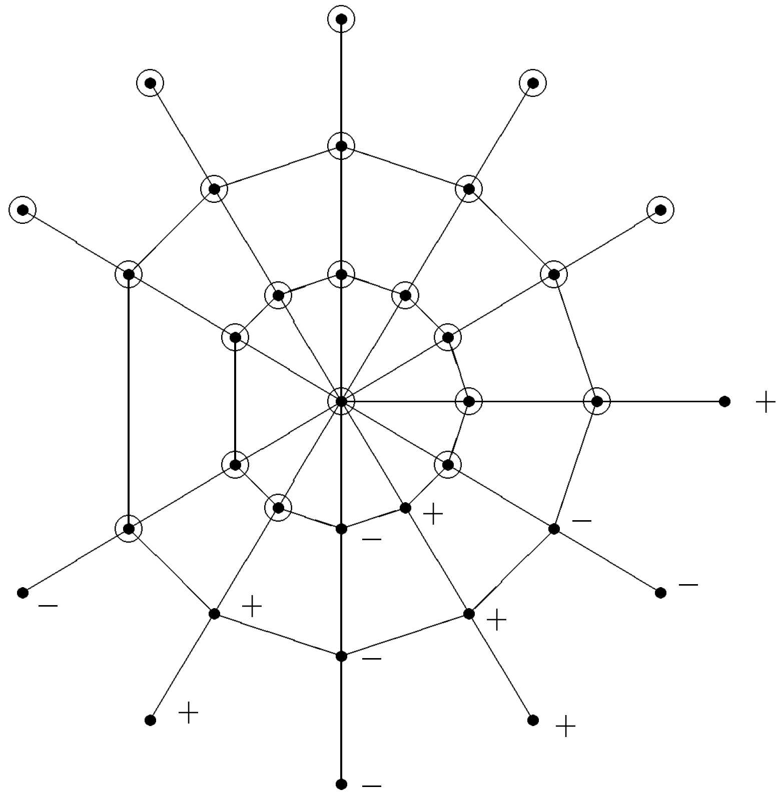

FIG 2.1

The result is that the values $u(p_{2m+2}), u(p_{2m+3}), \ldots, u(p_{4m+3})$ must alternate in sign. □

For any sequence of $2m+1$ consecutive nodes where both the function u and the current I_u are to be 0, there is a similar pattern. We will use these special γ-harmonic functions in the reconstruction algorithm of Section 5.

3. The Dirichlet-to-Neumann Map. Throughout this section, $\Omega = (\Omega_0, \Omega_1)$ is a circular network of type $C(m,n)$ and γ is a positive function on Ω_1. Let Λ be the Dirichlet-to-Neumann map for Ω as defined in Section 2. The boundary nodes are numbered sequentially by $p_1, p_2, \ldots, p_n$. As always, the convention is that $p_0 = p_n$. We put the inner product on boundary functions:

$$< \phi, \psi >= \sum_{j=1}^{n} \phi(p_j)\psi(p_j)$$

A bilinear form $Q(\,.\,,\,.\,)$ on boundary functions is defined by

$$Q(\phi, \psi) =< \phi, \Lambda\psi >$$

For each index $j = 1, 2, \ldots, n$, let ϕ_j be the boundary function which is 1 at node p_j and 0 at all other boundary nodes. The Dirichlet-to-Neumann map Λ is represented by a matrix $A = \{A_{i,j}\}$ as follows. The entries $A_{i,j}$ are given by:

$$A_{i,j} = Q(\phi_i, \phi_j)$$

The entry $A_{i,j}$ may be interpreted as the current at node p_i resulting from the boundary potential which is 1 at node p_j, and 0 at all other boundary nodes. It follows from Corollary 2.3 that if the boundary potential has value 1 at all boundary nodes, then the potential will have value 1 at all interior nodes, and hence the current is 0. This implies the sum relations: for each $i = 1, 2, \ldots, n$,

$$\sum_{j=1}^{n} A_{i,j} = 0$$

From 2.9, it follows immediately that the matrix A is symmetric; that is, $A_{i,j} = A_{j,i}$.

Before stating the remaining property of the matrix A, we need a definition.

DEFINITION 3.1. *A k by k matrix B is said to have the* Right Sign*, if*
(1) $k \equiv 1$ or 2 mod 4, then $\det B < 0$
(2) $k \equiv 3$ or 0 mod 4, then $\det B > 0$

Let A be the matrix representing the Dirichlet-to-Neumann map Λ for a circular network of type $C(m,n)$. Let B be a k by k submatrix of A formed by choosing k rows and k columns which correspond to $2k$ distinct nodes which occur in sequence (not necessarily consecutive) around the boundary of Ω. Such a matrix B is said to be *sequentially obtained* from A. By a rotation of Ω, we may assume that the rows are $i_1, \ldots, i_k$, and the columns are $j_1, \ldots, j_k$ with $1 \leq i_1 < \ldots < i_k < j_1 < \ldots < j_k \leq n$. In this situation, the matrix B lies strictly above the diagonal of A. Let $r_1, r_2, \ldots, r_k$ be the boundary nodes corresponding to the rows $i_1, i_2, \ldots, i_k$, and let $q_1, q_2, \ldots, q_k$ be the boundary nodes corresponding to the columns $j_1, j_2, \ldots, j_k$. The matrix B has the following interpretation. Let $v = (v_1, v_2, \ldots, v_k)$. Let ϕ be the boundary function with $\phi(q_i) = v_i$ for $1 \leq i \leq k$ and $\phi(p) = 0$ for all other boundary nodes p. Let $\Lambda(\phi)$ be the boundary current corresponding to ϕ. Then Bv is a vector whose entries are the values of $\Lambda(\phi)$ at nodes $r_1, r_2, \ldots, r_k$.

THEOREM 3.2. *Let A be the matrix representing the Dirichlet-to-Neumann map for a circular network of type $C(m,n)$, with $n \geq 4m+3$. Let k be an positive integer with $k \leq 2m+1$ and let B be a k by k submatrix sequentially obtained from A. Then B is nonsingular, and has the Right Sign.*

The proof will follow several lemmas.

LEMMA 3.3. *Let u be a γ-harmonic function on a circular network of type $C(m,n)$. Suppose u has value 0 and current 0 at k consecutive boundary nodes, where $k \leq 2m+1$. Then either $u \equiv 0$ or there is sequence of $k+1$ boundary nodes at which the values of u are non-zero and alternate in sign.*

Proof. Denote the set of boundary nodes where u is assumed to have value 0 and current 0 by V. Let $W = \{p_1, \cdots, p_w\}$ be the largest set of connected boundary nodes where u has value 0 and current 0 and which contains V. For each $p_i \in W$, let R_i be the ray from 0 to p_i and let C be the largest connected set of nodes on R_i containing p_i for which the value of u is 0. Let c_i be the cardinality of C. Let $c_K = \max\{c_1, \cdots, c_w\}$. We consider the following cases.

Case 1. Suppose there is an adjacent pair $p_i, p_{i+1} \in W$ such that $c_i = c_{i+1} = m+2$. Then either $u \equiv 0$ or else there is a "trapezoidal" set of nodes where $u = 0$ which is bounded by at least $2m+2$ nodes where the values of u are non-zero and alternate in sign. Each of these nodes where u is positive has a neighbor where u has greater positive value. Such a node is connected by a chain of nodes of successively more positive value to a node on the boundary of positive value. Each node of negative value is connected by a chain of nodes of successively more negative value to a node on the boundary of negative value. These chains cannot cross. This leads to a set of $2m+2$ boundary nodes at which the values of u are non-zero and alternate in sign. Since $k+1 \leq 2m+2$ the lemma is true in Case 1.

Case 2. Suppose there is an adjacent pair $p_i, p_{i+1} \in W$ such that $c_i = c_{i+1} = c_K < m+2$. Then $c_{j-1} = c_j - 1$ for $j < i$ and $c_{j+1} = c_j - 1$ for $j > i+1$. It follows that there will be a node on each ray $R_j, j \neq i, i+1$ at which $u \neq 0$ and which is adjacent to a node on R_j at which $u = 0$. The sign of u alternates as we go from R_1 to R_{i-1} and as we go from R_{i+2} to R_w. In addition there must be non-zero values of u on R_i, R_{i+1}, at least one of which alternates with the signs of u on $R_1, \cdots, R_{i-1}, R_{i+2}, \cdots, R_w$. Consider the sign of u on ray R_1. On the ray to the left of R_1 at the node which is one circle closer to the boundary of the network, the sign of u is opposite to what it is on R_1. A similar statement holds at R_w. An argument similar to Case 1 allows us to conclude that there are at least $k+1$ non-zero boundary values of u which alternate in sign.

Case 3. Suppose that the maximum value c_K is assumed only once. Then as in Case 2 we must have $c_{j-1} = c_j - 1$ for $j < K$, and $c_{j+1} = c_j - 1$ for $j > K$. An argument similar to Case 1 allows us to conclude that there are at least $k+1$ non-zero boundary values of u which alternate in sign. $\square$

LEMMA 3.4. *Let $\Omega = (\Omega_0, \Omega_1)$ be a circular network of type $C(m,n)$ with $n \geq 4m+3$. Let S be a set of contiguous boundary nodes, and let T be the complementary set of boundary nodes. Let k be an integer with $k \leq 2m+1$. Suppose that u is a γ-harmonic function with $u(p) = 0$ for all p in S and for which the current $I_u(p) = 0$ at k distinct nodes p in S. Then either u is identically zero or there are at least $k+1$ boundary nodes p with $u(p) \neq 0$.*

Proof. Suppose that there are g non-contiguous sequences of nodes from S of lengths $k_1, k_2, \ldots, k_g$, with $\sum k_i = k$, and suppose that $I_u = 0$ at each of these nodes and that $I_u \neq 0$ at all other noses in S. There must be a total of at least k sign changes among the values at the nodes neighboring the regions of zeros.

Each of these nodes where u is positive has a neighbor where u has greater positive value. Such a node is connected by a chain of nodes of successively more positive value to a node in T of positive value. Each node of negative value is connected by a chain of nodes of successively more negative value to a node on the boundary of negative value. These chains cannot cross. Thus there must be at least k sign changes among the values of u at the nodes in T. $\square$

DEFINITION 3.5. *A k by k non-singular matrix B is said to have the* Alternating Property *if the following condition holds. Suppose that $c = Bv$ and that the signs in c alternate. Then the signs in v must be the negative of the reversal of the signs in c. That is, if k is even, and the pattern of signs in c is $(-,+,-,+,\ldots,+)$, the pattern of signs in v must also be $(-,+,-,+,\ldots,+)$. If k is odd, and the pattern of signs in c is $(-,+,-,+,\ldots,-)$, the pattern of signs in v must be $(+,-,+,-,\ldots,+)$.*

LEMMA 3.6. *Let A be the matrix representing the Dirichlet-to-Neumann map for a circular network of type $C(m,n)$. Let k be an positive integer with $k \leq 2m+1$ and let B be a k by k submatrix sequentially obtained from A. Then B has the Alternating Property.*

Proof. Let $q_1, q_2, \ldots, q_k$ be the nodes corresponding to the choice of the k columns of B, and let $r_1, r_2, \ldots, r_k$ be the nodes corresponding to the choice of the k rows of B. By a rotation of Ω if necessary, we may assume that $1 \leq r_1 < \ldots < r_k < q_1 < \ldots < q_k \leq n$. Let $v = (v_1, \ldots, v_k)$ be a vector of potentials at nodes $q_1, \ldots, q_k$. and let $c = Bv$. For each $i = 1, \ldots, k$, let s_i be the interior neighbor of r_i. Figure 3.1 illustrates the case of a circular network of type $C(2,12)$ and $k = 4$.

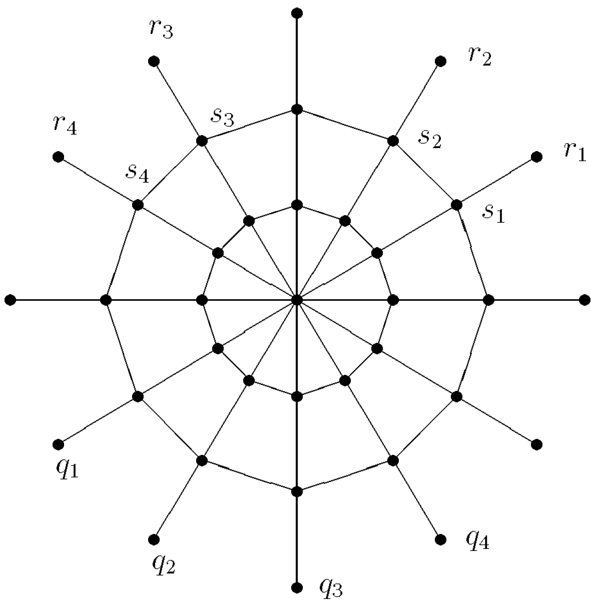

FIG 3.1

The sign of the potential at node s_i must be opposite to the sign of the current through r_i. By repeated use of Lemma 2.1, the node s_i can be connected by a chain of nodes with potential of the same sign and increasing magnitude to a boundary node also with potential of the same sign. These chains cannot cross. It follows that the potential at nodes $q_k, q_{k-1}, \ldots, q_2, q_1$ must have the same signs as potential at nodes $s_1, s_2, \ldots, s_k$. Thus the values of the potential at the nodes $q_1, q_2, \ldots, q_k$ must be the negatives of the reversal of the values of the current through nodes $r_1, r_2, \ldots, r_k$. $\square$

For any positive integer k let D be the k by k matrix with nonzero entries only on the diagonal, and $D_{i,i} = (-1)^{i+1}$.

LEMMA 3.7. *Let B be a k by k non-singular matrix which has the Alternating Property. Then each entry of the matrix $(-1)^k DB^{-1}D$ is non-negative. If in addition all of the $k-1$ by $k-1$ minors of B are nonsingular, then each entry of the matrix $(-1)^k DB^{-1}D$ is positive.*

Proof. If any entry of $(-1)^k DB^{-1}D$ were negative, then B would not be alternating. If each $k-1$ by $k-1$ minor of B is nonsingular, then every entry of B^{-1} is non-zero, and each entry of $(-1)^k DB^{-1}D$ must be positive. $\square$

We proceed with the proof of Theorem 3.2.

Proof. Let B be a k by k submatrix sequentially obtained from A. Suppose that $Bv = c$. Lemma 3.4 shows that if $c = 0$, then $v = 0$ also. Thus B is nonsingular. The one by one submatrices of B have the Alternating Property. This shows that the entries of B are negative. The proof that B has the right sign follows by Lemma 3.7 and induction on k, using Cramer's rule for B^{-1}. $\square$

REMARK 3.8. Let $\Omega = (\Omega_0, \Omega_1)$ be a circular network of type $C(m, 4m+3)$. Let u be the γ-harmonic function on Ω with the following boundary data: $u(p_j) = 0$ for $0 \le j \le 2m+1$; $I_u(p_j) = (-1)^j$ for $1 \le j \le 2m+1$. Theorem 3.6 shows that the voltages at the remaining nodes satisfy $u(p_j) > 0$ for j even and $2m+2 \le j \le 4m+2$ and $u(p_j) < 0$ for j odd and $2m+3 \le j \le 4m+1$. In this situation, the proof of Theorem 3.2 actually proves more. For each $1 \le j \le 2m+1$, boundary node p_j can be joined by a chain of nodes with potential of the same sign and of increasing magnitude to boundary node p_{4m+3-j} with potential of the same sign. The chain of edges joining these nodes will be called a *principal flow path*. Along a principal flow path the magnitude of the current is non-decreasing from boundary node p_j to the boundary node p_{4m+3}. The current along an edge joining a node of positive potential to a neighboring node of negative potential will be called *transverse* to the principal flow.

The principal flow paths for a circular network of type $C(2, 11)$ are illustrated in the following diagram. The boundary potentials (zero, positive or negative) are indicated by the symbols $(0, +, -)$ respectively, placed adjacent to the nodes.

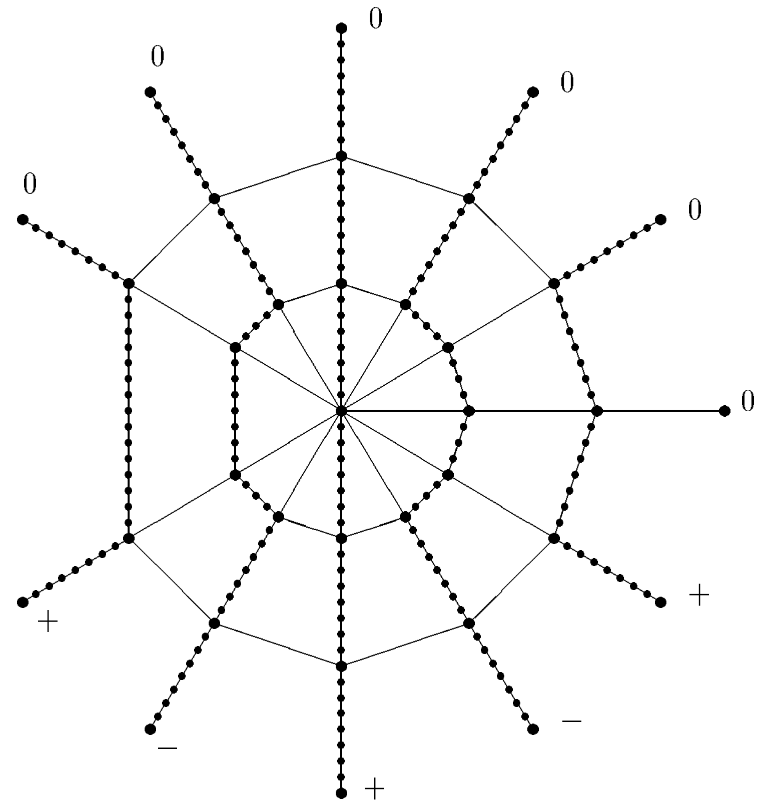

FIG 3.2

For any edge in Ω_1, there is a pattern of boundary data (obtained by a suitable rotation of Figure 3.2) that places the chosen edge along a principal flow path. Similarly, for any edge in Ω_1, there is a pattern of boundary data that places the chosen edge transverse to the principal flow.

4. The Differential of T. Let $\Omega = (\Omega_0, \Omega_1)$ be a circular network of type $C(m,n)$. The number of conductors is $N = n(2m+1)$. For each conductivity function γ on Ω_1, let $Q_\gamma(\, . \, , \, . \,)$ be the bilinear form in n variables as defined in Section 2. Let $\mathcal{F}(n)$ be the space of bilinear forms in n variables. Let

$$T \; : (R^+)^N \to \mathcal{F}(n)$$

be the function given by $T(\gamma) = Q_\gamma(\, . \, , \, . \,)$ We want to compute the differential of T. For this, we consider a small perturbation κ of γ, and calculate the difference $T(\gamma+\kappa) - T(\gamma)$. Let ϕ be a boundary function. Let f be the γ-harmonic function on Ω which takes on the boundary values ϕ and let h be the $(\gamma+\kappa)$-harmonic function on Ω which takes on the boundary values ϕ. Thus $L_\gamma f(p) = 0$ and $L_{\gamma+\kappa} h(p) = 0$ for all $p \in \text{int } \Omega$. Let $h = f + e$. Then $e(p) = 0$ for all $p \in \partial\Omega$, and $L_{\gamma+\kappa} e(p) = -L_\kappa f(p)$ for all $p \in \text{int } \Omega$. If c is a function on int Ω, $L^{-1}c$ is defined to be the solution v of $Lv = c$, with $v = 0$ on $\partial\Omega$. Then $L_\gamma^{-1} L_\gamma e = e$, and we have

$$(I + L_\gamma^{-1} L_\kappa) e = -L_\gamma^{-1} L_\kappa f$$

If $\|\kappa\|$ is small, $I + L_\gamma^{-1} L_\kappa$ is invertible, and

$$e = -(I + L_\gamma^{-1} L_\kappa) L_\gamma L_\kappa f$$

Thus e vanishes to order 1 in $\|\kappa\|$ because $L_\kappa f$ vanishes to order 1 in κ. Then

$$\begin{aligned}
Q_{\gamma+\kappa}(\phi,\phi) &= \sum_{pq\in\Omega_1} (\gamma(pq) + \kappa(pq)(h(p) - h(q))^2 \\
&= Q_\gamma(\phi,\phi) + 2 \sum_{pq\in\Omega_1} (\gamma(pq))(f(p) - f(q))(e(p) - e(q) \\
&+ \; 2 \sum_{pq\in\Omega_1} \kappa(pq)(f(p) - f(q))(e(p) - e(q) + \sum_{pq\in\Omega_1} \gamma(pq)(e(p) - e(q))^2 \\
&+ \; \sum_{pq\in\Omega_1} \kappa(pq)(f(p) - f(q))^2 + \sum_{pq\in\Omega_1} \kappa(pq)(e(p) - e(q))^2
\end{aligned}$$

Using that $e = 0$ on $\partial\Omega$ and Lemma 2.7, we have

$$Q_{\gamma+\kappa}(\phi,\phi) - Q_\gamma(\phi,\phi) = \sum_{pq\in\Omega_1} \kappa(pq)(f(p) - f(q))^2 + \mathcal{O}\|\kappa\|^2$$

Therefore the differential of T at the conductivity γ, perturbed by κ, and evaluated at (ϕ,ϕ) is

$$dT = \sum_{pq\in\Omega_1} \kappa(pq)(f(p) - f(q))^2$$

Considered as a linear map from $(R^+)^N$ to $\mathcal{F}(n)$, the differential dT is given by:

$$dT(\kappa)(\phi,\psi) = \sum_{pq\in\Omega_1} \kappa(pq)(f(p) - f(q)(g(p) - g(q))$$

where f and g are the γ-harmonic functions which take on the boundary values ϕ and ψ respectively.

LEMMA 4.1. *Let $\Omega = (\Omega_0, \Omega_1)$ be a circular network of type $C(m,n)$ with $n = 4m+3$. Let κ be any real-valued function on Ω_1. Suppose that for all γ-harmonic functions f and g, that*

$$\sum_{pq\in\Omega_1} \kappa(pq)(f(p) - f(q)(g(p) - g(q)) = 0$$

Then $\kappa \equiv 0$.

Proof. Order the edges of Ω from the outside inwards; that is, all the outermost edges come first, then the edges on the outer circle, etc. Recall from Lemma 2.11, that for each sequence of $2m+1$ consecutive boundary nodes of Ω_0, there is a (non-zero) special γ-harmonic function f which has value 0 and current 0 at these nodes. For each edge $\sigma \in \Omega$, there is a pair of such special functions f and g, such that the product $(f(p) - f(q)(g(p) - g(q) \neq 0$ when $pq = \sigma$, and this product is 0 for all edges pq which follow σ in the ordering. The proof that $\kappa(\sigma) = 0$ for all $\sigma \in \Omega_1$ follows readily by induction using the ordering on the edges. □

THEOREM 4.2. *Let $\Omega = (\Omega_0, \Omega_1)$ be a circular network of type $C(m,n)$ with $n = 4m+3$. Then the differential of T is one-to-one.*

Proof. This follows immediately from the expression for the differential dT and Lemma 4.1. □

5. An Algorithm for Computing Conductances. Let $\Omega = (\Omega_0, \Omega_1)$ be a circular network with of type $C(m, 4m+3)$. We use the results of Sections 2 and 3 to give an algorithm for computing γ from Λ_γ.

Let A be the matrix representing Λ, as in Section 3. We will use A to find the boundary values for the special γ-harmonic functions described by Lemma 2.11. Let w be the (column) vector whose entries are $w_i = A_{i,4m+3}$, for $i = 2m+2, \ldots, 4m+2$. Let B be the special submatrix of A whose entries are $B_{i,j} = A_{i,j}$, for $i = 2m+2, \ldots, 4m+2$ and $j = 1, \ldots, 2m+1$. Let v be the solution to the matrix equation $Bv + w = 0$, guaranteed by Lemma 3.6. Let ϕ be the boundary function whose values are

(1) $\phi(p_0) = 1$
(2) $\phi(p_j) = v_j$ for $j = 1, \ldots, 2m+1$.
(3) $\phi(p_j) = 0$ for $j = 2m+2, \ldots, 4m+2$.

Let $A(\phi) = I$ be the resulting current. By the construction, $I(p_i) = 0$ for $i = 2m+2, \ldots, 4m+2$. The pattern of zero voltages is indicated by the circled nodes by Figure 5.1.

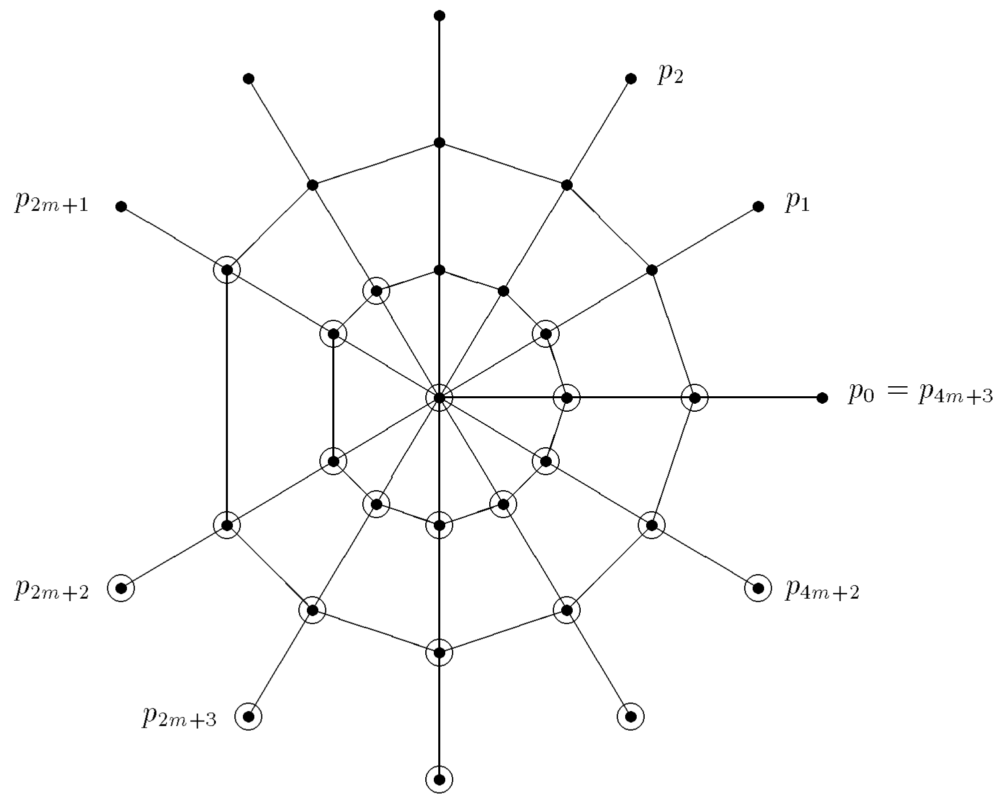

FIG 5.1

REMARK 5.1. By a rotation there is a similar voltage pattern with any other node p_j in the position of p_0.

The algorithm will proceed inwards by levels. The outermost boundary conductors are at level $m+1$. For each integer $i = m, m-1, \ldots, 1$, the circular conductors on the circle of radius i and the radial conductors between this circle and the circle of radius $i-1$ are at level i.

For each boundary node p_j, let q_j be its interior neighbor. We first use the boundary function ϕ and $I = A(\phi)$ to calculate the conductance $\gamma(p_0 q_0)$. The pattern of 0's shows that $u(q_0) = 0$. By Ohm's Law:

$$\gamma(p_0 q_0)(u(p_0) - u(q_0)) = I(p_0)$$

Then, using Remark 5.1, we can calculate $\gamma(p_j q_j)$ for all $j = 1, \ldots, 4m+3$.

Assuming now that we have calculated $\gamma(p_j q_j)$ for all boundary conductors, we calculate $\gamma(pq)$ for all circular conductors at level m as follows. From the potential ϕ and the current $I = A(\phi)$ We first calculate $u(q_1)$ by Ohm's Law:

$$\gamma(p_1 q_1)(u(p_1) - u(q_1)) = I(p_1)$$

All the current through $p_0 q_0$ must pass through $q_0 q_1$. Then we compute $\gamma(q_0 q_1)$ by Ohm's Law.

$$\gamma(q_0q_1)(u(q_o)-u(q_1))=I(p_0)$$

Using Remark 5.1 again, we can calculate all conductances on the outermost circle. We next calculate the radial conductors $\gamma(rs)$ at level $m-1$ as follows. The boundary potential ϕ and the boundary current $I=A(\phi)$ enables us to calculate the value of u at all nodes on the circle of radius m. We then calculate the current across edges $p(m,0)p(m,1)$, $p(m+1,1)p(m,1)$ and $p(m,1)p(m,2)$. Using Kirchhoff's Law, and the known value of 0 at $p(m-1,1)$ we can calculate $\gamma(p(m,1)p(m-1,1))$ by Ohm's Law. Using Remark 5.1 again, we calculate the conductances for all radial edges at level $m-2$. We then calculate the circular conductances at level $m-2$. Continuing inwards, in a similar way we calculate all the conductances.

THEOREM 5.2. *Let Ω be a circular network of type $C(m,n)$. The map which sends conductivity γ to the matrix representing Λ_γ is* 1 - 1. *Let γ and μ be two conductivities on Ω. If Λ_γ is sufficiently near to Λ_μ, then γ will be near to μ.*

Proof. The algorithm shows that the Dirichlet-to-Neumann map Λ_γ uniquely determines the conductivity γ. The algorithm also shows that each conductivity can be calculated by an algebraic formula which never involves division by 0. This shows the continuity of the inverse. □

REMARK 5.3. This method of special functions can be used to give an algorithm for computing conductances of a circular network of type $C(m,n)$ whenever $n\geq 4m+3$. The uniqueness and continuity of inverse also hold for such networks.

6. Characterization of Λ_γ. Let $\Omega=(\Omega_0,\Omega_1)$ be a circular network of type $C(m,4m+3)$. Suppose the conductivity is γ. The Dirichlet-to-Neumann map is represented by a n by n matrix $A=\{A_{i,j}\}$, as in Section 3. We showed that the matrix A has the following relations:

(R1) A is symmetric: $A_{i,j}=A_{j,i}$.

(R2) For each $i=1,2,\ldots,n$,

$$\sum_{j=1}^{n}A_{i,j}=0$$

In Section 3, we showed that the matrix A has the following property, which will be called the *Determinantal Property*.

(DP) Each square submatrix of A obtained by choosing k rows and k columns sequentially from A has the Right Sign (see Definition 3.1).

LEMMA 6.1. *Suppose that γ is a conductivity on a circular network with n boundary nodes. Then the values of the $n(n-1)/2$ entries of A above the diagonal determine uniquely the remaining entries of A.*

Proof. The entries below the diagonal are obtained from the symmetry relation: $A_{i,j} = A_{j,i}$. The diagonal entries are then obtained from the sum relation. □

THEOREM 6.2. *Let m be a non-negative integer, and let $n = 4m + 3$. Let A be a n by n matrix whose entries satisfy the relations R1, R2, and which has the DP. Then there is a unique conductivity function γ on a circular network of type $C(m, n)$ such that A is the matrix representing Λ_γ.*

The proof will follow several lemmas. An n by n matrix A will be called a λ-matrix if it satisfies the relations R1, R2, and has the DP. We will show (Lemma 6.7) that the set of n by n λ-matrices is path-connected. Thus the given λ-matrix A can be joined to the λ-matrix corresponding to $\gamma = 1$ by a path of n by n λ-matrices. The proof of the theorem will be completed by showing that every matrix on this path must be of the form Λ_γ.

We will need the following elementary facts from matrix algebra.

LEMMA 6.3. *Let $B^{(k)}$ be a sequence of n by n matrices with $\lim_{k\to\infty} B^{(k)} = B$. Let $v^{(k)}$ be a sequence of vectors of bounded norms. Then the norms of $B^{(k)}v^{(k)}$ and the magnitudes of $< v^{(k)}\ , B^{(k)}v^{(k)} >$ are bounded.*

LEMMA 6.4. *Let $B^{(k)}$ be a sequence of n by n matrices with $\lim_{k\to\infty} B^{(k)} = B$. Assume that B and each $B^{(k)}$ is nonsingular. Let c be a fixed vector, and let $v^{(k)}$ be a sequence of vectors with $B^{(k)}v^{(k)} = c$ for each $k = 1, 2, \ldots,$. Then the norms of $v^{(k)}$ are bounded.*

Let $M = \{M_{i,j}\}$ be a k by k matrix. For each (i, j), let $M(i, j)$ be the (i, j) -th *minor*, that is, the $(k-1)$ by $(k-1)$ matrix formed by deleting the i-th row and the j-th column of M. The expansion of $\det(M)$ by its first column gives

$$\det M = \sum_{i=1}^{i=k} (-1)^{i+1} M_{i,1} \det M(i, 1)$$

For each integer $k \geq 1$, we define a function f_k as follows. f_1 is defined to be the constant 0. For $k \geq 2$, f_k is a function of the entries of a k by k matrix M, defined by:

$$f_k(M) = \left(\sum_{i=1}^{i=k-1} (-1)^{i+k+1} M_{i,1} \det M(i, 1) \right) / \det M(k, 1)$$

Observe that $f_k(M)$ is a function of the $k^2 - 1$ entries

$$(M_{1,1}, \ldots, \widehat{M_{k,1}}, \ldots, M_{k,k})$$

That is, $f_k(M)$ is independent of the entry $M_{k,1}$; $f_k(M)$ is well defined if $\det M(k, 1) \neq 0$. Recall (Definition 3.1) that a k by k matrix M is said to have the Right Sign (RS) if:

(1) $\det M < 0$ when $k \equiv 1, 2 \bmod 4$

(2) $\det M > 0$ when $k \equiv 0, 3 \bmod 4$.

LEMMA 6.5. *Let M be a k by k matrix such that the minor $M(k,1)$ has the RS. Then if the entry $M_{k,1} < f_k(M)$, M will have the RS also.*

Proof. This follows by expanding $\det M$ by its first column, and using the definition of the function f_k. $\square$

Lemma 6.1 shows that we may take the $n(n-1)/2$ entries above the diagonal as parameters of A. Thus the total number of parameters is the same as the number of conductors. Let $N = 2m(4m+3) = n(n-1)/2$.

It is convenient to consider an extended matrix $\hat{A}$. For all integers p and q, the entries of $\hat{A}$ are given by $\hat{A}_{i+pn,j+qn} = A_{i,j}$.

The parameters are ordered as follows. For each integer h with $1 \leq h \leq N$, let

$$h = a + (4m+3)(b-1)$$

where $1 \leq a \leq 4m+3$ and $1 \leq b \leq 2m-1$. Then the h-th parameter is at position $(a, 2m+2+a-b)$ of $\hat{A}$. By means of the definition of the entries of $\hat{A}$, and symmetry, this corresponds to a unique entry of A above the diagonal. If $a - b \leq 2m+1$ the h-th parameter is the entry $(a, 2m+2+a-b)$ of A; if $a-b > 2m+1$ the h-th parameter is the entry $(a-b-2m-1, a)$ of A. Figure 6.1 shows the sequence of parameter entries in A for a circular network of type $C(1,7)$.

$$\begin{bmatrix} * & 15 & 8 & 1 & 5 & 13 & 21 \\ & * & 16 & 9 & 2 & 6 & 14 \\ & & * & 17 & 10 & 3 & 7 \\ & & & * & 18 & 11 & 4 \\ & & & & * & 19 & 12 \\ & & & & & * & 20 \\ & & & & & & * \end{bmatrix}$$

FIG 6.1

For $h > 4m+3$, let $h = a + (4m+3)(b-1)$, as above. Then the h-th parameter position is in the lower left corner of a b by b submatrix of the extended matrix $\hat{A}$, which will be denoted $B(h)$. The other entries of $B(h)$ correspond to parameters x_i for $i < h$. For each integer $1 \leq j \leq N$, we define a function $F_j(x_1, \ldots, x_{j-1})$ as follows. For $1 \leq j \leq 4m+3$, $F_j = 0$. Suppose inductively that F_i has been defined for $i < j$.

The domain of F_j will be the set of $x_1, x_2, \ldots, x_{j-1}$ such that for each $1 \leq i < j$, $x_i < F_i(x_1, x_2, \ldots, x_{i-1})$ Then

$$F_j(x_1, \ldots, x_{j-1}) = f_b(B(j))$$

Inductively, we see that $F_j(x_1, \ldots, x_{j-1})$ is well-defined for each $j = 1, \ldots, N$.

REMARK 6.6. Let S be the set of parameter values $x_1, x_2, \ldots, x_N$ such that for each $1 \leq j \leq N$, $x_j < F_j(x_1, x_2, \ldots, x_{j-1})$. An n by n matrix with relations R1 and R2 will have the DP if and only if its parameter values lie in the set S.

LEMMA 6.7. *S is a path-connected set in R^N.*

Proof. For each $h = 1, \ldots, N$, let S_h be the set of parameter values $(x_1, x_2, \ldots, x_h)$ such that $x_j < F_j(x_1, x_2, \ldots, x_{j-1})$, for each $1 \leq j \leq h$. We will show by induction on h that each S_h is path-connected set in R^h. $S_1 = \{x_1 : x_1 < 0\}$, and so is path-connected. Assume inductively that S_j is path-connected for $j < h$. Let $(x_1, \ldots, x_{h-1}, x_h)$ and $(y_1, \ldots, y_{h-1}, y_h)$ be two points in S_h. Take $\beta(t) = (\beta_1(t), \ldots, \beta_{h-1}(t))$ a path in S_{h-1} joining $(x_1, \ldots, x_{h-1})$ and $(y_1, \ldots, y_{h-1})$. Let $T = \min_t\{F_h(\beta(t))\}$. We have three paths:

(1) The straight line $(x_1, \ldots, x_{h-1}, x_h)$ to $(x_1, \ldots, x_{h-1}, T)$

(2) $(\beta(t), T)$

(3) The straight line $(y_1, \ldots, y_{h-1}, T)$ to $(y_1, \ldots, y_{h-1}, y_h)$

These three paths give a path from $(x_1, \ldots, x_{h-1}, x_h)$ to $(y_1, \ldots, y_{h-1}, y_h)$ in S_h. $\square$

DEFINITION 6.8. *A matrix whose entries satisfy the relations R1 and R2 and has the property DP will be called a λ-matrix.*

Lemma 6.7 implies that the set of λ-matrices is connected. Let A be a n by n λ-matrix. To prove Theorem 6.2 we need to show that there is a unique conductivity function γ such that A is the matrix representing Λ_γ. We denote by $L(n)$ the set of n by n matrices A which represent Λ_γ for some conductivity γ on Ω_1. It follows from Theorem 4.2, Lemma 6.5 and the open mapping theorem that $L(n)$ is an open subset of the set of n by n λ-matrices. Let $A(t)$ for $0 \leq t \leq 1$, be a path of λ-matrices joining $A(0)$ with $A(1)$, where $A(0)$ is the λ-matrix corresponding to $\gamma = 1$, and where $A(1) = A$ is the given λ-matrix. We will show that each matrix along this path is in $L(n)$. Suppose the contrary. Since the set of t for which $A(t)$ is in $L(n)$ is open, there is a the first value t_0 for which $A(t_0)$ is not in $L(n)$. For each $t < t_0$, let $\gamma(t)$ be the conductivity corresponding to $A(t)$. For each conductor pq, we pick a number $\mu(pq)$ which is zero, infinity, or a positive real number and a sequence $\{t_1, t_2, \ldots, t_k, \ldots\}$ with $\lim_{k\to\infty} t_k = t_0$, and such that $\lim_{k\to\infty}\gamma(t_k)(pq) = \mu(pq)$. We will write $\gamma^{(k)}$ for $\gamma(t_k)$ and $A^{(k)}$ for $A(t_k)$. We know that $\lim_{k\to\infty} A^{(k)} = A^{(0)}$ and each of these is a λ-matrix. It follows from Lemma 6.3 that for any fixed boundary potential ϕ, the magnitudes of $< \phi\ , A^{(k)}(\phi) >$ are bounded. Also, because of the conditions on the values of the parameters, each sequentially obtained square submatrix of $A^{(0)}$, and each sequentially obtained square submatrix of $A^{(k)}$ is non-singular.

We will make use of the principal flow patterns described in Lemma 3.8. Let c be the vector of currents $c = (-1, +1, \ldots, -1)$ at nodes $p_1, p_2, \ldots, p_{2m+1}$. For each k, let $M^{(k)}$ be the sequentially obtained $2m+1$ by $2m+1$ submatrix of $A^{(k)}$ consisting of the entries from rows $1, 2, \ldots, 2m+1$ and columns $2m+2, 2m+3, \ldots, 4m+2$. Let $v^{(k)}$ be the solution to $M^{(k)}v^{(k)} = c$. Let $\psi^{(k)}$ be the boundary potential given by $\psi^{(k)}(p_j) = 0$ for $0 \leq j \leq 2m+1$, and $\psi^{(k)}(p_j) = v^{(k)}(p_j)$ for $j = 2m+2, 2m+3, \ldots, 4m+2$. $\psi^{(k)}$ is the boundary potential which produces current c at nodes $p_1, p_2, \ldots, p_{2m+1}$. Let $u^{(k)}$ be the $\gamma^{(k)}$-harmonic function with boundary values $\psi^{(k)}(p_j)$. This situation is illustrated by the flow diagram FIG 3.1

LEMMA 6.9. *In this situation, there is an upper bound for the magnitudes of* $\mid u^{(k)}(p) \mid$ *for all* k *and all nodes* p. *There is also an upper bound for the currents* $\mid \gamma^{(k)}(u^{(k)}(p) - u^{(k)}(q)) \mid$ *for all edges* pq.

Proof. Lemma 6.4 shows that there is an upper bound for the values of $\mid \psi^{(k)}(p_j) \mid$ for all boundary nodes p_j and all k. By the maximum principle, this is also an upper bound for $\mid u^{(k)}(p) \mid$ for all nodes p and all k. Lemma 6.3 shows that there is an upper bound for the currents at all boundary nodes p_j and all k. This is also an upper bound for the current along any edge. □

We continue with the proof of Theorem 6.2.

(i) Assume that for some conductor pq, $\mu(pq) = 0$. Whether radial or circular, by a rotation of the figure, we may assume that pq lies along a principal flow line as in Figure 6.2.

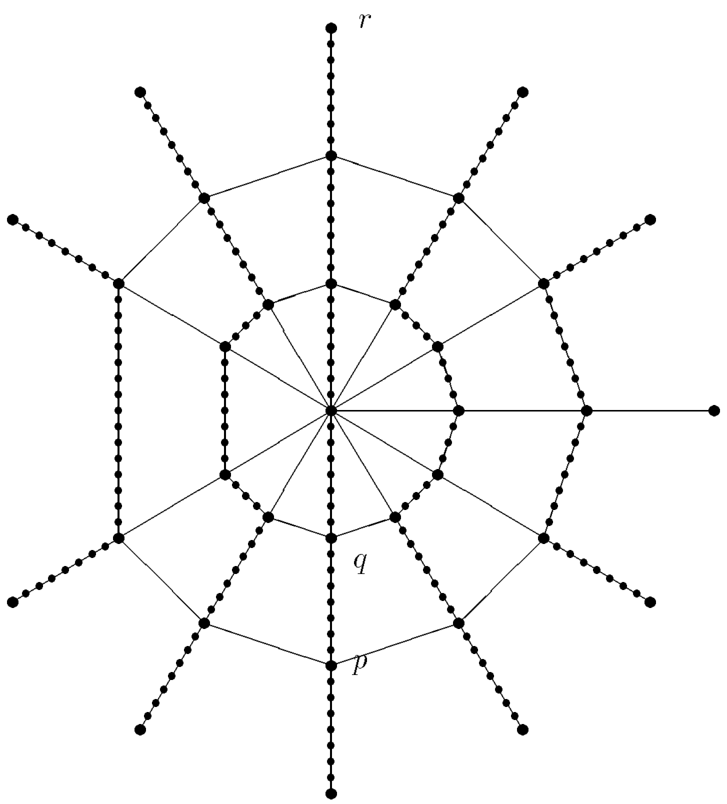

FIG 6.2

Let $\gamma^{(k)}(pq) = \varepsilon_k$, where $\lim_{k\to\infty}\varepsilon_k = 0$. Let the γ-harmonic functions $u^{(k)}$ be as in Lemma 3.8. Specifically, the boundary data is: $u^{(k)}(p_j) = 0$ for $0 \le j \le 2m+1$ and $I_{u^{(k)}}(p_j) = (-1)^j$ for $1 \le j \le 2m+1$.

Let r be the boundary node at the low end of the path of principal flow. Suppose that the current at r is -1 (a similar argument would apply if the current at r is $+1$). Then $u^{(k)}(q) > 0$, and the current across pq is at least 1. Then

$$u^{(k)}(p) - u^{(k)}(q) \ge 1/\varepsilon_k$$

This would imply that $\lim_{k\to\infty} u^{(k)}(p) = \infty$, contradicting Lemma 6.3.

(ii) Next suppose that $\mu(ab) = \infty$ for some boundary conductor ab. By a rotation of Ω we may assume that $\mu(p_0q_0) = \infty$. Refer to Figure 6.3 for the notation.

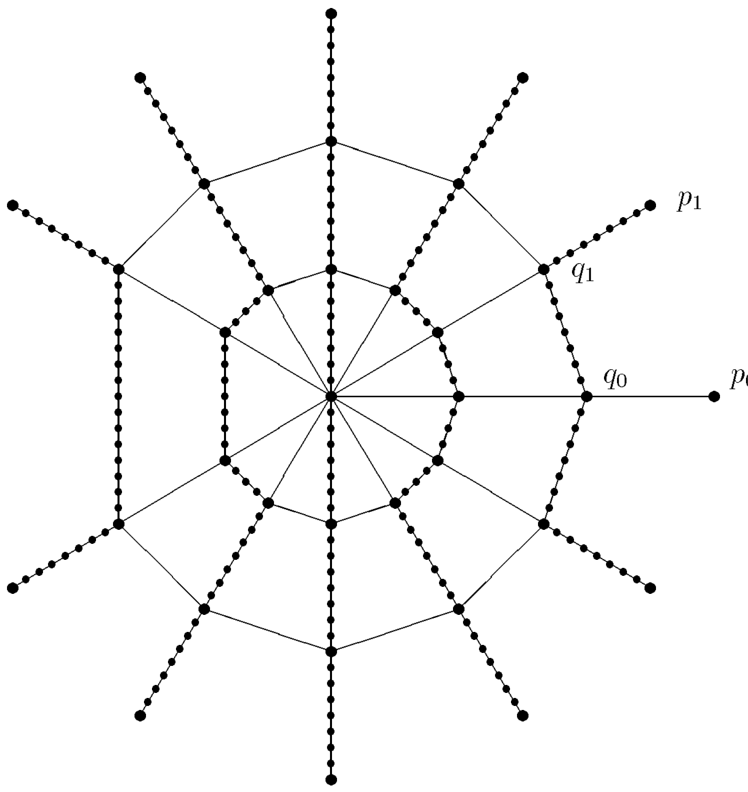

FIG 6.3

Given a positive real number R, choose a positive integer Z so that if $k \geq Z$, $\gamma^{(k)}(p_0 q_0) \geq R$. For each positive integer k, let $u^{(k)}$ be the $\gamma^{(k)}$-harmonic function on Ω as in Corollary 3.8. Let $\psi(k)$ be the function $u(k)$ restricted to the boundary of Ω. Let Y be an upper bound for all $\mid A^{(k)}(\psi^{(k)})(p_j) \mid$ Then

$$0 \leq u^{(k)}(q_1) \leq u^{(k)}(q_0) \leq Y/R$$

The current across conductor $q_1 p_1$ is 1, so the current across $q_0 q_1$ is at least 1. Then

$$\gamma^{(k)}(q_1 p_1) \geq R/Y$$

and

$$\gamma^{(k)}(q_0 q_1) \geq R/Y$$

Recall that ϕ_1 is the function on $\partial\Omega$ which is 1 at p_1 and 0 at all other boundary nodes. For each positive integer k, consider the network Ω with conductivity $\gamma^{(k)}$. Let $v^{(k)}$ be the potential on Ω which equals ϕ_1 on the boundary of Ω. Let W be an upper bound for $\mid A^{(k)}(\phi_1(p_j) \mid$ Then

$$v^{(k)}(q_0) \leq WY/R$$

$$v^{(k)}(q_1) \geq 1 - WY/R$$

The current $\gamma^{(k)}(q_0q_1)(v^{(k)}(q_1) - v^{(k)}(q_0))$ would tend to ∞ as $k \to \infty$. By the maximum principle, the value of $v^k(q_1)$ is $\geq$ the value of v^k at any node other than p_1. It follows that the current across p_1q_1 is greater than or equal to the current across q_1q_0. which contradicts the upper bound on the values of $A_{i,j}^{(k)}$.

From (i) and (ii), we can assume that $\varepsilon \leq \gamma^{(k)}(ab) \leq X$ for each boundary conductor ab and each $k \geq 0$.

(iii) Assume that for some interior conductor pq, $\mu(pq) = \infty$. Whether radial or circular, by a rotation of the figure, we may assume that the edge pq is transverse to the principal current flow. Let r be the boundary node at the low end of the principal path containing p. Similarly let s be the boundary node at the low end of the principal path containing q. Let r' be the interior neighbor of r and let s' be the interior neighbor of r. The situation is illustrated by Figure 6.4.

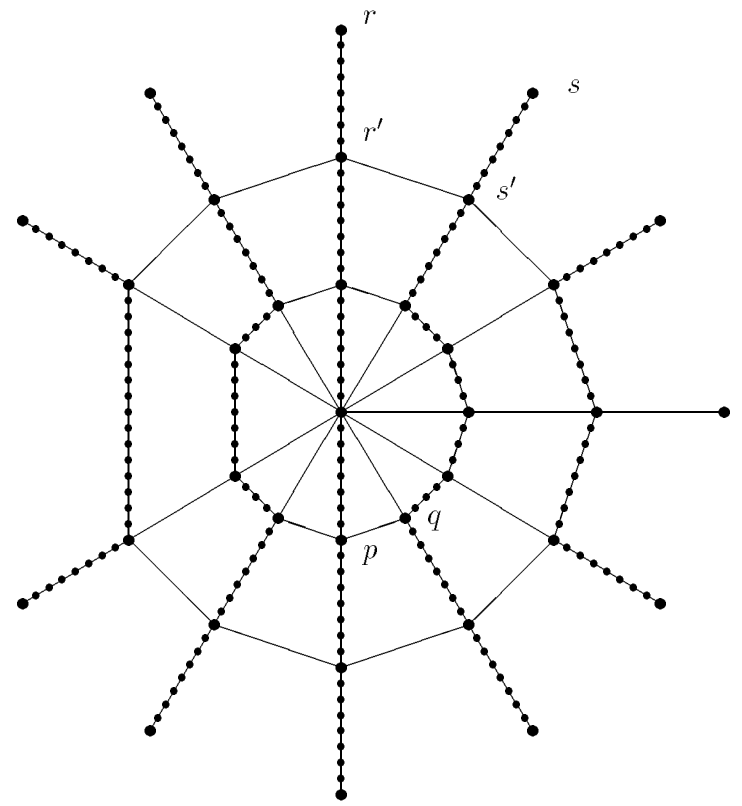

FIG 6.4

Let $\gamma^{(k)}(pq) = X^{(k)}$, where $\lim_{k\to\infty} X^{(k)} = \infty$. Let c and the γ-harmonic functions $u^{(k)}$ be as in Lemma 3.8. Again suppose that the current at r is -1. Then

$$u(p) \geq u(r') \geq 1/X$$

$$u(q) \leq u(s') \leq -1/X$$

This would give a current through pq which is

$$\gamma^{(k)}(pq)(u^{(k)}(p) - u^{(k)}(q)) \geq 2X^{(k)}/X$$

This has limit ∞, which contradicts Lemma 6.9.

Let A be a n by n matrix which is a λ-matrix. We have just shown that A is of the form Λ_γ. This completes the proof of Theorem 6.2.

7. The Neumann-to-Dirichlet map. Let $\Omega = (\Omega_0, \Omega_1)$ be a circular network of type $C_1(m, 4m+3)$. If boundary currents $f(p_j)$ are put at each boundary node p_j of Ω, with $\sum_{j=1}^n f(p_j) = 0$, there will result a potential u throughout Ω, which is unique to within an additive constant. Let $\phi(p_j)$ be the boundary potentials of u. The map which takes f to ϕ is called the Neumann-to-Dirichlet map Ψ. Ψ gives rise to a bilinear form F on the set of boundary functions with sum 0, by

$$F(f, g) = < f, \Psi(g) >$$

F is well defined, independent of the additive constants. From the bilinear form F, a matrix representation B of the Neumann-to-Dirichlet map is obtained as follows. For each $1 \le j \le n$, let f_j be the function on the boundary nodes of Ω, given by $f_j(p_j) = +1$, $f_j(p_{j+1}) = -1$ and $f_j(p_k) = 0$ for all $k \neq j, j+1$. Then the entries of B are given by

$$B_{i,j} = F(f_i, f_j)$$

The matrix B represents the Neumann-to-Dirichlet map on the network in the following way. Let a current of $+1$ be put at node p_j and -1 at node p_{j+1}. Then $B_{i,j}$ is the voltage difference between nodes p_i and p_{i+1}.

The relation between the matrix A for the Dirichlet-to-Neumann map Λ for Ω and the matrix B for the Neumann-to-Dirichlet map Ψ for Ω is the following. Let P be the matrix, whose entries are $P_{i,i} = +1$, $P_{i+1,i} = -1$, $P_{1,n} = -1$ and $P_{i,j} = 0$ for all other entries. Let P^T be the transpose of P.
Then

$$B = P^T A^{-1} P$$

and

$$A = P B^{-1} P^T$$

This requires some explanation, because A and B each have rank $n-1$. A^{-1} is defined on each column of P, and gives a column which is unique to within an additive constant. Multiplying on the left by P^T removes the ambiguity, so the product $P^T A^{-1} P$ is well defined and it is the matrix B. Similarly, $P B^{-1} P^T$ is well defined and is the matrix A.

A reconstruction algorithm which is similar to that given in Section 5 based on the Neumann-to-Dirichlet map can be given.

8. Effective resistances. Measurements at boundary nodes are made as follows. A current of $+1$ is put at node p_i, and a current of -1 is put at node p_j; at all other boundary nodes, the currents is 0. From this Neumann data, there will result a potential u throughout Ω, unique to within an additive constant. Let $R_{i,j}$ be the potential difference measured between node p_i and p_j; $R_{i,j}$ is called the *effective resistance* between nodes p_i and p_j. The set of measurenents $R_{i,j}$ may be used to reconstruct the matrix B which represents the Neumann-to-Dirichlet map as follows. For each pair of integers i and j between 1 and n, let $c_{i,j}$ be the current described above. That is $c_{i,j}(p_i) = 1$, $c_{i,j}(p_j) = -1$ and $c_{i,j}(p_k) = 0$ for all other boundary nodes p_k.

Then

$$R_{i,j+1} =< c_{i,j+1}, \Psi(c_{i,j+1}) >$$

using

$$c_{i,j+1} = f_i + f_{i+1} + \dots f_j$$

we see that

$$R_{i,j+1} = \sum_{p=i,q=i}^{p=j,q=j} B_{p,q}$$

From this, it follows that for each i,

$$B_{i,i} = R_{i,i+1}$$

and for $j > i$,

$$2B_{i,j} = R_{i,j+1} - \sum_{p=i,q=i}^{p=j-1,q=j-1} B_{p,q}$$

The entries of the matrix B may be computed from the entries of R, by induction on the difference $j - i$. From the reconstruction algorithm of Section 5, it follows that the effective resistances uniquely determine the values of the conductors in the network.

9. Other Circular Networks. In this Section, we will consider other types of networks with m circles and n rays. There are four types of circular networks labelled $C_1(m,n)$, $C_2(m,n)$, $C_3(m,n)$ and $C_4(m,n)$. The circular networks of type $C(m,n)$ defined in Section 1 will now be labelled $C_1(m,n)$.

For each pair of positive integers m, n, the circular network of type $C_2(m,n)$ has m circles and n rays. Figure 9.1 illustrates a circular network of type $C_2(3,12)$.

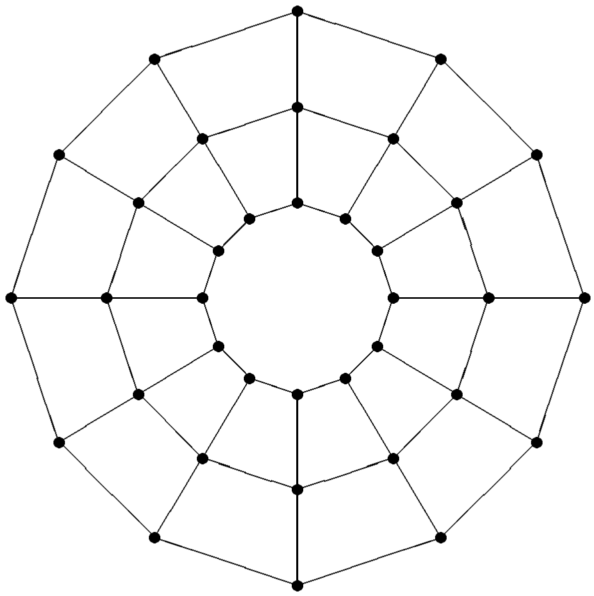

FIG 9.1

The nodes of $C_2(m,n)$ are the points $p_{i,j}$ for $1 \le i \le m$ and $0 \le j \le n-1$. The node $p_{i,j}$ is given in polar coordinates by $p_{i,j} = (i - 1/2, j2\pi/n)$. There are mn nodes. The interior consists of those nodes $p_{i,j}$ for $1 \le i < m$ and $0 \le j \le n-1$. This includes the nodes on innermost circle, but not on the outermost circle. The boundary consists of the nodes on the outermost circle, but not the nodes on the innermost circle. Each interior node not on the innermost circle has four neighboring nodes; each node on the innermost circle has three neighbors. Each boundary node has two neighbors which are also boundary nodes and one neighboring node which is an interior node. An edge is a radial line segment $p(i,j)p(i+1,j)$ or a circular arc $p(i,j)p(i,j+1)$. There are $n(2m-1)$ edges. A circular network of resistors of type $C_2(m,n)$ is such a network together with a conductivity function γ on the edges. An algorithm for recovering γ from Λ_γ like that of Section 5 can be given for circular networks of type $C_2(m,n)$ if $n \ge 4m+3$.

For each pair of positive integers m, n, a circular network of type $C_3(m,n)$ has m circles and n rays. Figure 9.2 illustrates a circular network of type $C_3(3,12)$.

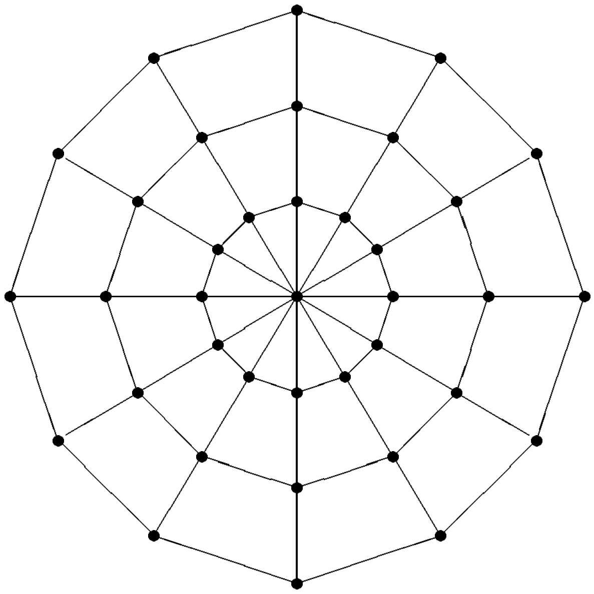

FIG 9.2

The nodes of $C_3(m,n)$ are the points $p_{i,j}$ for $0 \le i \le m$ and $0 \le j \le n-1$. The node $p_{i,j}$ is given in polar coordinates by $p_{i,j} = (i, j2\pi/n)$. There are $1+mn$ nodes. The interior consists of those nodes $p_{i,j}$ for $0 \le i < m$ and $0 \le j \le n-1$. The boundary consists of the nodes $p_{m,j}$ for $0 \le j \le n-1$. Each interior node, except the center node, has four neighboring nodes; the center node $p(0,0)$ has n neighbors. Each boundary node has three neighboring nodes: and two neighbors which are boundary nodes and one neighbor which is an interior node. An edge is a radial line segment $p(i,j)p(i+1,j)$ or a circular arc $p(i,j)p(i,j+1)$. There are $2mn$ edges. A circular network of resistors of type $C_3(m,n)$ is such a network together with a conductivity function γ on the set of edges. An algorithm for recovering γ from Λ_γ like that of Section 5 can be given for circular networks of type $C_3(m,n)$ if $n \ge 4m+1$.

For each pair of positive integers m, n, a circular network of type $C_4(m,n)$ has m circles and n rays. Figure 9.3 illustrates a circular network of type $C_4(2,8)$.

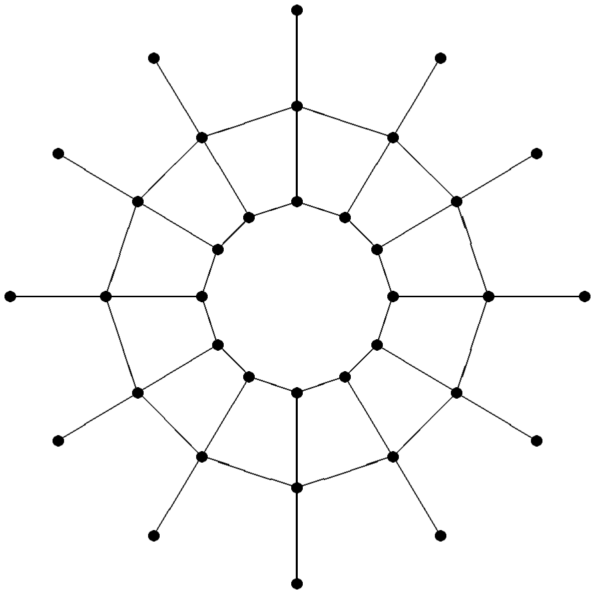

FIG 9.3

The nodes are the points in the plane $p_{i,j}$ for $1 \leq i \leq m$ and $0 \leq j \leq n-1$. The nodes $p_{i,j}$ are given in polar coordinates by $p_{i,j} = (i - 1/2, j2\pi/n)$. There are mn nodes. The interior consists of those nodes $p_{i,j}$ for $1 \leq i < m$ and $0 \leq j \leq n-1$. This includes the innermost circle. The boundary consists of those nodes $p_{m,j}$ for $0 \leq j \leq n-1$. Each interior node not on the innermost circle has four neighboring nodes; each node on the innermost circle has three neighbors. Each boundary node has one neighboring node which is an interior node. An edge is a radial line segment $p(i,j)p(i+1,j)$ or a circular arc $p(i,j)p(i,j+1)$. There are $2mn$ edges. A circular network of resistors of type $C_4(m,n)$ is such a network together with a conductivity function on the set of edges. An algorithm for recovering γ from Λ_γ like that of Section 5 can be given for circular networks of type $C_4(m,n)$ if $n \geq 4m+1$.

10. Dual Networks. Let Ω_1 be a network of type $C_1(m,n)$ and let Ω_2 be a network of type $C_2(m+1,n)$. Ω_1 is dual to the network Ω_2 as follows. Ω_2 is rotated clockwise by π/n so that each edge α in Ω_1 is perpendicular to an edge $\alpha^\perp$ in Ω_2. The orientation of $\alpha^\perp$ is to be that of α rotated clockwise by $\pi/2$. Figure 10.1 shows Ω_2, a network of type $C_2(2,8)$ (solid lines), and Ω_1 (dotted lines), a network of type $C_1(1,8)$.

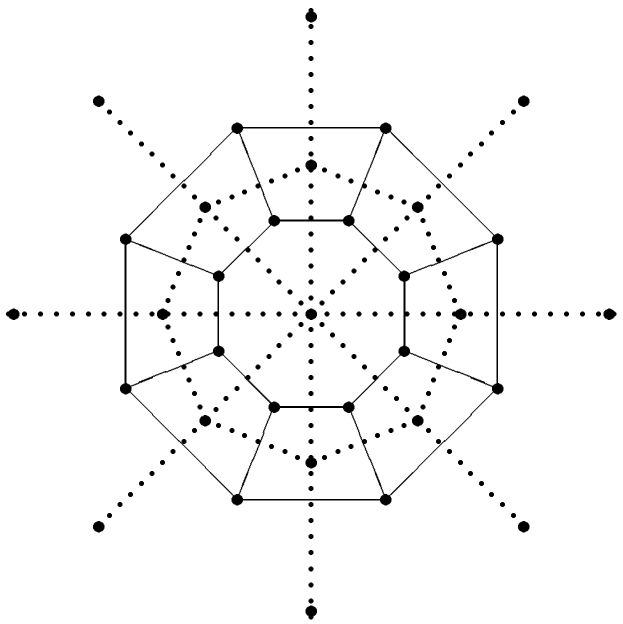

FIG 10.1

If γ_1 is a conductivity on Ω_1, the dual conductivity γ_2 on Γ_2 is defined by $\gamma_2(\alpha^\perp) = 1/\gamma_1(\alpha)$. For each γ_1 - harmonic function u on Ω_1, let v be the γ_2 - harmonic function on Γ_2, where

$$\Delta v(\alpha^\perp) = I_u(\alpha)$$

$$I_v(\alpha^\perp) = \Delta u(\alpha)$$

The function v is well defined to within additive constant. Each boundary node p_i of Ω_1 lies between two boundary nodes of Γ_2, which will be numbered q_i and q_{i+1} (with $q_{n+1} = q_1$). For each $1 \leq j \leq n$, let f_j be the function on the boundary nodes of Γ_2, given by $f_j(q_j) = +1$, $f_j(q_{j+1}) = -1$ and $f_j(q_l) = 0$ for all $l \neq j, j+1$. Let v_j be the γ_2 - harmonic function on Γ_2 with boundary current f_j. The Neumann-to-Dirichlet map for Γ_2 is represented by a matrix Ψ, where

$$\Psi_{i,j} =< f_i, \Psi(f_j) >$$

The matrix Ψ for the Neumann-to-Dirichlet map on the network Γ_2 is the *same* as the matrix Λ for the Dirichlet-to-Neumann map on Ω which was constructed in Section 2. Thus the matrix Ψ has the same properties as Λ.
Similarly, each network of type $C_3(m,n)$ is dual to a network of type $C-4(m,n)$.

11. The Inverse Conductivity Problem for Continuua. Let Ω be a compact, connected region in R^k with boundary $\partial\Omega$. Let γ be a positive C^∞ scalar-valued function on Ω; γ is called conductivity. The conductivity equation is:

$$\bigtriangledown(\gamma \bigtriangledown u) = 0$$

The (forward) Dirichlet problem is the following. Given a function f on $\partial\Omega$, find a function u on Ω such that:

$$\begin{aligned} \bigtriangledown(\gamma \bigtriangledown u) &= 0 \\ u &= f \quad \text{on} \quad \partial\Omega \end{aligned}$$

Similarly, the (forward) Neumann problem is the following. Given a function g on $\partial\Omega$, find a function u on Ω such that:

$$\begin{aligned} \bigtriangledown(\gamma \bigtriangledown u) &= 0. \\ \gamma\frac{\partial u}{\partial n} &= g \quad \text{on} \quad \partial\Omega \end{aligned}$$

If $\gamma(x)$ is the constant function $\gamma(x) = 1$, the conductivity equation is the Laplace Equation $\bigtriangledown \cdot \bigtriangledown u = 0$, and we have the ordinary Dirichlet or Neumann Problem.

Example 1 Let a material with electric conductivity $\gamma(x)$ occupy the region Ω. If a potential f is imposed on $\partial\Omega$ there will be a potential u throughout Ω which satisfies the conductivity equation. This potential u gives rise to a current $I = \gamma\frac{\partial u}{\partial n}$ at the boundary of Ω.

The Dirichlet-to-Neumann map

$$\Lambda : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$$

is defined by

$$\Lambda(f) = \gamma\frac{\partial u}{\partial n}$$

where u solves the conductivity equation in Ω with $u = f$ on $\partial\Omega$. In Example 1, the Dirichlet-to-Neumann map takes a potential on the boundary of Ω to the resulting current at the boundary of Ω.

The linear map $\Lambda = \Lambda_\gamma$ depends on γ. The Inverse Problem is to determine γ from Λ. Physically, this means to use measurements of potentials and currents at the boundary of Ω to determine the conductivity inside Ω. As in the discrete case, the Inverse Problem breaks into four problems:

1. Uniqueness: Does $\Lambda_\gamma = \Lambda_\mu$ imply $\gamma = \mu$?
2. Reconstruction: calculate γ from the map

$$\Lambda_\gamma : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$$

3. Continuity of Inverse: If Λ_γ is near to Λ_μ, does it follow that γ is near to μ?
4. Characterization: Which linear maps

$$\Lambda : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$$

are of the form $\Lambda = \Lambda_\gamma$ for some γ.

For compact, connected regions Ω in R^k with $k \geq 2$, and piecewise analytic conductivity γ, the uniqueness (1) was shown by Kohn and Vogelius [5]. In the case of a conductivity γ which is assumed to be C^∞ , the uniqueness was shown for dimensions $k \geq 3$, by Sylvester and Uhlmann [6]. For $k = 2$, the result is unknown. For $k \geq 3$, the continuity of the inverse (3) was shown by Allesandrini (1988). Some work on the reconstruction has been done by Wexler, Kohn and Vogelius and others (1983-1988). See [5], [3]. characterization problem in the continuum case.

Our methods show that there is an Alternating Property for the Dirichlet-to-Neumann map in the continuum case, which is analagous to the Alternating Property (Theorem 3.2) for the discrete case. Let Ω be a compact, connected and simply connected region with conductivity γ. The outward normal to $\partial\Omega$ will be called n. Let P and Q be distinct points on $\partial\Omega$. Assume that $\partial\Omega$ is homeomorphic to a circle, so that P and Q separate $\partial\Omega$ into two arcs which we call C_1 and C_2.

THEOREM 11.1. *Let ϕ be a function on $\partial\Omega$ which is identically 0 on C_1 and for which $\Lambda(\phi)$ changes sign k times on C_1. Then ϕ must change sign at least k times in C_2.*

Proof. Let u be the γ-harmonic function which solves the conductivity equation with boundary values ϕ. Then $\Lambda(\phi) = \gamma\frac{\partial u}{\partial n}$ is the resulting boundary current. Suppose that $p_1, p_2, \ldots, p_{k+1}$ is a sequence of points in order along C_1, for which the values $\Lambda\phi(p_1), \Lambda\phi(p_2), \ldots, \Lambda\phi(p_{k+1})$ alternate in sign. We will show that there are points $s_1, s_2, \ldots, s_{k+1}$ in C_2 where the function ϕ alternates in sign. Suppose that at some point $p_i \in C_1$, $\gamma\frac{\partial u}{\partial n}(p_i) < 0$. There is a line segment $p_i q_i$ in Ω along which u is monotone increasing. Suppose that $u(q_i) = \epsilon$. Let $U_i = \{x \in \Omega : u(x) > \epsilon/2\}$. Let V_i be the connected component of U_i in Ω which contains q_i. By the maximum principle, V_i must contain a point s_i on $\partial\Omega$ with $u(s_i) > 0$; necessarily $s_i \in C_2$. Similarly, for a point $p_i \in C_1$, where $\gamma\frac{\partial u}{\partial n}(p_i) > 0$ there is a line segment $p_i q_i$ along which u is monotone decreasing, and there is a connected open set V_i containing q_i and a point s_i on the boundary where $u(s_i) < 0$. The values of ϕ at the points $s_1, s_2, \ldots s_{k+1}$ must be the negatives of the reversal of the signs of the values $\Lambda(\phi)$ at the points $p_1, p_2, \ldots, p_{k+1}$. □

12. Complex impedances. In this Section, we consider networks where each edge has a complex frequency-dependent impedance $z(pq\,;\omega)$. The *admittance* $y(pq\,;\omega)$ is defined by $y(pq\,;\omega) = 1/z(pq\,;\omega)$. We assume that the real part of each $z(pq\,;\omega)$ is positive; then the real part of $y(pq\,;\omega)$ will also be positive. For each frequency ω, we consider functions on Ω_0 which have the form $u(p\,;\omega) = f(p\,;\omega)e^{i\omega t}$. That is, for each node $p \in \Omega_0$, $f(p;\omega)$ is a complex number, depending on ω. The identity of Lemma 2.8 can be used to show that the analogue to Lemma 2.5 is valid in the case of complex admittances with positive real part. Thus the Dirichlet-to-Neumann map is a well-defined linear map which takes (steady-state) boundary potentials of frequency ω to (steady-state) boundary currents of frequency ω. An inner product on complex boundary functions is defined by:

$$< \phi, \psi >= \sum_{j=1}^{n} \phi(p_j)\overline{\psi(p_j)}$$

where the bar stands for complex conjugate. For each index $j = 1, 2, \ldots, n$, let ϕ_j be the boundary function which is $e^{i\omega t}$ at node p_j and 0 at all other boundary nodes. The Dirichlet-to-Neumann map Λ is represented by a matrix $A = \{A_{k,j}\}$ of complex numbers. The entries $A_{k,j}$ are given by:

$$A_{k,j} =< \Lambda(\phi_j), \phi_k >$$

The algorithm of Section 5 applies to show that, for each frequency ω, measurements of the steady-state potentials and currents at the boundary of the network can be used to calculate the (frequency dependent) impedance along each interior edge. Section 9 shows that it is sufficient to measure the effective impedance between each pair of boundary nodes to determine the interior impedances.

13. Some numerical results. A program based on the algorithm for reconstructing the network of type $C_1(3, 15)$ has been written and several numerical experiments have been performed. Here we will report on the results of reconstructing the network in which all conductors have value 1. All computations were made in double precision using Fortran on a Decstation 5000. The largest error was approximately 1.5×10^{-10}, which means that roughly 6 digits were lost in the computation. If the entries of the lambda matrix were perturbed randomly by terms of magnitude 10^{-8} then the largest error in the computation of the conductors was approximately .5. If the lambda matrix was perturbed randomly by terms of magnitude 10^{-7} then some of the conductors were computed to have negative values. This would indicate that the reciprocal condition number of this problem is about 10^{-8}. Linpack estimates the reciprocal condition number of the derivative of the Dirichlet-to-Neumann map, considered as a map of R^{105} to R^{105}, to be 5.3×10^{-9}. The following graph shows a plot of the logarithm to base ten of the singular values of the derivative of the Dirichlet to Neumann map. Notice the values seem to occur in families of $15, 30, 30$ and 30 elements. The families of 30 are further subdivided into subfamilies of 15 elements.

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