THE DIRICHLET TO NEUMANN MAP FOR A RESISTOR NETWORK *

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Abstract. Relations that characterize the Dirichlet to Neumann map for a square network of resistors are given. Using these relations, a parametrization of the set of Dirichlet to Neumann maps is given.

Key words. network of resistors, inverse problem, conductivity

AMS(MOS) subject classifications. 31A25, 31B25, 35R30, 90B10

1. Introduction. We consider resistor networks in the plane. For each positive integer n, a square network Ω is constructed as follows. The *nodes* of Ω are the integer lattice points in the plane $p = (i, j)$ for which $0 \leq i \leq n+1$ and $0 \leq j \leq n+1$, with the four corner points $(0,0), (n+1,0), (0,n+1)$, and $(n+1,n+1)$ excluded. The set of nodes is denoted Ω_0. The *interior* of Ω_0, called int Ω_0, consists of those nodes $p = (i,j)$ with $1 \leq i \leq n$ and $1 \leq j \leq n$. The *boundary* of Ω_0, called $\partial\Omega_0$, is $\Omega_0 - \text{int}\Omega_0$. Each interior node p has four neighboring nodes which are the nodes at unit distance from p; the set of four neighbor nodes is called $\mathcal{N}(p)$. Each interior node has all of its neighbors in Ω_0. Each boundary node p has exactly one neighboring node which is the interior node at unit distance from p. An *edge* pq of Ω is the horizontal or vertical line segment which connects a pair of neighboring nodes p and q in int Ω_0, or which connects a boundary node p to its neighboring interior node q. The set of edges is denoted Ω_1. An edge pq, where p is a boundary node and q is its neighboring interior node will be called a boundary edge.

A *network of resistors* is a network $\Omega = (\Omega_0, \Omega_1)$ together with a positive real-valued function γ on Ω_1. For each edge pq in Ω_1, the number $\gamma(pq)$ is called the *conductance* of pq, and $1/\gamma(pq)$ is the *resistance* of pq. The function γ is called the *conductivity*. A function $u : \Omega_0 \to R$ gives a *current* across each conductor pq by *Ohm's law*: $I = \gamma(pq)(u(p) - u(q))$. A function $u : \Omega_0 \to R$ is called γ-*harmonic* if for each interior node p,

$$\sum_{q \in \mathcal{N}(p)} \gamma(pq)(u(q) - u(p)) = 0.$$

This property of a γ-harmonic function, which asserts that the sum of the currents flowing out of each interior node is zero, is *Kirkhoff's law*. If a function ϕ is defined at the boundary nodes, the network Ω will acquire a unique γ-harmonic function u, with $u(p) = \phi(p)$ for each boundary node p. (See [3].) The function u is called the *potential* due to ϕ. The potential drop across each conductor pq is $\Delta u(pq) = u(p) - u(q)$. The function u determines a current $I_\phi(p)$ through each boundary node p, by $I_\phi(p) = \gamma(pq)(u(p) - u(q))$, where q is the interior neighbor of p. For each conductivity γ on Ω_1, the linear map Λ_γ from boundary functions to boundary functions is defined by $\Lambda_\gamma \phi = I_\phi$. The boundary function ϕ is called Dirichlet data, the boundary current I_ϕ is called Neumann data, and the map Λ_γ is called the Dirichlet-to-Neumann map.

The inverse problem is to recover the conductivity γ from Λ_γ. In our situation, this leads to four problems.

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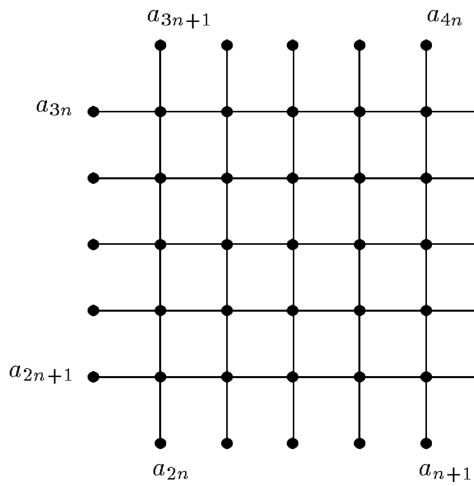

FIG. 1.

(1) Uniqueness: If $\Lambda_\gamma = \Lambda_\mu$, does it necessarily follow that $\gamma = \mu$?

(2) Continuity: If Λ_γ is near to Λ_μ, does it necessarily follow that γ is near to μ?

(3) Reconstruction: Give an algorithm for using Λ_γ to compute γ.

(4) Characterization: For each integer n, which $4n$ by $4n$ matrices are of the form Λ_γ for some γ?

In [3], we showed that the Neumann-to-Dirichlet map uniquely determines the conductivity γ, and we gave an algorithm for computing γ from the Neumann-to-Dirichlet map.

We modify the methods of [3] to show that the Dirichlet-to-Neumann map also uniquely determines the conductivity. In §4, we give an algorithm for computing the conductivity γ from the Dirichlet-to-Neumann map Λ_γ. The algebraic formulas of the algorithm show the continuity of the inverse. In [3] we showed that for square resistor networks with n nodes on each side, the set of Dirichlet-to-Neumann maps forms a manifold of dimension $2n(n+1)$ in the space of $4n$ by $4n$ matrices. In §5, we give a set of $2n(n+1)$ entries of Λ which parametrize this manifold. In §6, we describe by inequalities the domain over which these parameters may vary. The main result of this paper is Theorem 6.1 which gives a characterization of Dirichlet-to-Neumann maps for square resistor networks. By considerations of duality, there is a similar characterization of Neumann-to-Dirichlet maps. Some numerical results based on the reconstruction algorithm of §4 are given in [2]. For related work on the inverse conductivity problem see [1],[4],[5] and [6].

2. Properties of the Dirichlet-to-Neumann Map. Let $\Omega = (\Omega_0, \Omega_1)$ be a square network with n nodes on each side, and let γ be a conductivity function on Ω_1. The Dirichlet-to-Neumann map is represented by a matrix $\Lambda = \{\lambda_{i,j}\}$ in the standard way. We number the boundary nodes $a_1, a_2, \cdots, a_{4n}$, clockwise starting from the upper right corner, as illustrated in Fig. 1 with $n = 5$.

For each index $j = 1, 2, \cdots, 4n$, let ϕ_j be the boundary function which is 1 at node a_j and 0 at all other boundary nodes. Let u_j be the γ-harmonic extension of ϕ_j to all of Ω_0 and let I_{ϕ_j} be the resulting current. The entries $\lambda_{i,j}$ of the matrix Λ are given by

$$\lambda_{i,j} = I_{\phi_j}(a_i).$$

The $4n$ boundary nodes are grouped into four blocks of n nodes each; we denote by N, E, S, and W the nodes on the North, East, South, and West faces, respectively.

LEMMA 2.1. *Let u be a γ-harmonic function on Ω_0, and let p be an interior node. Then either $u(p) = u(q)$ for all nodes $q \in \mathcal{N}(p)$ or else there is at least one node $q \in \mathcal{N}(p)$ for which $u(p) > u(q)$ and there is at least one node $r \in \mathcal{N}(p)$ for which $u(p) < u(r)$.*

Proof. Kirkhoff's law may be rewritten as

$$\left\{ \sum_{q \in \mathcal{N}(p)} \gamma(pq) \right\} u(p) = \sum_{q \in \mathcal{N}(p)} \gamma(pq) u(q).$$

Thus the value of u at each interior node is the weighted average of the values at the neighboring points. □

COROLLARY 2.2. *Let u be a γ-harmonic function on Ω_0. Then the maximum and minimum values of u occur on the boundary of Ω_0.*

COROLLARY 2.3. *Let u be a γ-harmonic function on Ω_0 such that $u(p) = 0$ for all $p \in \partial\Omega_0$. Then $u(p) = 0$ for all $p \in \Omega_0$*

The following lemma provides a way to construct γ-harmonic functions with prescribed data.

LEMMA 2.4. *Let $\Omega = (\Omega_0, \Omega_1)$ be a network with a conductivity γ. Suppose given the boundary values $\phi(p)$ for all nodes p on the N, W and S faces, and suppose given the current $I_\phi(p)$ for all nodes p on the W face. Then there is a unique γ-harmonic function u with this boundary data.*

Proof. For each $j = 0, 1, \cdots, n+1$, let C_j be the jth column of nodes in Ω_0. The values of $u(p)$ and the values of the current $I_\phi(p)$ are given for each node p in C_0. Using Ohm's law, we obtain the values of u for each node in C_1. Kirkhoff's law is a five-point formula, and the positivity of γ implies that each coefficient is nonzero. Thus the value of u at four of the points determines the value at the fifth point. Proceeding from W to E, and using this five-point formula, we obtain the values of u for each node in columns $C_2, C_3, \cdots, C_{n+1}$. □

LEMMA 2.5. *Let $\Omega = (\Omega_0, \Omega_1)$ be a network with a conductivity γ. Suppose that ϕ is a boundary function which is zero at all nodes on the N, E, and S faces, and arbitrary at the nodes on the W face. Suppose that $I_\phi = 0$ at k distinct nodes on the E face. Then either ϕ is identically zero, or there are at least $k+1$ nodes p on the W face for which $\phi(p) \neq 0$. In the latter case, there must be at least k changes of sign among the values of $\phi(p)$ on the W face.*

Proof. Let u be the γ-harmonic extension of ϕ to int Ω_0. Consider first the case $k = n$; that is, $I_\phi(p) = 0$ for all nodes p on the E face. Then Lemma 2.4 implies that $u = 0$, so also $\phi(p) = 0$ for all p. Next suppose that $k < n$, and consider first the case where the k nodes on the E face are contiguous. Suppose that $p_0, p_1, \cdots, p_k, p_{k+1}$ are nodes on the E face with $I_\phi(p_0) \neq 0$, $I_\phi(p_i) = 0$ for $1 \leq i \leq k$ and $I_\phi(p_{k+1}) \neq 0$. For each $i = 1, 2, \cdots, k$, the node p_i has an interior neighbor at which the value of u is zero. These zero values are part of a wedge of zero values as illustrated in Fig. 2.

Each of the zero values on the boundary of the wedge must have a neighbor with positive value and a neighbor with a negative value by Lemma 2.1. For each index $0 \leq i \leq k+1$, let q_i be the first node to the left of p_i for which $u(q_i) \neq 0$. The

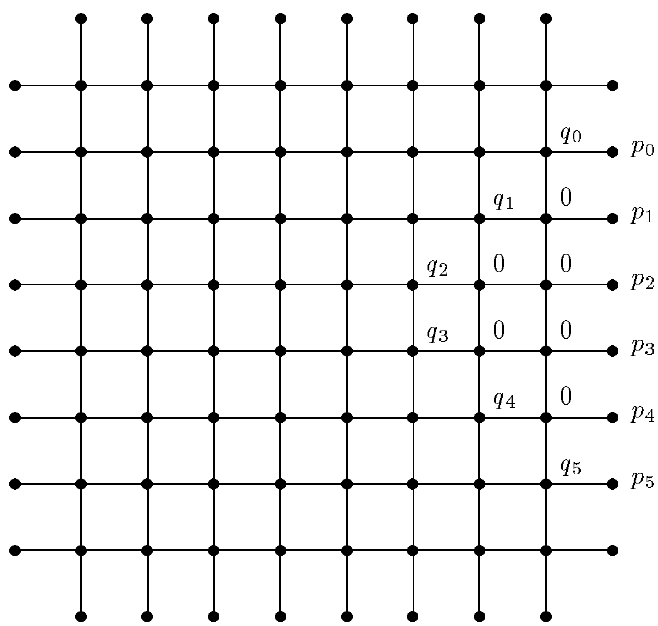

FIG. 2.

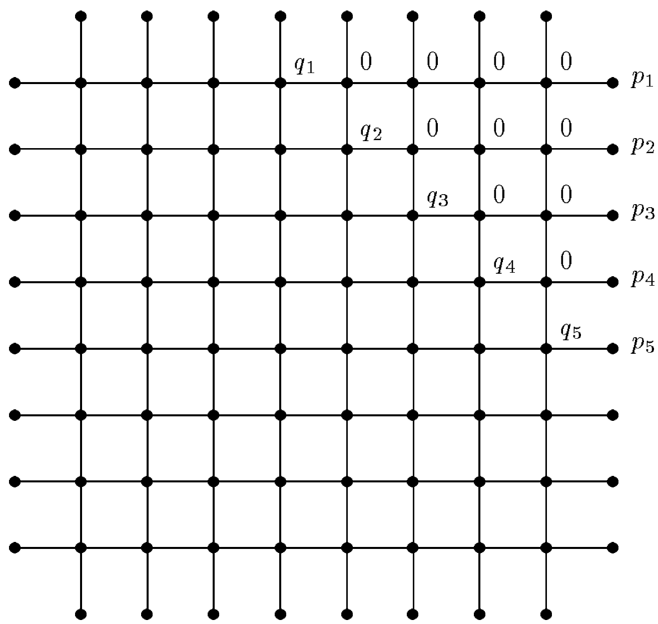

FIG. 3.

sequence of values $u(q_0), u(q_1), \cdots, u(q_{k+1})$ are all nonzero, and there must be at least k sign changes among them. (The middle pair of nodes might have values of the same sign.) By Lemma 2.1, each of these nodes where u is positive has a neighbor where u has even greater positive value. Such a node is connected by a chain of nodes of successively more positive value to a node on the W face of positive value. Each node of negative value is connected by a chain of successively more negative value to a node on the W face of negative value. (If the middle pair of nodes have values of the same sign, they contribute only one chain.) Thus there must be at least $k+1$ nonzero values on the W face, with at least k sign changes.

We must also consider the situation where the nodes with zero current are the k nodes at the top of the E face, as illustrated in Fig. 3. $I_\phi(p_i) = 0$ for $i = 1, \cdots, k$ and $I_\phi(p_{k+1}) \neq 0$.

The sequence of values $u(q_1), \cdots, u(q_{k+1})$ are all nonzero, and there must be at least k sign changes among them. As before, there must be at least $k+1$ nonzero values on the W face, with at least k sign changes. A similar situation occurs where $p_1, \cdots, p_k$ are the last k nodes of the E face.

Finally suppose that there are m noncontiguous sequences of nodes on the E face of lengths $k_1, k_2, \cdots, k_m$, with $\sum k_i = k$, and suppose that $I_\phi = 0$ at each of these nodes. Along each row of the network, going from right to left, there is a first node q with $u(q) \neq 0$. There must be a total of at least k sign changes among these values. Thus there must be at least $k+1$ nonzero values on the W face, with at least k sign

$$\Lambda = \begin{array}{|c|c|c|c|} \hline G & A & B & C \\ \hline A^T & H & E & D \\ \hline B^T & E^T & I & F \\ \hline C^T & D^T & F^T & J \\ \hline \end{array}$$

FIG. 4.

changes. □

The matrix Λ has a block structure as in Fig. 4.

The $n \times n$ matrix B has the following interpretation. Let $v = (v_1, v_2, \cdots, v_n)$. We consider boundary data ϕ, with $\phi(p) = 0$ except on the W face, where $\phi(a_{2n+i}) = v_i$, for $i = 1, 2, \cdots, n$. Then $c = Bv$ is the vector of currents that results on the E face.

THEOREM 2.6. *Let k be an integer with $k \leq n$ and let M be the $k \times k$ matrix formed by choosing any k columns and any k rows from B. Then M is non-singular.*

Proof. First consider the case $k = n$, when M is B itself. Let v be a vector of potentials on the W face, and $c = Bv$ the resulting vector of currents on the E face. Lemma 2.4 implies that if $c = 0$, then $v = 0$ also. Thus B must be nonsingular. Next suppose that $k < n$. Let $v = (v_1, \cdots, v_k)$ be a vector of potentials at the chosen nodes $q_1, \cdots, q_k$ on the W face. Consider $c = Mv$ which is the resulting vector of currents on the E face. If $c = 0$, Lemma 2.5 implies that $v = 0$, also, because there cannot be $k + 1$ sign changes among the k values $v_1, \cdots, v_k$. Thus M must be nonsingular. □

For any integer k let P be the square matrix with nonzero entries only on the diagonal, and $P_{i,i} = (-1)^i$.

DEFINITION 2.7. A $k \times k$ nonsingular matrix M is said to have the *Alternating Property* if the matrix $(-1)^k PM^{-1}P^{-1}$ is totally positive, that is all of its entries are positive.

LEMMA 2.8. *A $k \times k$ nonsingular matrix M has the Alternating Property if and only if the following condition holds. Suppose that $c = Mv$ and that the signs in c alternate. Then the signs in v are the negative of the reversal of the signs in c. That is, If k is even, and the pattern of signs in c is $(-, +, -, +, \cdots, +)$, the pattern of signs in v must also be $(-, +, -, +, \cdots, +)$. If k is odd, and the pattern of signs in c is $(-, +, -, +, \cdots, -)$, the pattern of signs in v must be $(+, -, +, -, \cdots, +)$.*

Proof. The proof is elementary matrix algebra. □

THEOREM 2.9. *Each square submatrix of B has the Alternating Property,*

Proof. Let M be a $k \times k$ submatrix of B. Let $v = (v_1, \cdots, v_k)$ be a vector of potentials at the nodes on the W face corresponding to the chosen columns of M, and let $c = Mv$. Let $p_1, p_2, \cdots, p_k$ be the nodes on the W face corresponding to the choice of the k columns of M, and let $q_1, q_2, \cdots, q_k$ be the nodes on the E face corresponding

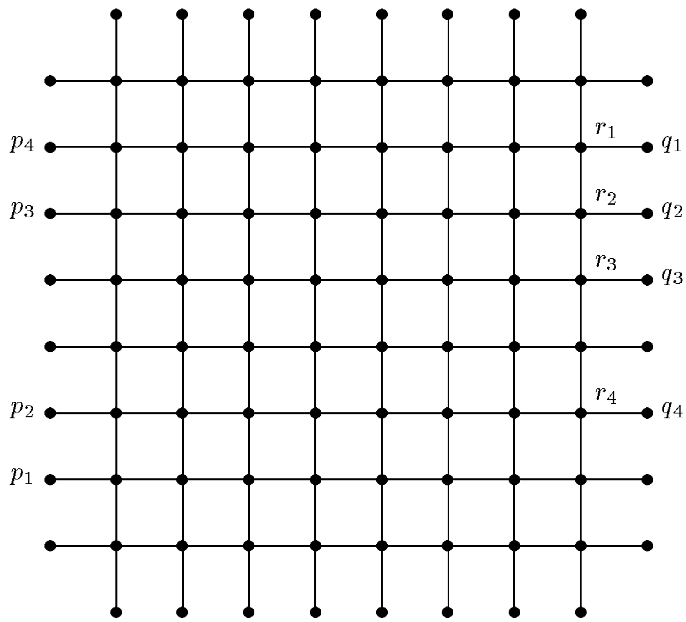

FIG. 5.

to the choice of the k rows of M. For each $i = 1, \cdots, k$, let r_i be the interior neighbor of q_i. Fig. 5 illustrates the situation with $n = 8$ and $k = 4$.
The sign of the potential at r_i must be opposite to the sign of the current c_i. Thus the potential at $r_1, r_2, \cdots, r_k$ must have $k-1$ sign changes. By the proof of Lemma 2.5, each node r_i can be connected by a chain of nodes of the same sign to a node of the same sign on the W face. Thus the potentials at $p_k, p_{k-1}, \cdots, p_2, p_1$ must have the same $k-1$ sign changes as the potentials at $r_1, r_2, \cdots, r_k$. ◻

COROLLARY 2.10. *Suppose* $c = (c_1, c_2, \cdots, c_n) = (-1, 1, -1, 1, \cdots, (-1)^n)$ *is a vector of currents on the E face, and suppose that the potential is zero on the N, E, and S faces. The values of the potential along each row of the network are all of the same sign and are increasing in magnitude from E to W.*

Proof. The argument is the same as for Theorem 2.9. ◻

DEFINITION 2.11. A $k \times k$ matrix M is said to have the *Right Sign*, if

(1) If $k \equiv 1$ or 2 mod 4, then $\det(M) < 0$

(2) If $k \equiv 3$ or 0 mod 4, then $\det(M) > 0$

DEFINITION 2.12. A matrix A is said to have the *Determinant Property* if every square submatrix M of A has the Right Sign.

THEOREM 2.13. *Let* A *be an* $n \times n$ *matrix, and assume that each square submatrix of* A *is nonsingular. The Alternating Property on the submatrices of* A *is equivalent to the Determinantal Property on the submatrices of* A*.*

Proof. Assume that the submatrices of A have the Alternating Property. Let M

be any $k \times k$ submatrix of A and let $c = Mv$. Now consider $Pv = PM^{-1}P^{-1}Pc$. Lemma 2.8 shows that the matrix $(-1)^k PM^{-1}P^{-1}$ must be totally positive, that is all of its entries are positive. Thus the (i,j) entry of M^{-1} has the same sign as $(-1)^{k+i+j}$. For $k = 1$, each entry of A is negative. By induction on k, and Cramer's rule for M^{-1} we find that $\det(M)$ has the Right Sign. The argument may be reversed: if each $k \times k$ sub-matrix M of A has $\det(M)$ of the Right Sign, Cramer's rule implies that the matrix $(-1)^k PM^{-1}P^{-1}$ is totally positive, and so M has the Alternating Property. □

3. Relations in Λ. Let $\Omega = (\Omega_0, \Omega_1)$ be a square network with n nodes on each side, and let γ be a conductivity function on Ω_1. The Dirichlet-to-Neumann map is represented by a matrix $\Lambda = \{\lambda_{i,j}\}$ as in §2.

THEOREM 3.1. *Let k be an integer with $1 \le k \le n$, and take $m = 4n - k + 1$. Then there is a unique set of numbers $\alpha_1, \alpha_2, \cdots, \alpha_k$ such that the relation*

$$\lambda_{i,m} + \sum_{j=1}^{k} \lambda_{i,j}\alpha_j = 0$$

holds for each i with $k < i < m$,

Proof. Let u be the γ-harmonic function with boundary data: $\phi(a_m) = 1$; $\phi(p) = 0$, for all other nodes p on the N, W, and S faces; and the current $I_\phi(p) = 0$ at all nodes p on the W face. Lemma 2.4 implies that u is uniquely determined by these conditions. The proof of Lemma 2.4 shows that the only nonzero boundary values of u are at nodes $a_1, a_2, \cdots, a_k$. Call the values $u(a_i) = \alpha_i$. Thus there is a unique solution $\alpha_1, \alpha_2, \cdots, \alpha_k$ to the overdetermined system. □

Remark 3.2. In the situation of Theorem 3.1, the potential u is zero for all nodes on or below the diagonal connecting a_{m-1} to a_{k+1}. We will make use of this in the next section.

Theorem 3.1 and the remark are illustrated in Fig. 6. The numbers indicate the values of the potential u at the nodes. The diagonal is indicated by ◯'s.

Suppose that the values of $\lambda_{i,j}$ are known for $i = k+1, \cdots, m-1$, $j = 1, \cdots, k$. Let $M = \{m_{i,j}\}$ be the k by k matrix with

$$m_{i,j} = \lambda_{3n-k+i,j}.$$

Then M is nonsingular by Theorem 2.6. Suppose also that for $i = 3n-k+1, \cdots, 3n$, the entries of $\lambda_{i,m}$ are known. The values $\alpha_1\, \alpha_2, \cdots, \alpha_k$ can be found by solving the system. Then the other values of $\lambda_{i,m}$ for $i = k+1$ to $m-1$ are found by substituting into the equations. A similar relation holds for any node in any face, and columns from faces either clockwise or anticlockwise from that node.

LEMMA 3.3. *Let $\Omega = (\Omega_0, \Omega_1)$ be a network with conductivity γ. Let f and g be γ-harmonic functions on Ω_0 with $g = \psi$ and $f = \phi$ on $\partial\Omega_0$, then*

$$\sum_{p \in \partial\Omega_0} \psi(p) I_\phi(p) = \sum_{p \in \partial\Omega_0} \phi(p) I_\psi(p)$$

Proof. The is given in [3]. □

From this, it follows immediately that the matrix Λ is symmetric; that is, $\lambda_{i,j} = \lambda_{j,i}$. Thus there are relations involving the rows of Λ similar to those of Theorem 3.1.

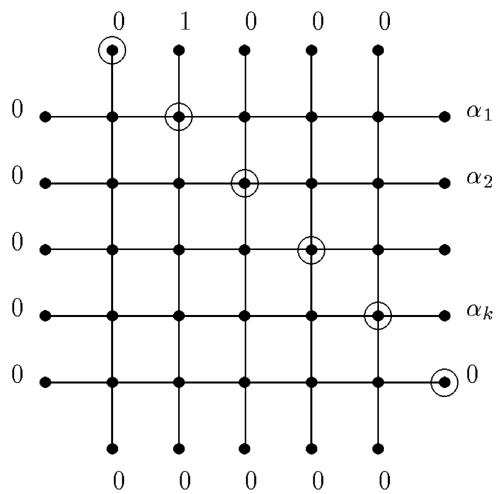

FIG. 6.

Corollary 2.2 implies that if the boundary potential ϕ has value 1 at all boundary nodes, then the potential u has value 1 at all interior nodes, and hence the current $I_\phi = 0$. From this follow the sum relations: for each $j = 1, 2, \cdots, 4n$,

$$\sum_{i=1}^{4n} \lambda_{i,j} = 0.$$

As we shall see in §6, the symmetry relations, the sum relations and the relations given by Theorem 3.1 are the only relations in Λ.

4. An algorithm for computing conductance. We use the results of §3 to give an algorithm for computing γ from Λ_γ. This algorithm follows the same ideas given in [3], and is included here because the presentation is somewhat simpler. The corner conductances are computed first. Refer to Fig. 7, where a_1 and a_{4n} are the two nodes at the NE corner, with interior neighbor q. We want to compute $\gamma(a_{4n}q)$ and $\gamma(a_1 q)$.

Theorem 3.1 shows that there is a value α such that there is a relation

$$\lambda_{i,4n} + \lambda_{i,1}\alpha = 0$$

that holds for every $i = 2$ to $4n-1$. The value of α can be obtained as $-\lambda_{3n,4n}/\lambda_{3n,1}$. Consider the boundary potential ϕ, where $\phi(a_1) = \alpha$, $\phi(a_{4n}) = 1$, and $\phi = 0$ at all other boundary nodes. The resulting potential u is zero at all interior nodes. In particular, $u(q) = 0$. The boundary current I_ϕ is nonzero only at the two nodes a_1 and a_{4n}, with

$$I_\phi(a_1) = \lambda_{1,4n} + \lambda_{1,1}\alpha,$$

$$I_\phi(a_{4n}) = \lambda_{4n,4n} + \lambda_{4n,1}\alpha.$$

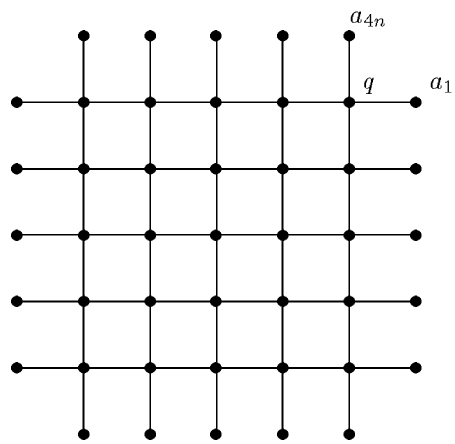

FIG. 7.

The corner conductances are calculated by

$$\gamma(a_{4n}q) = I_\phi(a_{4n}),$$

$$\gamma(a_1 q) = I_\phi(a_1)/\alpha.$$

Inductively, suppose that we have computed all conductances above the ladder which joins node a_{4n-k+1} to node a_k. Refer to Fig. 8, which illustrates the case $n = 5$ and $k = 4$. The conductances to be computed are those indicated by $\times$ along the ladder.

Theorem 3.1 shows that there is a relation:

$$\lambda_{i,4n-k+1} + \sum_{j=1}^{k} \lambda_{i,j}\alpha_j = 0$$

that holds for each $i = k+1, \cdots, 4n-k$. Using these equations for $i = 3n-k+1, \cdots, 3n$, this (overdetermined) system may be solved for $\alpha_1, \alpha_2, \cdots, \alpha_k$. Consider the boundary potential ϕ given by $\phi(a_i) = \alpha_i$ for $1 \le i \le k$, and $\phi(a_{4n-k+1}) = 1$; $\phi = 0$ at all other boundary nodes. The resulting potential u is zero at all nodes on or below the diagonal line joining a_{4n-k} to a_{k+1}, as indicated by the $\bigcirc$'s in Fig. 9.

The values of ϕ and $\Lambda\phi$ are known at all boundary nodes. The value of u at the interior neighbors of $a_{4n-k+2}, \cdots, a_{4n}$ is computed by using Ohm's law and already calculated conductances. Proceeding row by row (downwards), using the value of ϕ at $a_1, a_2, \cdots, a_k$, the conductances already calculated, and Kirkhoff's law, we calculate the value of u at all nodes marked with a $\star$. The current along each boundary conductor, in particular the current along the conductor joining node a_{4n-k+1} to its interior neighbor, is known from $\Lambda\phi$. Going downward and to the right, the current along each conductor in the ladder is computed by using Kirkhoff's law. Thus the value of u is computed at every node, as well as the current along each conductor

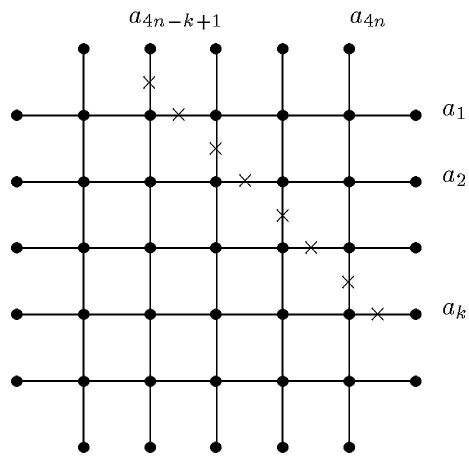

Fig. 8.

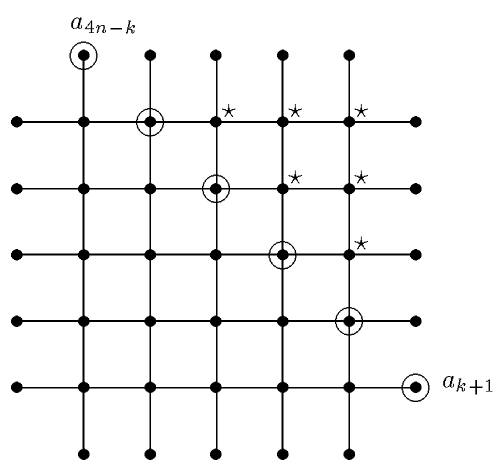

Fig. 9.

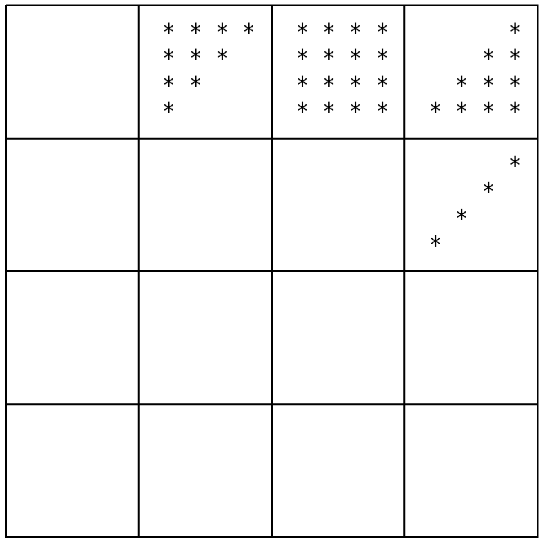

FIG. 10.

in the ladder. From this we calculate the value of each conductor in the ladder. We do this successively for each of the ladders above the main diagonal, and each of the ladders below the main diagonal, and we have computed all the conductances.

5. Parameters for Λ. Let $\Omega = (\Omega_0, \Omega_1)$ be a square network with n nodes on each side, with conductivity γ. The Dirichlet-to-Neumann map is represented by a $4n \times 4n$ matrix $\Lambda = \{\lambda_{i,j}\}$, as in §2. Refer to Fig. 4 for the block structure of the Dirichlet-to-Neumann map. The parameters of Λ we take are the following:

(1) All the entries of B.
(2) All the entries of A on or above the main antidiagonal.
(3) All the entries of C on or below the main antidiagonal.
(4) All the entries of D on the main antidiagonal.

Thus the total number of parameters is

$$n^2 + n(n+1)/2 + n(n+1)/2 + n = 2n(n+1)$$

which is the same as the number of conductors. The positions of these parameters are indicated by the $*$'s in Fig. 10 (illustrated with $n = 4$).

THEOREM 5.1. *Suppose that γ is a conductivity on a square network with n nodes on each side. Then the values of the $2n(n+1)$ parameters of Λ determine uniquely the remaining entries of Λ.*

Proof. For $k = 1, 2, \cdots, n$, consider successively the columns of A from the right. Let $m = 2n + 1 - k$. Consider the system of equations

$$\lambda_{i,m} + \sum_{j=1}^{k} \lambda_{i,2n+j}\alpha_j = 0$$

for $i = 1, \cdots, k$. Let $M = \{m_{i,j}\}$ be the $k \times k$ matrix where $m_{i,j} = \lambda_{i,2n+j}$. Then by Theorem 2.6, M is nonsingular. Thus the system may be solved for $\alpha_1, \alpha_2, \cdots, \alpha_k$. The values of $\lambda_{i,m}$ for $i = k+1$ to n are found by substituting into these equations. A similar argument gives all the entries of C. Using the symmetry relations, we have all the entries of A, B, C, A^T, B^T, C^T. Consider the columns C_{4n} and C_1. There is a relation: for $i = 2, \cdots, 4n-1$,

$$\lambda_{i,1} + \lambda_{i,4n}\beta = 0.$$

The value of β is found from this relation when $i = n+1$. The remaining nondiagonal entries of C_{4n} and C_1 are found by using this relation for other values of i. By the symmetry relations, all non-diagonal entries of R_{4n} and R_1 are determined. Similarly, consider the rows R_n and R_{n+1}. There is a relation: for $j \neq n, n+1$,

$$\lambda_{n,j} + \lambda_{n+1,j}\delta = 0.$$

The value of δ is found from this relation when $j = 4n$. The remaining nondiagonal entries of R_n and R_{n+1} are found by using this relation for other values of j. By the symmetry relations, all nondiagonal entries of C_n and C_{n+1} are determined. Using columns C_{4n+1-j}, C_j, and rows R_{n+j}, R_{n+1-j} for $j = 1, 2, \cdots, n$, and submatrices of A^T and C, we find the nondiagonal entries in these columns and rows. Similarly, the nondiagonal values of block I can be found from blocks B, C, E and F. Finally, the diagonal entries are found from the sum relation: for each $i = 1, 2, \cdots, 4n$,

$$\sum_{j=1}^{4n} \lambda_{i,j} = 0.$$

□

6. Characterization of the Λ_γ. Let $\Omega = (\Omega_0, \Omega_1)$ be a square network with n nodes on each side, and with conductivity function γ. The Dirichlet-to-Neumann map is represented by a matrix $\Lambda = \{\lambda_{i,j}\}$. We have seen that there are relations among the entries of Λ as follows.

(R1) Let k be an integer with $1 \leq k \leq n$, and take $m = 4n - k + 1$. Then there is a unique set of numbers $\alpha_1, \alpha_2, \cdots, \alpha_k$ such that for each i with $k < i < m$,

$$\lambda_{i,m} + \sum_{j=1}^{k} \lambda_{i,j}\alpha_j = 0.$$

A similar relation holds for any node in any face, and columns from faces either clockwise or anticlockwise from that node.

(R2) Λ is symmetric: $\lambda_{i,j} = \lambda_{j,i}$.
Thus, there are relations similar to (R1) involving the rows of Λ.

(R3) For each $i = 1, 2, \cdots, 4n$,

$$\sum_{j=1}^{4n} \lambda_{i,j} = 0.$$

Also, the matrix Λ has the following property.

(DP) Each of the blocks A, B, C, D, E, F, has the Determinant Property (see Definition 2.12).

THEOREM 6.1. *Let Λ be a $4n \times 4n$ matrix whose entries satisfy the relations (R1), (R2), and (R3), and which has Determinant Property. Then there is a unique conductivity function γ such that $\Lambda = \Lambda_\gamma$.*
The proof will follow several lemmas. We define a $4n \times 4n$ matrix Λ to be a λ-matrix if it satisfies the relations (R1), (R2), (R3), and if its parameter values satisfy certain inequalities (see Definition 6.4). We show that if a matrix Λ has relations (R1), (R2), (R3), and property (DP), then it is a λ-matrix. We show (Lemma 6.5) that the set of $4n \times 4n$ λ-matrices is connected. We consider a path of $4n \times 4n$ λ-matrices connecting the λ-matrix corresponding to $\gamma = 1$ to the given λ-matrix Λ. The proof of the theorem will be completed by showing that every matrix on this path must be of the form Λ_γ.

LEMMA 6.2. *Let B_k be a sequence of $n \times n$ matrices with $\lim_{k\to\infty} B_k = B$. Assume that B and each B_k is nonsingular. Let c be a fixed vector, and suppose that $c = B_k v_k$ for each $k = 1, 2, \cdots,$. Then the norms of v_k are bounded.*

Proof. The proof is elementary matrix algebra. □

Let $M = \{m_{i,j}\}$ be a $k \times k$ matrix. For each (i,j), let $M_{i,j}$ be the (i,j)th *minor*, that is, the $k-1 \times k-1$ matrix formed by deleting the ith row and the jth column of M. The expansion of $\det(M)$ by its first column gives

$$\det(M) = \sum_{i=1}^{i=k} (-1)^{i+1} m_{i,1} \det(M_{i,1}).$$

We define functions $f_1, f_2, \cdots$ as follows. f_1 is defined to be the constant 0. For $k \geq 2$,

$$f_k(M) = \sum_{i=1}^{i=k-1} (-1)^{i+k} m_{i,1} \det(M_{i,1}) / \det(M_{k,1}).$$

Observe that $f_k(M)$ is a function of the $k^2 - 1$ entries

$$(m_{1,1}, \cdots, \widehat{m_{k,1}}, \cdots, m_{k,k}),$$

that is, $f_k(M)$ is independent of the entry $m_{k,1}$. $f_k(M)$ is well defined if $\det(M_{k,1}) \neq 0$. There is a similar function $g_k(M)$:

$$g_k(M) = \sum_{i=2}^{i=k} (-1)^{i+1} m_{i,k} \det(M_{i,k}) / \det(M_{1,k}).$$

$g_k(M)$ is a function of the $k^2 - 1$ entries

$$(m_{1,1}, \cdots, \widehat{m_{1,k}}, \cdots, m_{k,k})$$

and is well-defined if $\det(M_{1,k}) \neq 0$. Recall (Definition 2.11) that a $k \times k$ matrix M is said to have the Right Sign (RS) if:

(1) $\det(M) < 0$ for $k \equiv 1, 2 \bmod 4$.

(2) $\det(M) > 0$ for $k \equiv 0, 3 \bmod 4$.

LEMMA 6.3. *Let M be a $k \times k$ matrix such that* $\det(M_{k,1})$ *has the RS. Then if* $m_{k,1} < f_k(M)$, $\det(M)$ *will have the RS also. Similarly, if* $\det(M_{1,k})$ *has the RS and if* $m_{1,k} < g_k(M)$, $\det(M)$ *will be have the RS also.*

Proof. This follows by expanding $\det(M)$ by its first column or last column respectively. □

Now consider the $n \times n$ blocks B, A, C, D of Λ. Order the parameters by

(a) Columns of B, upward from the left,

(b) Columns of A, downward from the right,

(c) Columns of C, upward from the left,

(d) Entries of D, downward to the left along the main antidiagonal.

We define functions $F_j(x_1, \cdots, x_{j-1})$ for $j = 1, 2, \cdots, 2n(n+1)$ as follows. $F_1 = 0$. Suppose inductively that F_j has been defined for $j < k$, and suppose that $x_1, x_2, \cdots, x_{k-1}$ are parameter values such that for each $1 \leq j < k$, $x_j < F_j(x_1, x_2, \cdots, x_{j-1})$. Then $F_k(x_1, \cdots, x_{k-1})$ is defined as follows.

(1) Suppose that $k \leq n^2$; that is, the parameter x_k is in B, say at the (a, b) position of the matrix B. Then let

$$F_k(x_1, \cdots, x_{k-1}) = \min_M \{g_h(M)\}$$

where M varies over all $h \times h$ submatrices of B where $h \leq n - a + 1$, and which have the parameter x_k as the upper right entry.

(2) Suppose that $n^2 < k \leq n^2 + n(n+1)/2$; that is, the parameter x_k is in A, say at the (a, b) position of the matrix A. Use the relations (R1) to fill in the entries in columns $b + 1, \cdots, n$ of A. Let $h = a$ and let M be the $h \times h$ submatrix of A which has the parameter x_k as the lower left entry. Then let

$$F_k(x_1, \cdots, x_{k-1}) = f_h(M).$$

(3) Suppose that $n^2 + n(n+1)/2 < k \leq n^2 + n(n+1)$; that is, the parameter x_k is in C, say at the (a, b) position of the matrix C. Use the relations (R1) to fill in the entries in columns $1, \cdots, b-1$ of C. Let $h = n - a + 1$, and let M be the $h \times h$ submatrix of C which has the parameter x_k as the upper right entry. Then let

$$F_k(x_1, \cdots, x_{k-1}) = g_h(M).$$

(4) Suppose that $n^2 + n(n+1) < k \leq 2n(n+1)$; that is, the parameter x_k is in D, say at the $(a, n-a+1)$ position of the matrix D. Use the relations (R1) to fill in the entries of D above and to the right of $(a, n-a+1)$. Let $h = n - a + 1$, and let M be the $h \times h$ submatrix of D which has the parameter x_k as the lower left entry. Then let

$$F_k(x_1, \cdots, x_{k-1}) = f_h(M).$$

Inductively, we see that for each $k = 1, \cdots, 2n(n+1)$, $F_k(x_1, \cdots, x_{k-1})$ is well-defined.

DEFINITION 6.4. Let S be the set of parameter values $x_1, x_2, \cdots, x_{2n(n+1)}$ such that for each $1 \leq j \leq 2n(n+1)$, $x_j < F_j(x_1, x_2, \cdots, x_{j-1})$.

LEMMA 6.5. *S is path-connected set in $R^{2n(n+1)}$.*

Proof. For each $k = 1, \cdots, 2n(n+1)$, let S_k be the set of parameter values $(x_1, x_2, \cdots, x_k)$ such that for each $1 \leq j \leq k$, $x_j < F_j(x_1, x_2, \cdots, x_{j-1})$. We will show by induction that each S_k is path-connected set in R^k.

$S_1 = \{x_1 : x_1 < 0\}$, and so is path-connected. Assume inductively that S_j is path-connected for $j < k$. Let $(x_1, \cdots, x_{k-1}, x_k)$ and let $(y_1, \cdots, y_{k-1}, y_k)$ be two points in S_k. Take $\beta(t) = (\beta_1(t), \cdots, \beta_{k-1}(t))$ a path in S_{k-1} joining $(x_1, \cdots, x_{k-1})$ and $(y_1, \cdots, y_{k-1})$. Let T be any number less than $\min_t\{F_k(\beta(t))\}$. We have three paths:

(1) The straight line $(x_1, \cdots, x_{k-1}, x_k)$ to $(x_1, \cdots, x_{k-1}, T)$.

(2) $(\beta(t), T)$.

(3) The straight line $(y_1, \cdots, y_{k-1}, T)$ to $(y_1, \cdots, y_{k-1}, y_k)$.

These three paths give a path from $(x_1, \cdots, x_{k-1}, x_k)$ to $(y_1, \cdots, y_{k-1}, y_k)$ in S_k. □

Definition 6.6. A matrix whose entries satisfy the relations (R1), (R2), and (R3) whose parameter values lie in S will be called a λ-matrix. The set of $4n \times 4n$ λ-matrices will be called $L(4n)$.

Lemma 6.7. *Let Λ be a $4n \times 4n$ λ-matrix. Then there is a unique conductivity function γ such that $\Lambda = \Lambda_\gamma$.*

Proof. The set of $4n \times 4n$ matrices of the form Λ_γ will be called $M(4n)$. It follows from Theorem 5.1 of [3], §2, §3, Lemma 6.3 and Invariance of Domain that $M(4n)$ is an open subset of $L(4n)$. Lemma 6.5 implies that $L(4n)$ is a connected set. Let Λ be a λ-matrix and take $\Lambda(0) = \Lambda_\gamma$ where $\gamma(pq) = 1$ for all pq. Let $\Lambda(t)$ for $0 \le t \le 1$, be a path of λ-matrices joining $\Lambda(0)$ with Λ. We will show that each matrix along this path is in $M(4n)$. Suppose the contrary; let t_0 be the first value for which $\Lambda(t_0)$ is not in $M(4n)$. For each conductor pq, we pick a number $\mu(pq)$ which is zero, infinity, or a positive real number and a sequence $\{t_1, t_2, \cdots, t_k, \cdots\}$ with $\lim_{k\to\infty} t_k = t_0$, and such that $\lim_{k\to\infty}\gamma(t_k)(pq) = \mu(pq))$.

We know that $\lim_{k\to\infty}\Lambda(t_k) = \Lambda(t_0)$ and each of these is a λ-matrix. In particular, because of the conditions in Definition 6.4 on the values of the parameters x_j for $j = n^2$ and $j = 2n(n+1)$, each of the blocks B and D of $\Lambda(t_0)$ and the blocks B_k and D_k of $\Lambda(t_k)$ are nonsingular.

(i) Suppose that $\mu(pq) = 0$ for some boundary conductor pq where p s the boundary node a_i. Corresponding to this index i, and any other index j we would have $\mid \lambda_{i,j}(t_k) \mid \le \gamma(t_k)(pq)$ and hence $\lim_{k\to\infty}\Lambda_{i,j}(t_k) = 0$ which would contradict the assumption that $\Lambda(t_0)$ is a λ-matrix. Thus we can assume that there is a positive real number ε such that for each boundary conductor pq and each $k \ge 0$, that $\gamma(t_k)(pq) \ge \varepsilon$.

(ii) Next suppose that $\mu(pq) = \infty$ for some boundary conductor pq. First suppose there are two boundary conductors pq and rq at a corner with $\gamma(t_k)(pq) = X_k$ and $\gamma(t_k)(rq) = Y_k$ and $\lim_{k\to\infty}X_k = \infty$ and $\lim_{k\to\infty}Y_k = \infty$. Consider the Dirichlet problem with 1 at p and zero at all other boundary nodes. A potential $v(t_k)$ results at q. By choosing a subsequence of the $\{t_k\}$, for which the potential $v(t_k)$ has a limit, we would have $X_k(1 - v(t_k)) \to$ a finite limit and $Y_k(0 - v(t_k)) \to$ a finite limit. Then we would have both $v(t_k) \to 0$ and $v(t_k) \to 1$, a contradiction.

From the above, without loss of generality, we can assume that at each corner, at least one conductance is bounded as $k \to \infty$. Suppose that pq is a boundary conductor on the N face such that $\gamma(t_k)(pq) = X_k$, with $\lim_{k\to\infty}X_k = \infty$. Consider Fig. 11.

Assume that the values of $\gamma(t_k)(rr')$ and $\gamma(t_k)(tt')$ are all bounded by X. (If necessary interchange the corner conductors.) Let $c = (-1, 1, -1, \cdots, (-1)^n)$. For each $k = 1, 2, \cdots$, let u_k be the $\gamma(t_k)$-harmonic function on Ω_0 such that $u_k = 0$ on the N, E, and S faces, and with current c on the E face. Then

$$u_k(q) \ge u_k(r') \ge 1/X.$$

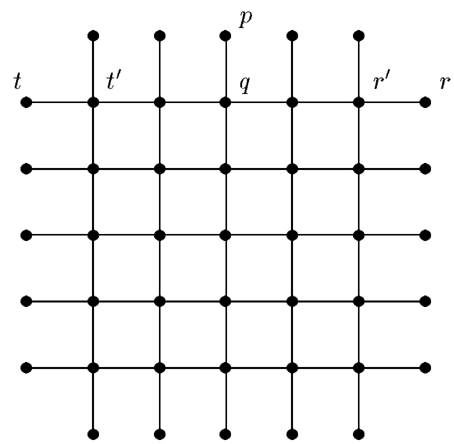

FIG. 11.

Hence the current across pq is $\geq X_k/X$ and the current at t must also be $\geq X_k/X$. Hence

$$u_k(t) \geq X_k/X^2.$$

Then we would have $||c||_\infty = 1$, and $||v_k||_\infty \to \infty$, contradicting Lemma 6.2.
Thus we can assume that for each boundary conductor pq and each $k \geq 0$, that that $\varepsilon \leq \gamma(t_k)(pq) \leq X$.
(iii) Assume that for some interior conductor pq, $\mu(pq) = 0$. without loss of generality, we may assume that pq is horizontal, as in Fig. 12.

Suppose that $\gamma(t_k)(pq) = \varepsilon_k$, where $\lim_{k\to\infty}\varepsilon_k = 0$. Let c and the γ-harmonic functions u_k be as in (ii). Suppose that the current at r is -1 (a similar argument would apply if the current at r is $+1$). Then

$$\varepsilon_k u_k(p) \geq u_k(q) \geq u_k(r') \geq 1/X$$

and

$$u_k(s) \geq u_k(p) \geq 1/X\varepsilon_k.$$

Then we would have $||c||_\infty = 1$, and $||v_k||_\infty \to \infty$, contradicting Lemma 6.2

(iv) Finally, suppose that $\mu(pq) = \infty$ for some interior conductor pq. without loss of generality, we may assume that pq is vertical as in Fig. 13.

Suppose that $\gamma(t_k)(pq) = X_k$, where $\lim_{k\to\infty}X_k = \infty$. Let c and the γ-harmonic functions u_k be as in (2). Again suppose that the current at r is -1. Then

$$u(p) \geq u(r'_1) \geq 1/X.$$

The current across pq would be $\geq X_k/X$ and the current at s would also be $\geq X_k/X$. Then

$$u(s) \geq X_k/X^2.$$

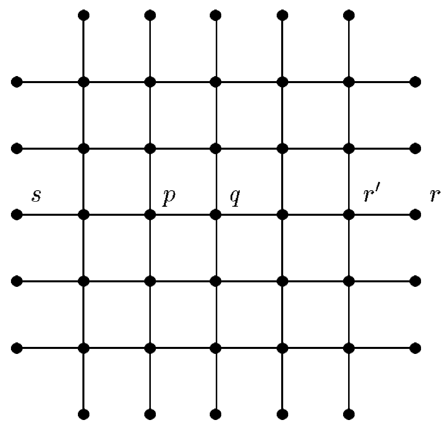

Fig. 12.

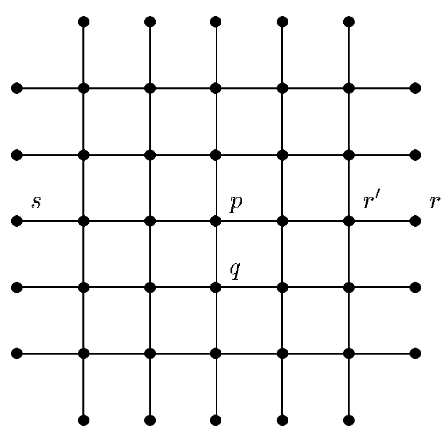

Fig. 13.

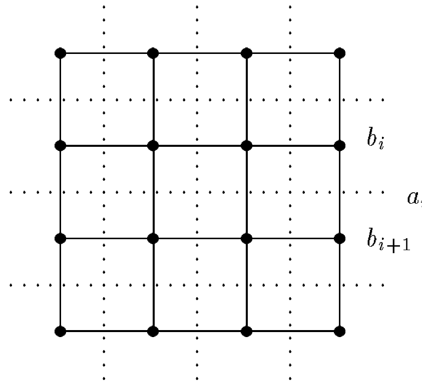

Figure 7.1

FIG. 14.

Then we would have $||c||_\infty = 1$, and $||v_k||_\infty \to \infty$, contradicting Lemma 6.2. □

Proof. of Theorem 6.1. Let Λ be a $4n \times 4n$ matrix which has the relations (R1), (R2), (R3), and the Determinant Property. Then by Lemma 6.3 and Definition 6.4, Λ is a λ-matrix. Lemma 6.7 shows that Λ is of the form Λ_γ. □

7. The Dual Network. For each positive integer n, a square network $\Pi = (\Pi_0, \Pi_1)$, with n edges on each side, is constructed as follows. The nodes of Π are the half-integer lattice points in the plane $p = (i + 1/2, j + 1/2)$ for which $0 \leq i \leq n$ and $0 \leq j \leq n$. The edges of Π are the horizontal and vertical line segments which connect pairs of neighboring nodes p and q in Π.

The network Π is dual to the network Ω of §1, as follows. For each edge α in Ω, let $\alpha^\perp$ be the edge in Π which is the perpendicular bisector of α. For $n = 3$, Fig. 14 shows Π (solid lines), and Ω (dotted lines).

If γ is a conductivity on Ω, the dual conductivity π on Π is defined by $\pi(\alpha^\perp) = \gamma(\alpha)^{-1}$. For each γ - harmonic function u on Ω, let v be the π-harmonic function on Π, where

$$\Delta v(\alpha^\perp) = I_u(\alpha), \;\; I_v(\alpha^\perp) = \Delta u(\alpha).$$

Each boundary node a_i of Ω lies between two boundary nodes of Π, which will be numbered b_i and b_{i+1} (with $b_{4n+1} = b_1$). For each $1 \leq j \leq 4n$, let f_j be the function on the boundary nodes of Π, given by $f_j(b_j) = +1$, $f_j(b_{j+1}) = -1$ and $f_j(b_m) = 0$ for all $m \neq j, j+1$. Let v_j be the π-harmonic function on Π with boundary current f_j. The Neumann-to-Dirichlet map for Π is represented by a matrix Ψ, where

$$\Psi_{i,j} = v_j(b_i) - v_j(b_{i+1}).$$

The matrix Ψ for the Neumann-to-Dirichlet map on the network Π is the *same* as the matrix Λ for the Dirichlet-to-Neumann map on Ω which was constructed in §2. Thus the matrix Ψ has the same properties as Λ.

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