The Ends Justify the Means

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1 Introduction

In the 1780s, Lagrange invented a function that we now refer to as the arithmetic-geometric mean. Here is the definition:

Definition 1. *Fix two nonnegative real numbers a_0 and b_0. Consider the sequences $\{a_n\}$ and $\{b_n\}$, defined by the rule*

$$\begin{cases} a_{n+1} &= \frac{a_n+b_n}{2}, \\ b_{n+1} &= \sqrt{a_n b_n}. \end{cases} \tag{1}$$

If $\{a_n\}$ and $\{b_n\}$ converge to a common limit, then we call this limit the arithmetic-geometric mean of a and b and denote it by $M(a,b)$.

It is an exercise to show that the sequences $\{a_n\}$ and $\{b_n\}$ are convergent to the same limit, so that the arithmetic-geometric mean is well-defined. This will be proven in the next section as Theorem 3.

A few years later, apparently independently, the mathematician Gauss, then 14 years old, re-invented the arithmetic-geometric mean ("AGM"). Gauss apparently then stopped work on the AGM for nearly a decade, until he began experimenting with it in the 1790s. He soon discovered, by numerical experiment, that

$$\frac{1}{\mathrm{M}\left(1,\sqrt{2}\right)} = G \stackrel{\text{def}}{=} \frac{2}{\pi}\int_0^1 \frac{dx}{\sqrt{1-x^4}}. \tag{2}$$

The number G, referred to as *Gauss' constant*, is closely related to the arc length of the lemniscate, a figure-8 curve that is, in turn, closely related to the arc length of the ellipse. Calculating the arc length of the ellipse was an enduring thread of study in mathematics throughout the late 1700s and the first half of the 1800s, largely because the motion of celestial bodies is elliptic. Gauss found this connection so striking as to write that the AGM "would surely open up a whole new field of analysis," and many of the entries in Gauss' well-known diary during the years 1799 and 1800 concerned the AGM [14]. Gauss eventually managed to prove the equation

$$\frac{1}{\mathrm{M}_2(a,b)} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}, \tag{3}$$

relating the AGM with a complete elliptic integral of the first kind. (By modern standards, Gauss never proved this rigorously, though a rigorous proof of (3) was produced by Landen only a few years after Gauss' conjectured formula, by a trigonometric substitution known as the Landen transformation.) Though Gauss completed in roughly 1800 a long paper containing many results on the AGM, this paper was not published until 1866, when a large number of Gauss' unpublished works were published posthumously [13]. A much more detailed exposition of the history behind the AGM is given in [1].

For more than one and a half centuries, AGM theory lay dormant. However, interest in the arithmetic-geometric mean has been rejuvenated: in the early 1980s, a number of numerical analysts, including Jonathan and Peter Borwein [8] [7], Eugene Salamin [16], Richard Brent [9], David Bailey [2], and Yasumasa Kanada [18], introduced the AGM as an extremely effective method for numerically computing elementary functions and constants, such as $\sin x$, $\cos x$, e^x, $\log x$, and π. The fast computation of elementary functions and constants is at the heart of numerical analysis, and AGM-like iterations are emerging as one of the most powerful tools in this field.

The first goal of this thesis is to introduce two methods of studying iterative means that have come to prominence in the last three decades. In §2, we introduce the *theta-function approach*, in which carefully-chosen compositions of special functions are introduced to normalize an iterative mean — that is, to reduce the mean to a much simpler iteration. In §3, we note the fundamental connection between the arithmetic-geometric mean and isogenies of elliptic curves that was first proposed by Jean-Benoît Bost and Jean-François Mestre in [15], and which is developed in greater generality by Eleanor Farrington in [12]. This connection suggests an approach to iterative means from the point of view of algebraic geometry, not analysis; for evidence of this, the reader is directed to [11].

A number of generalizations of the AGM and means related to it have been proposed. Examples include the Schlömilch mean, described in [17], a quadratically-convergent Gaussian iteration defined as the common limit of the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ defined by

$$\begin{cases} a_{n+1} &= \frac{a_n+b_n+c_n}{3}, \\ b_{n+1} &= \sqrt{\frac{a_nb_n+a_nc_n+b_nc_n}{3}}, \\ c_{n+1} &= \sqrt[3]{a_nb_nc_n}, \end{cases} \tag{4}$$

which is the $m=3$ case of the generalized AGM ("gAGM") that will be presented in the next section and which, as is claimed in [8], "does not appear to have a closed form;" and another cubic analogue of the AGM, proposed in [6], defined as the limit of the sequences $\{a_n\}$ and $\{b_n\}$ defined by

$$\begin{cases} a_{n+1} &= \frac{a_n+2b_n}{3}, \\ b_{n+1} &= \sqrt[3]{b_n\left(\frac{a_n^2+a_nb_n+b_n^2}{3}\right)}. \end{cases} \tag{5}$$

In [4], Jonathan and Peter Borwein note that it is precisely because the AGM is a quadratically convergent algebraic iteration with an identifiable nonelementary limit that it occupies such a central role in the computation of elementary functions and constants, so any new iteration related to the AGM is potentially of interest. In the next sections, we will define a family of compound means, indexed by $m \in \mathbb{N}$, in which the $m=2$ case is the AGM and the $m=3$ case is the Schlömilch mean. A different type of generalization is formulated in [10], in which the AGM is extended to be meromorphic on the entire complex plane.

2 The theta-function approach

The foremost problem in studying iterative means is the lack of a unified method for characterizing an iteration in terms of known special functions — or even of determining whether such a characterization exists. One promising avenue is the so-called theta-function approach, which Jonathan and Peter Borwein apply in [8], to give another proof of (3), and in [5], to characterize a certain specialization of the Borchardt mean. The method is, in principle, simple:

- Uniformize the iteration — *i.e.*, make a simplifying change of variable in terms of theta functions (or, more generally, in terms of some known special functions). The difficulty in this step is cobbling together theta-function identities to yield a workable uniformization.
- Produce a functional equation involving the iterative mean by applying this uniformization.
- Characterize the iterative mean in terms of the theta functions used in the first step by applying the inverse function theorem.

This procedure will likely seem vague and murky to the reader, so after deriving the necessary theta-function identities, we illustrate the applications of the theta-function approach mentioned above. Finally, we illustrate an unusual application of the method, in which the first step is reversed: instead of beginning with an iteration and finding the correct identity to produce an appropriate uniformization, Borwein and Borwein begin with an identity and construct an iteration that yields to this identity!

2.1 Some theta-function identities

For a complex variable $z \in \mathbb{C}$ and a real variable $q \in (0,1)$, define three theta functions by

$$\theta_1(z,q) \overset{\text{def}}{=} -\imath \sum_{n=-\infty}^{\infty} (-1)^n q^{\left(n+\frac{1}{2}\right)^2} e^{(2n+1)\imath z}, \qquad \theta_2(z,q) \overset{\text{def}}{=} \sum_{n=-\infty}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} e^{(2n+1)\imath z}, \tag{6}$$
$$\theta_3(z,q) \overset{\text{def}}{=} \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\imath z}, \qquad \theta_4(z,q) \overset{\text{def}}{=} \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2n\imath z}.$$

With one exception, we work only with $\theta_1(0,q)$, $\theta_2(0,q)$, $\theta_3(0,q)$, and $\theta_4(0,q)$; for convenience, these quantities will be denoted by $\theta_1(q)$, $\theta_2(q)$, $\theta_3(q)$, and $\theta_4(q)$, respectively. By the usual complex-analytic arguments, these functions are complex-analytic when treated as functions of z, and real-analytic when treated as functions of q. We now prove some theta-function identities that we will need for the analysis of a number of iterative means.

Identity 1. *For all* $q \in (0,1)$, $\theta_3(q) + \theta_4(q) = 2\theta_3\left(q^4\right)$.

Proof. This is nearly immediate:

$$\theta_3(q) + \theta_4(q) = \sum_{n=-\infty}^{\infty} \left(q^{n^2} + (-1)^n q^{n^2}\right) = 2 \sum_{n=-\infty}^{\infty} q^{(2n)^2} = 2\theta_3\left(q^4\right). \tag{7}$$

$\mathcal{W}$

Identity 2. *For all* $q \in (0,1)$, $\theta_3(q)^2 + \theta_4(q)^2 = 2\theta_3\left(q^2\right)^2$.

Proof. To prove this identity, we use a small amount of number theory. Let $r_2(n)$ denote the number of ways to write a nonnegative integer n as a sum of squares, where we distinguish both sign and permutation (so there are eight different way to write $5 = (\pm 2)^2 + (\pm 1)^2 = (\pm 1)^2 + (\pm 2)^2$ as a sum of squares). Then

$$\theta_3(q)^2 = \sum_{m,n=-\infty}^{\infty} q^{m^2+n^2} = \sum_{n=0}^{\infty} r_2(n) q^n, \qquad \theta_4(q)^2 = \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{m^2+n^2} = \sum_{n=0}^{\infty} (-1)^n r_2(n) q^n. \tag{8}$$

It follows that

$$\theta_3(q)^2 + \theta_4(q)^2 = \sum_{n=-\infty}^{\infty} \left(r_2(n) q^n + (-1)^n r_2(n) q^n\right) = 2 \sum_{n=-\infty}^{\infty} r_2(2n) q^{2n}. \tag{9}$$

To finish the proof, we need an invariance property of r_2.

Lemma 1. *For all nonnegative integers n, $r_2(n) = r_2(2n)$.*

Proof of lemma. Fix some $n \geq 0$. Define A to be the set of all integer pairs (a,b) with $a^2+b^2=n$, and B to be the set of all integer pairs (c,d) with $c^2+d^2=2n$. We define two functions Θ, Φ from A to B and from B to A, respectively. Let Θ send (a,b) to $(a+b,a-b)$; since $(a+b)^2+(a-b)^2 = 2\left(a^2+b^2\right)$, the image of Θ really does lie in B. Let Φ send (c,d) to $\left(\frac{c+d}{2}, \frac{c-d}{2}\right)$. The equation $c^2+d^2=2n$ gives $c \equiv d \bmod 2$, and $\left(\frac{c+d}{2}\right)^2 + \left(\frac{c-d}{2}\right)^2 = \frac{c^2+d^2}{2}$, so the image of Φ really does lie in A. Finally,

$$(\Phi \circ \Theta)(a,b) = \Phi(a+b,a-b) = \left(\frac{(a+b)+(a-b)}{2}, \frac{(a+b)-(a-b)}{2}\right) = (a,b), \tag{10}$$

$$(\Theta \circ \Phi)(c,d) = \Theta\left(\frac{c+d}{2}, \frac{c-d}{2}\right) = \left(\frac{c+d}{2}+\frac{c-d}{2}, \frac{c+d}{2}-\frac{c-d}{2}\right) = (c,d), \tag{11}$$

so A and B are in bijective correspondence. This proves that $r_2(n) = |A|$ is equal to $r_2(2n) = |B|$. $\mathcal{W}$

By this lemma and the work above,

$$\theta_3(q)^2 + \theta_4(q)^2 = 2\sum_{n=-\infty}^{\infty} r_2(2n)q^{2n} = 2\sum_{n=-\infty}^{\infty} r_2(n)q^{2n} = 2\theta_3\left(q^2\right)^2. \tag{12}$$

$\mathcal{W}$

Identity 3. *For all $q \in (0,1)$, $\theta_3(q)\theta_4(q) = \theta_4\left(q^2\right)^2$.*

Proof. This is a result of Identities 1 and 2:

$$\theta_3(q)\theta_4(q) = \frac{1}{2}\left(\theta_3(q)+\theta_4(q)\right)^2 - \frac{1}{2}\left(\theta_3(q)^2+\theta_4(q)^2\right) \overset{1,2}{=} 2\theta_3\left(q^4\right)^2 - \theta_3\left(q^2\right)^2 \overset{2}{=} \theta_4\left(q^2\right)^2. \tag{13}$$

$\mathcal{W}$

Identity 4 ("Jacobi's identity")**.** *For all $q \in (0,1)$, $\theta_4(q)^4 + \theta_2(q)^4 = \theta_3(q)^4$.*

Proof. As in the proof of Identity 2,

$$\begin{aligned}
\theta_3(q)^2 - \theta_3\left(q^2\right)^2 &= \sum_{n=-\infty}^{\infty} r_2(n)q^n - \sum_{n=-\infty}^{\infty} r_2(n)q^{2n} \\
&= \sum_{n=-\infty}^{\infty} r_2(n)q^n - \sum_{n=-\infty}^{\infty} r_2(2n)q^{2n} \\
&= \sum_{n=-\infty}^{\infty} r_2(2n+1)q^{2n+1} \\
&= \sum_{k+l\equiv 1 \bmod 2} q^{k^2+l^2}.
\end{aligned} \tag{14}$$

By setting $i \overset{\text{def}}{=} \frac{l+k-1}{2}$ and $j \overset{\text{def}}{=} \frac{l-k-1}{2}$, we can write k and l as $i-j$ and $i+j+1$, respectively. Moreover, if $(i,j) \neq (i',j')$, then $(i-j, i+j+1) \neq (i'-j', i'+j'+1)$, so by the above work,

$$\theta_3(q)^2 - \theta_3\left(q^2\right)^2 = \sum_{k+l\equiv 1 \bmod 2} q^{k^2+l^2} = \sum_{i,j=-\infty}^{\infty} q^{(i-j)^2+(i+j+1)^2} = \sum_{i,j=-\infty}^{\infty} \left(q^2\right)^{\left(i+\frac{1}{2}\right)^2+\left(j+\frac{1}{2}\right)^2} = \theta_2\left(q^2\right)^2. \tag{15}$$

Upon rearranging, we have

$$\theta_3\left(q^2\right)^2 + \theta_2\left(q^2\right)^2 = \theta_3(q)^2. \tag{16}$$

Multiplying this equation by -1 and replacing the right-hand side via Identity 2 yields $-\theta_3\left(q^2\right)^2-\theta_2\left(q^2\right)^2=\theta_4(q)^2-2\theta_3\left(q^2\right)^2$; rearranging, we have

$$\theta_3\left(q^2\right)^2-\theta_2\left(q^2\right)^2=\theta_4(q)^2. \tag{17}$$

Multiplying these last two displayed equations together and replacing q by $q^{\frac{1}{2}}$ gives $\theta_3(q)^4-\theta_2(q)^4=\theta_3\left(q^{\frac{1}{2}}\right)^2\theta_3\left(q^{\frac{1}{2}}\right)^2$; applying Identity 3 to the right-hand side and rearranging terms finally yields

$$\theta_4(q)^4+\theta_2(q)^4=\theta_3(q)^4. \tag{18}$$

$\mathcal{W}$

Identity 5 ("The cubic modular equation")**.** *For all* $q\in(0,1)$, $\theta_4(q)\theta_4\left(q^3\right)+\theta_2(q)\theta_2\left(q^3\right)=\theta_3(q)\theta_3\left(q^3\right)$.

Proof. We begin by separating even and odd powers in a certain difference of theta functions:

$$\begin{aligned}\theta_3(q)\theta_3\left(q^3\right)-\theta_4(q)\theta_4\left(q^3\right)&=\left(\sum_{k+l\equiv 0 \bmod 2}q^{k^2+3l^2}+\sum_{k+l\equiv 1 \bmod 2}q^{k^2+3l^2}\right)-\left(\sum_{k+l\equiv 0 \bmod 2}q^{k^2+3l^2}-\sum_{k+l\equiv 1 \bmod 2}q^{k^2+3l^2}\right)\\&=2\sum_{i,j=-\infty}^{\infty}q^{(i+j+1)^2+3(i-j)^2}.\end{aligned} \tag{19}$$

Note that $(i+j+1)^2+3(i-j)^2=\left(2i-j+\frac{1}{2}\right)^2+3\left(j+\frac{1}{2}\right)^2$. Every integer pair (m,n) with $m+n\equiv 0 \bmod 2$ can be written in exactly one way as $(2i-j,j)$, and the equivalence $(2i-j)+j\equiv 0 \bmod 2$ is always satisfied, so in fact

$$\theta_3(q)\theta_3\left(q^3\right)-\theta_4(q)\theta_4\left(q^3\right)=2\sum_{i,j=-\infty}^{\infty}q^{(i+j+1)^2+3(i-j)^2}=2\sum_{m+n\equiv 0 \bmod 2}q^{\left(m+\frac{1}{2}\right)^2+\left(n+\frac{1}{2}\right)^2}. \tag{20}$$

Since

$$\begin{aligned}\theta_2(q)\theta_2\left(q^3\right)&=\sum_{m,n=-\infty}^{\infty}q^{\left(m+\frac{1}{2}\right)^2+3\left(n+\frac{1}{2}\right)^2}\\&=\sum_{m+n\equiv 0 \bmod 2}q^{\left(m+\frac{1}{2}\right)^2+3\left(n+\frac{1}{2}\right)^2}+\sum_{m+n\equiv 1 \bmod 2}q^{\left(m+\frac{1}{2}\right)^2+3\left(n+\frac{1}{2}\right)^2}\\&=\sum_{m+n\equiv 0 \bmod 2}q^{\left(m+\frac{1}{2}\right)^2+3\left(n+\frac{1}{2}\right)^2}+\sum_{m+n\equiv 1 \bmod 2}q^{\left((m-1)+\frac{1}{2}\right)^2+3\left(n+\frac{1}{2}\right)^2}\\&=2\sum_{m+n\equiv 0 \bmod 2}q^{\left(m+\frac{1}{2}\right)^2+3\left(n+\frac{1}{2}\right)^2},\end{aligned} \tag{21}$$

we have $\theta_3(q)\theta_3\left(q^3\right)-\theta_4(q)\theta_4\left(q^3\right)=\theta_2(q)\theta_2\left(q^3\right)$; rearranging, we may conclude that

$$\theta_4(q)\theta_4\left(q^3\right)+\theta_2(q)\theta_2\left(q^3\right)=\theta_3(q)\theta_3\left(q^3\right). \tag{22}$$

$\mathcal{W}$

Identity 6 ("The septic modular equation")**.** *For all* $q\in(0,1)$,

$$\sqrt{\theta_4(q)\theta_4\left(q^7\right)}+\sqrt{\theta_2(q)\theta_2\left(q^7\right)}=\sqrt{\theta_3(q)\theta_3\left(q^7\right)}. \tag{23}$$

Proof. $\mathcal{W}$

Identity 7. *Define $K(k)$ for $k \in [0,1)$ by*

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}. \tag{24}$$

Then

$$\frac{2}{\pi} K\left(\frac{\theta_2(q)^2}{\theta_3(q)^2}\right) = \theta_3(q)^2 \tag{25}$$

for all $q \in (0,1)$.

To prove this identity, we will need to consider the functions $\theta_1(z,q)$, $\theta_2(z,q)$, $\theta_3(z,q)$, and $\theta_4(z,q)$, which are dependent on both the argument z and the nome q. For the rest of the section, the dependence of the theta functions on the nome will often be suppressed. From now on, we will let τ, a complex number lying on the ray $[0,+\imath\infty)$, be defined by the equation $q = e^{\pi\imath\tau}$. The four theta functions behave nicely when the argument is translated by one of the periods π and $\pi\tau$, and for this reason they are referred to as quasi doubly periodic functions. For instance,

$$\begin{aligned}
\theta_3(z+\pi,q) &= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\imath(z+\pi)} = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\imath z} = \theta_3(z,q), \\
\theta_3(z+\pi\tau,q) &= \sum_{n=-\infty}^{\infty} e^{n^2\pi\imath\tau} e^{2n\imath(z+\pi\tau)} = e^{-\pi\imath\tau} e^{-2\imath z} \sum_{n=-\infty}^{\infty} e^{(n+1)^2\imath\tau} e^{2(n+1)\imath z} = q^{-1} e^{-2\imath z} \theta_3(z,q).
\end{aligned} \tag{26}$$

We refer to the quantities 1 and $q^{-1}e^{-2\imath z}$ as the two multipliers of θ_3, corresponding to the periods π and $\pi\tau$. Similar calculations produce the following table of multipliers.

	θ_1	θ_2	θ_3	θ_4
π	-1	-1	1	1
$\pi\tau$	$-q^{-1}e^{-2\imath z}$	$q^{-1}e^{-2\imath z}$	$q^{-1}e^{-2\imath z}$	$-q^{-1}e^{-2\imath z}$

Using these quasiperiodicity properties, we prove a lemma about the relationship between the theta functions and their derivatives.

Lemma 2. *If θ is any one of the four theta functions, then for all $z \in \mathbb{C}$,*

$$\frac{\theta'(z+\pi)}{\theta(z+\pi)} = \frac{\theta'(z)}{\theta(z)}, \qquad \frac{\theta'(z+\pi\tau)}{\theta(z+\pi\tau)} = \frac{\theta'(z)}{\theta(z)} - 2\imath. \tag{27}$$

Proof. We prove the claim for θ_1; the proofs for the other functions are nearly identical. For any $z \in \mathbb{C}$,

$$\begin{aligned}
\frac{\theta_1'(z+\pi)}{\theta_1(z+\pi)} &= \frac{-\theta_1'(z)}{-\theta_1(z)} = \frac{\theta_1'(z)}{\theta_1(z)}, \\
\frac{\theta_1'(z+\pi\tau)}{\theta_1(z+\pi\tau)} &= \frac{\left(-q^{-1}e^{-2\imath z}\theta_1(z)\right)'}{-q^{-1}e^{-2\imath z}\theta_1(z)} = \frac{2\imath q^{-1}e^{-2\imath z}\theta_1(z) - q^{-1}e^{-2\imath z}\theta_1'(z)}{-q^{-1}e^{-2\imath z}\theta_1(z)} = \frac{\theta_1'(z)}{\theta_1(z)} - 2\imath.
\end{aligned} \tag{28}$$

$\mathcal{W}$

Rational compositions of theta functions can be formed that have multipliers 1 and 1, in which case these compositions are doubly periodic meromorphic endomorphisms of $\mathbb{C}$, or *elliptic functions*. Elliptic functions have many special properties; we collect some of these properties here. Given an elliptic function with periods $2\omega_1$ and $2\omega_2$, the *cells* of this function are those parallelograms with vertices z ("the base point"), $z+2\omega_1$, $z+2\omega_1+2\omega_2$, and $z+2\omega_2$ for $z\in\mathbb{C}$ and $n,m\in\mathbb{Z}$, with the restriction that poles of f must not lie on the boundary of a cell. Note that every elliptic function has at least one cell, since the period parallelogram with base point 0 can always be perturbed to yield a cell, using the fact that the singularities of a meromorphic function have no accumulation point.

Lemma 3. *Each of the four theta functions has exactly one zero inside each cell.*

Proof. Let θ be one of the theta functions and U be a cell with base point z. It follows from Lemma 2 and the residue calculus that the number of zeroes of f inside U is equal to

$$\begin{aligned}\frac{1}{2\pi\imath}\int_{\partial U}\frac{f'(z)}{f(z)}\,dz &= \frac{1}{2\pi\imath}\left(\int_z^{z+\pi}+\int_{z+\pi}^{z+\pi+\pi\tau}+\int_{z+\pi+\pi\tau}^{z+\pi\tau}+\int_{z+\pi\tau}^{z}\right)\frac{f'(z)}{f(z)}\,dz \\ &= \frac{1}{2\pi\imath}\int_z^{z+\pi}\left(\frac{f'(z)}{f(z)}-\frac{f'(z+\pi\tau)}{f(z+\pi\tau)}\right)dz-\frac{1}{2\pi\imath}\int_z^{z+\pi\tau}\left(\frac{f'(z)}{f(z)}-\frac{f'(z+\pi)}{f(z+\pi)}\right)dz \\ &= \frac{1}{2\pi\imath}\int_z^{z+\pi}2\imath=1.\end{aligned}\tag{29}$$

$\mathcal{W}$

In fact, we can be more explicit about the zeroes of the four theta functions. Note that

$$\begin{aligned}\theta_1(0,q) &= -\imath\sum_{n=-\infty}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2} \\ &= -\imath\sum_{n=-\infty}^{-1}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}-\imath\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2} \\ &= -\imath\sum_{m=0}^{\infty}(-1)^{-m-1} q^{\left((-m-1)+\frac{1}{2}\right)^2}-\imath\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}=0,\end{aligned}\tag{30}$$

and that, as an easy computation shows,

$$\theta_1(z)=-\theta_2\left(z+\frac{1}{2}\pi\right)=-\imath q^{\frac{1}{4}}e^{\imath z}\theta_3\left(z+\frac{1}{2}\pi+\frac{1}{2}\pi\tau\right)=-\imath q^{\frac{1}{4}}e^{\imath z}\theta_4\left(z+\frac{1}{2}\pi\tau\right).\tag{31}$$

It follows from the last lemma that the zeroes of θ_1, θ_2, θ_3, and θ_4 are, respectively, those points congruent to 0, $\frac{1}{2}\pi$, $\frac{1}{2}\pi+\frac{1}{2}\pi\tau$, and $\frac{1}{2}\pi\tau$ modulo the periods π and $\pi\tau$.

Corollary 1. *On the real line, the zeros of $\theta_1(u,q)$ are the points $n\pi, n\in\mathbb{Z}$ and the zeros of $\theta_2(u,q)$ are the points $\left(n+\frac{1}{2}\right)\pi, n\in\mathbb{Z}$. The theta functions $\theta_3(u,q)$ and $\theta_4(u,q)$ have no real zeros. For every $q\in(0,1)$, $\theta_1(q)$ is zero, while $\theta_2(q)$, $\theta_3(q)$, and $\theta_4(q)$ are strictly positive.*

Proof of corollary. The characterization of the zeroes of the theta functions is an immediate consequence of the previous paragraph. To see that $\theta_1(q)=0$, simply note that

$$\begin{aligned}\theta_1(z,q) &= -\imath\sum_{n=-\infty}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}e^{(2n+1)\imath z} \\ &= -\imath\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}e^{(2n+1)\imath z}-\imath\sum_{n=-\infty}^{-1}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}e^{(2n+1)\imath z} \\ &= -\imath\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}e^{(2n+1)\imath z}+\imath\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}e^{-(2n+1)\imath z} \\ &= 2\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}\sin((2n+1)z).\end{aligned}\tag{32}$$

That $\theta_2(q)$ and $\theta_3(q)$ are strictly positive is obvious. Since $\theta_4(q)$ has no zeros, is continuous on $[0,1)$, and satisfies $\theta_4(0) = 1$, $\theta_4(q)$ is positive on $(0,1)$. $\mathcal{W}$

Lemma 4. *The sum of the resides of an elliptic function $f(z)$ at its poles in any one cell is zero.*

Proof of lemma. Fix a cell with base point z; call its boundary C, and assign to C the usual orientation. By Cauchy's theorem, the sum of the residues of f inside C is equal to

$$\begin{aligned}\frac{1}{2\pi\imath}\int_C f(z)\,dz &= \frac{1}{2\pi\imath}\left(\int_z^{z+2\omega_1} + \int_{z+2\omega_1}^{z+2\omega_1+2\omega_2} + \int_{z+2\omega_1+2\omega_2}^{z+2\omega_2} + \int_{z+2\omega_2}^{z}\right) f(z)\,dz \\ &= \frac{1}{2\pi\imath}\int_z^{z+2\omega_1}\left(f(z) - f(z+2\omega_2)\right)dz - \frac{1}{2\pi\imath}\int_z^{z+2\omega_2}\left(f(z) - f(z+2\omega_1)\right) \\ &= 0.\end{aligned} \tag{33}$$

$\mathcal{W}$

Lemma 5. *If $f(z)$ is an elliptic function with the property that for some cell U, f is bounded in modulus on $\overline{U}$, then f is constant on $\mathbb{C}$.*

Proof. This is an immediate (but useful!) consequence of Liouville's theorem. $\mathcal{W}$

Corollary 2. *Say that $f(z)$ is an elliptic function with the property that for some cell U has at most one pole within ∂U, counting multiplicity. Then $f(z)$ is constant.*

Our first application of these general properties of elliptic functions is the derivation of several relations among the squares of the four theta functions. The functions $\theta_1(z)^2$, $\theta_2(z)^2$, $\theta_3(z)^2$, and $\theta_4(z)^2$ are quasi doubly periodic functions with periods π and $\pi\tau$ and corresponding multipliers 1 and $q^{-2}e^{-4\imath z}$, so for all $a,b,c,d \in \mathbb{C}$, the quotients

$$\frac{a\theta_1(z)^2 + b\theta_4(z)^2}{\theta_2(z)^2}, \qquad \frac{c\theta_1(z)^2 + d\theta_4(z)^2}{\theta_3(z)^2} \tag{34}$$

are elliptic functions. Moreover, by Lemma 3, in every cell the denominators have exactly two zeros, counting multiplicity. It follows that for the correct choice of a,b,c,d, with at least one of a,b and and least one of c,d nonzero, these quotients will be elliptic functions with at most one zero in each cell, thus, by Corollary 2, constant.

The multipliers of $\theta_1(z)$ and of $\theta_4(z)$ are different, so these two theta functions are not linearly dependent over $\mathbb{C}$. It follows that $a\theta_1(z)^2 + b\theta_4(z)^2$ and $c\theta_1(z)^2 + d\theta_4(z)^2$, for $(a,b),(c,d)$ nonzero, are not identically zero, so the quotients displayed above are each equal to a nonzero constant. It follows that there exist relations of the form

$$\theta_2(z)^2 = a\theta_1(z)^2 + b\theta_4(z)^2, \qquad \theta_3(z)^2 = c\theta_1(z)^2 + d\theta_4(z)^2. \tag{35}$$

Straightforward manipulations yield the identities

$$\theta_1\left(\frac{1}{2}\pi\tau\right) = \imath q^{-\frac{1}{4}}\theta_4(0), \qquad \theta_2\left(\frac{1}{2}\pi\tau\right) = q^{-\frac{1}{4}}\theta_3(0), \qquad \theta_3\left(\frac{1}{2}\pi\tau\right) = q^{-\frac{1}{4}}\theta_2(0), \qquad \theta_4\left(\frac{1}{2}\pi\tau\right) = 0. \tag{36}$$

Setting $z = \frac{1}{2}\pi\tau$ in (35) and applying these formulas yields

$$a = -\frac{\theta_3(0)^2}{\theta_4(0)^2}, \qquad c = -\frac{\theta_2(0)^2}{\theta_4(0)^2}. \tag{37}$$

Setting $z = 0$ (35) and noting that $\theta_1(0) = 0$ yields

$$b = \frac{\theta_2(0)^2}{\theta_4(0)^2}, \qquad d = \frac{\theta_3(0)^2}{\theta_4(0)^2}. \tag{38}$$

We have now derived the equations

$$\theta_4(0)^2\theta_2(z)^2 = \theta_2(0)^2\theta_4(z)^2 - \theta_3(0)^2\theta_1(z)^2, \qquad \theta_4(0)^2\theta_3(z)^2 = \theta_3(0)^2\theta_4(z)^2 - \theta_2(0)^2\theta_1(z)^2. \tag{39}$$

Next, we use these relations to construct a solution of a differential equation that is intimately related to the elliptic integral K. It follows from the table of multipliers recorded above that the function $\theta_1(z)/\theta_4(z)$ has multipliers -1 and 1 with respect to the periods π and $\pi\tau$; the derivative of this quotient must, therefore, have the same periodicity factors. The quotient $\theta_2(z)\theta_3(z)/\theta_4(z)^2$ has the same multipliers, so if

$$\phi(z) \stackrel{\text{def}}{=} \frac{\theta_1'(z)\theta_4(z) - \theta_1(z)\theta_4'(z)}{\theta_2(z)\theta_3(z)}, \tag{40}$$

then $\phi(z)$ is an elliptic function. Moreover applying the identities

$$\begin{aligned} \theta_1\left(z+\frac{1}{2}\pi\tau\right) &= \imath q^{-\frac{1}{4}}e^{-\imath z}\theta_4(z), \quad & \theta_2\left(z+\frac{1}{2}\pi\tau\right) &= q^{-\frac{1}{4}}e^{-\imath z}\theta_3(z), \\ \theta_3\left(z+\frac{1}{2}\pi\tau\right) &= q^{-\frac{1}{4}}e^{-\imath z}\theta_2(z), \quad & \theta_4\left(z+\frac{1}{2}\pi\tau\right) &= \imath q^{-\frac{1}{4}}e^{-\imath z}\theta_1(z) \end{aligned} \tag{41}$$

and the consequent identities

$$\begin{aligned} \theta_1'\left(z+\frac{1}{2}\pi\tau\right) &= q^{-\frac{1}{4}}e^{-\imath z}\left(\theta_4(z)+\imath\theta_4'(z)\right), \quad & \theta_2'\left(z+\frac{1}{2}\pi\tau\right) &= q^{-\frac{1}{4}}e^{-\imath z}\left(-\imath\theta_3(z)+\theta_3'(z)\right), \\ \theta_3'\left(z+\frac{1}{2}\pi\tau\right) &= q^{-\frac{1}{4}}e^{-\imath z}\left(-\imath\theta_2(z)+\theta_2'(z)\right), \quad & \theta_4'\left(z+\frac{1}{2}\pi\tau\right) &= q^{-\frac{1}{4}}e^{-\imath z}\left(\theta_1(z)+\imath\theta_1'(z)\right) \end{aligned} \tag{42}$$

yields

$$\phi\left(z+\frac{1}{2}\pi\tau\right) = \phi(z). \tag{43}$$

On the other hand, we know from the work immediately following Lemma 3 that the only possible poles of $\phi(z)$ are simple poles at points congruent to $\frac{1}{2}\pi$ and $\frac{1}{2}\pi+\frac{1}{2}\pi\tau$ modulo π and $\pi\tau$. It follows that $\phi(z)$ is periodic with periods π and $\frac{1}{2}\pi\tau$, and that the only possible poles of $\phi(z)$ are simple poles at points congruent to $\frac{1}{2}\pi$ modulo $\frac{1}{2}\pi$. We may conclude from Lemma 2 that $\phi(z)$ is constant. Since $\theta_1(0)=0$, this constant must be equal to $\theta_1'(0,q)\theta_4(0,q)/\theta_2(0,q)\theta_3(0,q)$. We prove another identity, another remarkable result of Jacobi, in order to work this constant into a simple form.

Identity 8. *The equation*

$$\theta_1'(0,q) = \theta_2(0,q)\theta_3(0,q)\theta_4(0,q) \tag{44}$$

holds.

Proof. Soon to come! $\mathcal{W}$

It follows that $\phi(z)$ is identically equal to $\theta_4(0,q)^2$. We may conclude that

$$\frac{d}{dz}\left(\frac{\theta_1(z)}{\theta_4(z)}\right) = \theta_4(0,q)^2\frac{\theta_2(z,q)}{\theta_4(z,q)}\frac{\theta_3(z,q)}{\theta_4(z,q)}. \tag{45}$$

An application of (39) yields the differential equation

$$\left(\frac{d}{dz}\frac{\theta_1(z,q)}{\theta_4(z,q)}\right)^2 = \left(\theta_2(0)^2-\theta_3(0)^2\left(\frac{\theta_1(z)}{\theta_4(z)}\right)^2\right)\left(\theta_3(0)^2-\theta_2(0)^2\left(\frac{\theta_1(z)}{\theta_4(z)}\right)^2\right). \tag{46}$$

This equation can be put into a cleaner form by the obvious change of variables. Indeed, we see that if we define

$$\operatorname{sn}(u,q) \stackrel{\text{def}}{=} \frac{\theta_3(0,q)}{\theta_2(0,q)} \frac{\theta_1\left(\frac{u}{\theta_3(0,q)^2},q\right)}{\theta_4\left(\frac{u}{\theta_3(0,q)^2},q\right)}, \tag{47}$$

then

$$\left(\frac{d}{du}\operatorname{sn}(u,q)\right)^2 = \left(1-\operatorname{sn}(u,q)^2\right)\left(1-\frac{\theta_2(0,q)^4}{\theta_3(0,q)^4}\cdot\operatorname{sn}(u,q)^2\right). \tag{48}$$

Our notation here is not entirely standard: most authors write the Jacobi sine function $\operatorname{sn}(u,k)$ as a function with parameter $k \stackrel{\text{def}}{=} \theta_2(0,q)^2/\theta_3(0,q)^2$. For explicitness, we regard $\operatorname{sn}(u,q)$ as a function with the nome q as its parameter.

As the last step in our preparation for the proof of Identity 7, we prove that the Jacobi sine function inverts the incomplete elliptic integral of the first kind.

Proposition 1. *For all* $0 \le u < \frac{1}{2}\pi\theta_3(q)^2$,

$$u = \int_0^{sn(u,q)} \frac{dt}{\left((1-t^2)\left(1-\frac{\theta_2(q)^4}{\theta_3(q)^4}t^2\right)\right)^{\frac{1}{2}}}. \tag{49}$$

Proof. This is a direct consequence of (48). We begin by proving that $\operatorname{sn}(u,q)$ maps $\left[0,\frac{1}{2}\pi\theta_3(q)^2\right]$ onto $[0,1]$, and that $\frac{d}{du}\operatorname{sn}(u,q)$ does not vanish on the first interval, except at the right endpoint. By (45),

$$\frac{d}{du}\operatorname{sn}(u,q) = \frac{d}{du}\left(\frac{\theta_3(q)}{\theta_2(q)}\frac{\theta_1\left(\frac{u}{\theta_3(q)^2},q\right)}{\theta_4\left(\frac{u}{\theta_3(q)^2},q\right)}\right) = \frac{\theta_2\left(\frac{u}{\theta_3(q)^2},q\right)\theta_3\left(\frac{u}{\theta_3(q)^2},q\right)\theta_4(q)^2}{\theta_2(q)\theta_3(q)\theta_4\left(\frac{u}{\theta_3(q)^2},q\right)^2}. \tag{50}$$

It follows from Corollary 1 that $\frac{d}{du}\operatorname{sn}(u,q)$ is continuous and nonzero on $\left[0,\frac{1}{2}\pi\theta_3(q)^2\right)$. Moreover, $\frac{d}{du}\operatorname{sn}(u,q)$ is obviously equal to 1 at 0, so by the intermediate value theorem, $\frac{d}{du}\operatorname{sn}(u,q)$ is strictly positive on $\left[0,\frac{1}{2}\pi\theta_3(q)^2\right)$. Similarly to (36), the identities

$$\begin{aligned} \theta_1\left(z+\frac{1}{2}\pi,q\right) &= \theta_2(z,q), & \theta_2\left(z+\frac{1}{2}\pi,q\right) &= -\theta_1(z,q), \\ \theta_3\left(z+\frac{1}{2}\pi,q\right) &= \theta_4(z,q), & \theta_4\left(z+\frac{1}{2}\pi,q\right) &= \theta_3(z,q) \end{aligned} \tag{51}$$

follow from a short computation, so $\operatorname{sn}\left(\frac{1}{2}\pi\theta_3(q)^2,q\right)=1$. Since $\theta_1(0,q)=0$, $\operatorname{sn}(0,q)=0$. This shows that on the interval $\left[0,\frac{1}{2}\pi\theta_3(q)^2\right]$, $\operatorname{sn}(u,q)$ increases from 0 to 1, and has positive derivative everywhere on this interval but at the right endpoint.

It follows from this work, along with (48), that

$$\frac{d}{du}\operatorname{sn}(u,q) = \left(\left(1-\operatorname{sn}(u,q)^2\right)\left(1-\frac{\theta_2(q)^4}{\theta_3(q)^4}t^2\right)\right)^{\frac{1}{2}} \tag{52}$$

for all $u \in \left[0,\frac{1}{2}\theta_3(q)^2\right)$, where the right-hand side is, by (48), the square root of a nonnegative quantity, and is, by the previous paragraph, to be interpreted as a nonnegative quantity. It follows that

$$\frac{d}{du}\int_0^{\operatorname{sn}(u,q)} \frac{dt}{\left((1-t^2)\left(1-\frac{\theta_2(q)^4}{\theta_3(q)^4}t^2\right)\right)^{\frac{1}{2}}} = \left(\frac{d}{du}\operatorname{sn}(u,q)\right)\left(\frac{1}{\left(\left(1-\operatorname{sn}(u,q)^2\right)\left(1-\frac{\theta_2(q)^4}{\theta_3(q)^4}\operatorname{sn}(u,q)^2\right)\right)^{\frac{1}{2}}}\right) = 1. \tag{53}$$

Since $\operatorname{sn}(u,q)=1$, we may conclude that for all $u\in\left[0,\frac{1}{2}\pi\theta_3(q)^2\right)$,

$$u=\int_0^{\operatorname{sn}(u,q)}\frac{dt}{\left((1-t^2)\left(1-\frac{\theta_2(q)^4}{\theta_3(q)^4}t^2\right)\right)^{\frac{1}{2}}}. \tag{54}$$

$\mathcal{W}$

By Proposition 1,

$$\begin{aligned}\frac{1}{2}\pi\theta_3(q)^2&=\lim_{u\to\frac{1}{2}\pi\theta_3(q)^{2^-}}u \\ &=\lim_{u\to\frac{1}{2}\pi\theta_3(q)^{2^-}}\int_0^{\operatorname{sn}(u,q)^2}\frac{dt}{\left((1-t^2)\left(1-\frac{\theta_2(q)^4}{\theta_3(q)^4}t^2\right)\right)^{\frac{1}{2}}} \\ &=\lim_{y\to1^-}\int_0^{y}\frac{dt}{\left((1-t^2)\left(1-\frac{\theta_2(q)^4}{\theta_3(q)^4}t^2\right)\right)^{\frac{1}{2}}} \\ &=\int_0^{1}\frac{dt}{\left((1-t^2)\left(1-\frac{\theta_2(q)^4}{\theta_3(q)^4}t^2\right)\right)^{\frac{1}{2}}}.\end{aligned} \tag{55}$$

A change of variables shows that

$$\int_0^{1}\frac{dt}{\left((1-t^2)\left(1-\frac{\theta_2(q)^4}{\theta_3(q)^4}t^2\right)\right)^{\frac{1}{2}}}=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\left(1-\frac{\theta_2(q)^4}{\theta_3(q)^4}\sin^2\theta\right)^{\frac{1}{2}}}, \tag{56}$$

so we have now proven Identity 7.

2.2 The arithmetic-geometric mean

Recall that the AGM iteration is defined by

$$\begin{cases} a_{n+1} &= \frac{a_n+b_n}{2}, \\ b_{n+1} &= \sqrt{a_nb_n}. \end{cases} \tag{57}$$

Denote the limit of the AGM iteration with initial inputs $a,b>0$ by $\mathrm{M}(a,b)$. It follows from Identities 2 and 3 that

$$\frac{\theta_3(q)^2+\theta_4(q)^2}{2}=\theta_3\left(q^2\right)^2, \qquad \sqrt{\theta_3(q)^2\theta_4(q)^2}=\theta_4\left(q^2\right)^2 \tag{58}$$

for all $q\in(0,1)$, so for all such q, $\mathrm{M}\left(\theta_3(q)^2,\theta_4(q)^2\right)=\mathrm{M}\left(\theta_3\left(q^2\right)^2,\theta_4\left(q^2\right)^2\right)$. Applying this rule n times, it follows that $\mathrm{M}\left(\theta_3(q)^2,\theta_4(q)^2\right)=\mathrm{M}\left(\theta_3\left(q^{2^n}\right)^2,\theta_4\left(q^{2^n}\right)^2\right)$. Since $\theta_3(0)=\theta_4(0)=1$, it follows from the homogeneity of M that for all $q\in[0,1)$,

$$\mathrm{M}\left(1,\frac{\theta_4(q)^2}{\theta_3(q)^2}\right)=\frac{1}{\theta_3(q)^2}. \tag{59}$$

By Corollary 1, $\theta_2(q)$, $\theta_3(q)$, and $\theta_4(q)$ are positive, so it follows from this equation, along with Identity 4, that

$$\mathrm{M}\left(1,\frac{\theta_4(q)^2}{\theta_3(q)^2}\right)=\frac{1}{\theta_3(q)^2}=\frac{\pi}{2K\left(\sqrt{1-\frac{\theta_4(q)^4}{\theta_3(q)^4}}\right)}. \tag{60}$$

I claim that as q ranges from 0 to 1, $\frac{\theta_4(q)^2}{\theta_3(q)^2}$ parameterizes the interval $(0,1]$. To see this, first note that by Corollary 1, $0 \leq \theta_4(q) \leq \theta_3(q)$, so $\frac{\theta_4(q)^2}{\theta_3(q)^2}([0,1)) \subset (0,1]$. Since $\theta_4(0) = \theta_3(0) = 1$, to prove equality in this containment it suffices to show that as q tends to 1, $\frac{\theta_4(q)^2}{\theta_3(q)^2}$ tends to 0. It is apparent that $\lim_{q\to 1^-} \theta_3(q) = \lim_{q\to 1^-} \sum_{n\in\mathbb{Z}} q^{n^2} = +\infty$, so it suffices to prove that $\theta_4(q)$ is bounded on $[0,1)$. This follows from the inequality

$$\theta_4(q) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} = 1 + 2\sum_{n\in\mathbb{N}} q^{(2n)^2} - 2\sum_{n\in\mathbb{N}} q^{(2n-1)^2} \leq 1. \tag{61}$$

We may conclude that for all $x \in [0,1]$,

$$\mathrm{M}(1,x) = \frac{\pi}{2K\left(\sqrt{1-x^2}\right)}. \tag{62}$$

This is the fundamental integral formula of Gauss.

2.3 Another specialization of the Borchardt mean

In [3], Carl Borchardt proposed the four-term iteration defined by the rule

$$\begin{cases} a_{n+1} &= \frac{a_n+b_n+c_n+d_n}{4}, \\ b_{n+1} &= \frac{\sqrt{a_nb_n}+\sqrt{c_nd_n}}{2}, \\ c_{n+1} &= \frac{\sqrt{a_nc_n}+\sqrt{b_nd_n}}{2}, \\ d_{n+1} &= \frac{\sqrt{a_nd_n}+\sqrt{b_nc_n}}{2}. \end{cases} \tag{63}$$

This mean can be characterized in terms of a hyperelliptic integral, and we will proceed with the analysis in the next subsection. To motivate the study of this mean, note that when $a_0 = b_0$ and $c_0 = d_0$, the Borchardt iteration reduces to the AGM iteration. When, on the other hand, $b_0 = c_0 = d_0$, the Borchardt iteration reduces to the iteration

$$\begin{cases} a_{n+1} &= \frac{a_n+3b_n}{4}, \\ b_{n+1} &= \frac{\sqrt{a_nb_n}+b_n}{2}. \end{cases} \tag{64}$$

Following the notation of [5], we denote the limit of (64) by $B(a_0,b_0)$ and, abusing notation, define $B(x)$ to be $B(1,x)$. Because of its asymptotic behavior, (64) is not a convenient iteration to uniformize. Instead, we begin with the iteration

$$\begin{cases} a_{n+1} &= \left(\frac{a_n^2+3b_n^2}{4}\right)^{\frac{1}{2}}, \\ b_{n+1} &= \left(\frac{a_nb_n+b_n^2}{2}\right)^{\frac{1}{2}}, \end{cases} \tag{65}$$

the limit of which we denote by $B_2(a_0,b_0)$. As for $B(a_0,b_0)$, we define $B_2(x)$ to be $B_2(1,x)$. Observe that $B_2(x) = B\left(x^2\right)^{\frac{1}{2}}$: we have modified the asymptotic behavior of $B(x)$ for small x while preserving homogeneity.

We now characterize $B(x)$.

Theorem 1. *For* $0 < h \leq 1$,

$$B(h) = \frac{\pi^2}{16\left(1+\sqrt{h}\right)^2 K\left(\frac{\sqrt{1-h}}{(1+\sqrt{h})^2}\left(\sqrt{1+3h}+2\sqrt{h}\right)\right) K\left(\frac{\sqrt{1-h}}{(1+\sqrt{h})^2}\left(\sqrt{1+3h}-2\sqrt{h}\right)\right)}. \tag{66}$$

Proof. We begin by using the results of Subsection 2.1 to prove that

$$L(q) \stackrel{\text{def}}{=} \theta_3(q)\theta_3\left(q^3\right) + \theta_2(q)\theta_2\left(q^3\right), \qquad M \stackrel{\text{def}}{=} \theta_3(q)\theta_3\left(q^3\right) - \theta_2(q)\theta_2\left(q^3\right), \tag{67}$$

for $q \in [0,1)$, really do uniformize (65). By Identities 3 and 5,

$$\begin{aligned}\left(\frac{L(q)M(q)+M(q)^2}{2}\right)^{\frac{1}{2}} &= \left(\frac{L(q)+M(q)}{2}\right)^{\frac{1}{2}} M(q)^{\frac{1}{2}} \\ &\overset{5}{=} \left(\theta_3(q)\theta_3\left(q^3\right)\theta_4(q)\theta_4\left(q^3\right)\right) \\ &\overset{3}{=} \theta_4\left(q^2\right)\theta_4\left(q^6\right) \\ &\overset{5}{=} M\left(q^2\right).\end{aligned} \tag{68}$$

Next, note that by (16), (17), and Identity 4,

$$4\theta_2\left(q^2\right)^2\theta_3\left(q^2\right)^2 = \left(\theta_3\left(q^2\right)^2+\theta_2\left(q^2\right)^2\right)^2 - \left(\theta_3\left(q^2\right)^2-\theta_2\left(q^2\right)^2\right)^2 \overset{(16),(17)}{=} \theta_3(q)^4-\theta_4(q)^4 \overset{4}{=} \theta_2(q)^4, \tag{69}$$

so

$$L\left(q^2\right)^2 - M\left(q^2\right)^2 = 4\theta_3\left(q^2\right)\theta_3\left(q^6\right)\theta_2\left(q^2\right)\theta_2\left(q^6\right) = \theta_2(q)^2\theta_2\left(q^3\right)^2 = \left(\frac{L(q)-M(q)}{2}\right)^2. \tag{70}$$

It follows from the last displayed equation along with (68) that

$$\left(\frac{L(q)^2+3M(q)^2}{4}\right)^2 = \left(\frac{(L(q)-M(q))^2+2L(q)M(q)+2M(q)^2}{4}\right)^{\frac{1}{2}} \overset{(68)}{=} \left(\left(\frac{L(q)-M(q)}{2}\right)^2+M\left(q^2\right)^2\right)^{\frac{1}{2}} \overset{(70)}{=} L\left(q^2\right). \tag{71}$$

To sum up, we have now proven that

$$\left(\frac{L(q)^2+3M(q)^2}{4}\right)^{\frac{1}{2}}, \qquad \left(\frac{L(q)M(q)+M(q)^2}{2}\right)^{\frac{1}{2}}, \tag{72}$$

which yields the uniformization

$$B_2\left(L(q),M(q)\right) = B_2\left(L\left(q^2\right),M\left(q^2\right)\right). \tag{73}$$

Since θ_3 and θ_4 are continuous on $[0,1)$ and since $\theta_2(0)=0$ and $\theta_3(0)=1$, applying (73) recursively and using the homogeneity of B_2 yields

$$B_2\left(\frac{M(q)}{L(q)}\right) = \frac{1}{L(q)}. \tag{74}$$

We may conclude that

$$B\left(\frac{M(q)^2}{L(q)^2}\right) = \frac{1}{L(q)^2}. \tag{75}$$

Define

$$a = \frac{\theta_2(q)^2}{\theta_3(q)^2}, \qquad b = \frac{\theta_2\left(q^3\right)^2}{\theta_3\left(q^3\right)^2}, \tag{76}$$

and note that by Identities 4 and 5,

$$\begin{aligned}\left(a^2b^2\right)^{\frac{1}{4}} + \left(\left(1-a^2\right)\left(1-b^2\right)\right)^{\frac{1}{4}} &= \frac{\theta_2(q)\theta_2\left(q^3\right)}{\theta_3(q)\theta_3\left(q^3\right)} + \left(\left(\frac{\theta_3(q)^4-\theta_2(q)^4}{\theta_3(q)^4}\right)\left(\frac{\theta_3\left(q^3\right)^4-\theta_2\left(q^3\right)^4}{\theta_3\left(q^3\right)^4}\right)\right)^{\frac{1}{4}} \\ &\overset{4}{=} \frac{\theta_2(q)\theta_2\left(q^3\right)}{\theta_3(q)\theta_3\left(q^3\right)} + \frac{\theta_4(q)\theta_4\left(q^3\right)}{\theta_3(q)\theta_3\left(q^3\right)} \\ &\overset{5}{=} 1.\end{aligned} \tag{77}$$

The formulas $L(q) = \theta_3(q)\theta_3\left(q^3\right)\left(1+(ab)^{\frac{1}{2}}\right)$ and $M(q) = \theta_3(q)\theta_3\left(q^3\right)\left(1-(ab)^{\frac{1}{2}}\right)$ yield the equation

$$B\left(\left(\frac{1-(ab)^{\frac{1}{2}}}{1+(ab)^{\frac{1}{2}}}\right)^2\right) = B\left(\frac{M(q)^2}{L(q)^2}\right) = \frac{1}{L(q)^2} = \frac{1}{\theta_3(q)^2\theta_3\left(q^3\right)^2\left(1+(ab)^{\frac{1}{2}}\right)^2} = \frac{\pi^2}{4\left(1+(ab)^{\frac{1}{2}}\right)^2 K(a)K(b)}. \quad (78)$$

Denote $\left(1-(ab)^{\frac{1}{2}}\right)^2/\left(1+(ab)^{\frac{1}{2}}\right)^2$ by h. Since $0 \le \theta_4(q) \le \theta_3(q)$, it follows from Jacobi's identity that $0 \le \theta_2(q) \le \theta_3(q)$, so a and b lie in the interval $[0,1]$. Using this observation, it follows from (77) and (78) that

$$\{a,b\} = \left\{\frac{\sqrt{1-h}}{\left(1+\sqrt{h}\right)^2}\left(\sqrt{1+3h}+2\sqrt{h}\right), \frac{\sqrt{1-h}}{\left(1+\sqrt{h}\right)^2}\left(\sqrt{1+3h}-2\sqrt{h}\right)\right\}, \quad (79)$$

where the specific bijection between these two sets depends on whether $a \le b$ or $b \le a$. (I can see numerically that $b \le a$ for all q, but I am not sure how to prove it.) Since (78) is symmetric and a and b, the explicit bijection is unnecessary, and (66) follows for every h that can be expressed in the form

$$h = \left(\frac{1-\frac{\theta_2(q)\theta_2\left(q^3\right)}{\theta_3(q)\theta_3\left(q^3\right)}}{1+\frac{\theta_2(q)\theta_2\left(q^3\right)}{\theta_3(q)\theta_3\left(q^3\right)}}\right)^2. \quad (80)$$

I claim that every $h \in (0,1]$ can be expressed in this form. For this it suffices to show that $\frac{\theta_2(q)}{\theta_3(q)}$ maps $[0,1)$ to $[0,1)$. This follows from the earlier observation that $0 \le \theta_2(q) \le \theta_3(q)$, from the fact that $\frac{\theta_2(0)}{\theta_3(0)} = 0$, and from the equation

$$\lim_{q\to 1^-}\frac{\theta_2(q)}{\theta_3(q)} \stackrel{4}{=} \lim_{q\to 1^-}\sqrt[4]{1-\frac{\theta_4(q)^4}{\theta_3(q)^4}} = 1. \quad (81)$$

Here we have used the fact, proven earlier, that $\lim_{q\to 1^-}\theta_4(q)\theta_3(q) = 0$. We may conclude that (66) holds for all $h \in (0,1]$.

$\mathcal{W}$

2.4 A mean generated by an identity

2.5 The Borchardt mean

Denote the limit of the four-term *Borchardt iteration* mentioned two sections previously by $G(a_0,b_0,c_0,d_0)$. As Borchardt proves in [3], and as is briefly discussed by the Borweins in [5], the Borchardt mean can be characterized by the theta-function method as follows.

Theorem 2. *For all $a_0,b_0,c_0,d_0 > 0$,*

$$\frac{\pi^2}{G(a_0,b_0,c_0,d_0)} = \int_0^{\alpha_3}\int_{\alpha_2}^{\alpha_1}\frac{x-y}{\sqrt{R(x)R(y)}}\,dx\,dy, \quad (82)$$

where $R(x) \stackrel{def}{=} x(x-\alpha_0)(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$ and where $\alpha_0,\alpha_1,\alpha_2,\alpha_3$ are determined by the following procedure. Let

$$A \stackrel{def}{=} a_) + b_0 + c_0 + d_0, \qquad B \stackrel{def}{=} a_0 + b_0 - c_0 - d_0, \qquad C \stackrel{def}{=} a_0 - b_0 + c_0 - d_0, \qquad D \stackrel{def}{=} a_0 - b_0 - c_0 + d_0, \quad (83)$$

and define $B_1, C_1, D_1, B_2, C_2, D_2$ by the equations

$$2B_1 \stackrel{def}{=} \sqrt{AB} + \sqrt{CD}, \qquad 2C_1 \stackrel{def}{=} \sqrt{AC} + \sqrt{BD}, \qquad 2D_1 \stackrel{def}{=} \sqrt{AD} + \sqrt{BC}, \tag{84}$$
$$2B_2 \stackrel{def}{=} \sqrt{AB} - \sqrt{CD}, \qquad 2C_2 \stackrel{def}{=} \sqrt{AC} - \sqrt{BD}, \qquad 2D_2 \stackrel{def}{=} \sqrt{AD} - \sqrt{BC}$$

and let $\Delta \stackrel{def}{=} \sqrt[4]{ABCDB_1C_1D_1B_2C_2D_2}$. Then

$$\alpha_0 \stackrel{def}{=} \frac{ACB_1}{\Delta}, \qquad \alpha_1 \stackrel{def}{=} \frac{CC_1D_1}{\Delta}, \qquad \alpha_2 \stackrel{def}{=} \frac{AC_2D_1}{\Delta}, \qquad \alpha_3 \stackrel{def}{=} \frac{B_1C_1C_2}{\Delta}. \tag{85}$$

The uniformizing functions for the Borchardt mean are not theta-functions but multidimensional theta functions; specifically, they are the four functions

$$\sum_{n,m\in\mathbb{Z}} (\pm 1)^n (\pm 1)^m q^{sm^2+tmn+un^2}. \tag{86}$$

3 Iterative means and elliptic curves

4 The generalized arithmetic-geometric mean

4.1 Definition

First, we show that the 2-dimensional AGM, given by Definition 1 and denoted from now on by M_2, is well-defined.

Theorem 3. *The 2-dimensional AGM is well-defined.*

Proof. Fix $a_0, b_0 \geq 0$ and assume without loss of generality that $a_0 \geq b_0$. By the arithmetic mean-geometric mean inequality, $a_n \geq b_n$ for all $n \geq 1$. Since the arithmetic and the geometric means are, in fact, means, it follows that the elements of the sequences $\{a_n\}$ and $\{b_n\}$ are nested:

$$b_0 \leq b_1 \leq b_2 \leq b_3 \leq \cdots \leq a_3 \leq a_2 \leq a_1 \leq a_0. \tag{87}$$

That is, $\{a_n\}$ is a decreasing sequence bounded from below by b_0, and $\{b_n\}$ is an increasing sequence bounded from above by a_0. By the monotone convergence theorem, $\{a_n\}$ and $\{b_n\}$ are convergent, say to the limits N_1 and N_2, respectively.

Letting n tend to infinity in the first line of (1), $N_1 = \frac{N_1}{2} + \frac{N_2}{2}$. It follows that $L = M$, so the sequences $\{a_n\}$ and $\{b_n\}$ converge to a common limit. $\mathcal{W}$

It turns out that there is a geometric interpretation of M_2:

> Consider a rectangle R with side lengths a and b. Two quantities are often thought of as "characteristic" of the rectangle: area and perimeter. To each of these quantities correspond a way of "combining" a and b. Namely, we can draw a square S with the same area (resp. perimeter) as R, and take the side length of S to be the mean of a and b with respect to area (resp. perimeter). We leave it to the reader to show that the mean of a and b with respect to area (resp. perimeter) is equal to the arithmetic (resp. geometric) mean of a and b, that is, equal to $\frac{a+b}{2}$ (resp. $\sqrt{ab}$).
>
> This formulation of the arithmetic and geometric means suggests a compromise between the two: given R, draw a rectangle R_1, with side lengths $\frac{a+b}{2}$ and $\sqrt{ab}$. By Theorem 3, iterating this process results in a square with side length $\mathrm{M}_2(a,b)$.

This interpretation gives a geometrically-motivated way of generalizing the 2-dimensional arithmetic-geometric mean to m dimensions, by beginning with an m-dimensional rectangular hyperprism and conserving the m "characteristic" quantities. For instance, in the $m = 3$ case — the Schlömilch mean, defined by the rule (4) — we conserve edge length, surface area, and volume.

When defining $\mathrm{M}_m\left(a^{(1)}, \ldots, a^{(m)}\right)$ for $m \geq 4$, we symbolically extrapolate from the $m = 2$ and $m = 3$ cases to arrive at the following definition:

Definition 2. *For $m \geq 1$, fix m nonnegative real numbers $a_0^{(1)}, \ldots, a_0^{(m)}$. Consider the sequences $\left\{a_n^{(1)}\right\}, \ldots, \left\{a_n^{(m)}\right\}$, defined by the rule*

$$\begin{cases} a_{n+1}^{(1)} &= L_1\left(a_n^{(1)}, \ldots, a_n^{(m)}\right), \\ & \vdots \\ a_{n+1}^{(m)} &= L_m\left(a_n^{(1)}, \ldots, a_n^{(m)}\right), \end{cases} \tag{88}$$

where

$$L_k\left(a_n^{(1)}, \ldots, a_n^{(m)}\right) \stackrel{\text{def}}{=} \left(\frac{1}{\binom{m}{k}} \sum_{0 \leq i_1 < \ldots < i_k \leq m} a_n^{(i_1)} \cdot \ldots \cdot a_n^{(i_k)}\right)^{\frac{1}{k}}. \tag{89}$$

If $\left\{a_n^{(1)}\right\}, \ldots, \left\{a_n^{(m)}\right\}$ converge to a common limit, then we call this limit the m-dimensional arithmetic-geometric mean of $a_0^{(1)}, \ldots, a_0^{(m)}$ and denote it as $M_m\left(a_0^{(1)}, \ldots, a_0^{(m)}\right)$.

Theorem 4. *The m-dimensional arithmetic-geometric mean is well-defined.*

Proof. Fix $a^{(1)}, \ldots, a^{(m)} \geq 0$ and assume without loss of generality that $a^{(1)} \geq \ldots \geq a^{(m)}$. Maclaurin's inequality states that $L_1 \geq \ldots \geq L_m$ for all nonnegative real inputs, so for all $n \geq 0$,

$$a_n^{(1)} \geq a_n^{(2)} \geq a_n^{(3)} \geq \ldots \geq a_n^{(m-2)} \geq a_n^{(m-1)} \geq a_n^{(m)}. \tag{90}$$

That is, the iterates $a_n^{(2)}, \ldots, a_n^{(m-1)}$ are bounded from above and below by $a_n^{(1)}$ and $a_n^{(m)}$, respectively.

By the inequality $L_1 \geq L_m$ and the fact that L_1 and L_m are means, we also have the nested inequality

$$a_0^{(1)} \geq a_1^{(1)} \geq a_2^{(1)} \geq \ldots \geq a_2^{(m)} \geq a_1^{(m)} \geq a_0^{(m)}. \tag{91}$$

That is, $\left\{a_n^{(1)}\right\}$ is a decreasing sequence bounded from below from a_0^m, and $\left\{a_n^{(m)}\right\}$ is an increasing sequence bounded from above by $a_0^{(1)}$. By the monotone convergence theorem, $\left\{a_n^{(1)}\right\}$ and $\left\{a_n^{(m)}\right\}$ is convergent, say to the limit N. Subtracting $a_n^{(1)}$ from both sides of the first line of (88) and letting n tend to infinity yields

$$\frac{m-1}{m} N = \lim_{n \to \infty} \frac{1}{m} \sum_{i=1}^{m-1} a_n^{(i)}. \tag{92}$$

With (90), this implies that the sequences $\left\{a_n^{(2)}\right\}, \ldots, \left\{a_n^{(m)}\right\}$ each converge to N. $\mathcal{W}$

4.2 Basic properties of the gAGM

A number of the properties of the classical AGM also hold for the gAGM, including the digit-doubling property of M_2 that leads to its usefulness in numerical computation.

Definition 3. *If $\lim_{n \to \infty} a_n = L$, then we say that the sequence $\{a_n\}$ exhibits p-th order convergence if there exists some positive constant C such that for all $n \geq 0$,*

$$\left|\frac{a_{n+1} - L}{(a_n - L)^p}\right| \leq C. \tag{93}$$

By Theorem 8.8.c in [8], each of the sequences $\left\{a_n^{(1)}\right\}, \ldots, \left\{a_n^{(m)}\right\}$ exhibit second-order convergence. Specifically, Theorem 8.8.c states that if $L_1, \ldots, L_m$ are twice continuously differentiable m-dimensional means, then convergence in the Gaussian iteration is quadratic, and uniformly so on compact subsets of the domain.

Theorem 5. *On $\mathbb{R}_+^m$, M_m is real-analytic.*

Proof. To prove this theorem, we extend the domain of the means $L_1, \ldots, L_m$ used in the definition of M_m to an open subset of $\mathbb{C}^m$, where $\sqrt[k]{\cdot}$ is defined to map the slit plane $\mathbb{C} \setminus (-\infty, 0]$ onto the sector $\left\{z : z \neq 0, |\arg z| < \frac{\pi}{k}\right\}$. Consider the following lemma.

Lemma 6. *For fixed $R > 0$, let $D_R = \left\{z \in \mathbb{C} : 0 < |z| < R, |\arg z| < \frac{\pi}{2m}\right\}$. If $z_1, \ldots, z_m \in D_R$, then $L_k(z_1, \ldots, z_m) \in D_R$ for all $1 \leq k \leq m$.*

Proof. Take $z_1, \ldots, z_m \in D$. First, note that

$$
\begin{aligned}
|L_k(z_1, \ldots, z_m)| &= \left| \left(\frac{1}{\binom{m}{k}} \sum_{1 \leq i_1 < \ldots < i_k \leq m} z_{i_1} \cdot \ldots \cdot z_{i_k} \right)^{\frac{1}{k}} \right| \\
&= \left| \frac{1}{\binom{m}{k}} \sum_{1 \leq i_1 < \ldots < i_k \leq m} z_{i_1} \cdot \ldots \cdot z_{i_k} \right|^{\frac{1}{k}} \\
&\leq \left(\frac{1}{\binom{m}{k}} \sum_{1 \leq i_1 < \ldots < i_k \leq m} \left| z_{i_1} \cdot \ldots \cdot z_{i_k} \right| \right)^{\frac{1}{k}} \\
&= R.
\end{aligned}
\tag{94}
$$

Second, note that for $1 \leq i_1 < \ldots < i_k \leq m$,

$$
\begin{aligned}
\left|\arg\left(z_{i_1} \cdot \ldots \cdot z_{i_k}\right)\right| &\leq \left|\arg z_{i_1}\right| + \ldots + \left|\arg z_{i_k}\right| \\
&< \frac{k\pi}{2m} \\
&\leq \frac{\pi}{2}.
\end{aligned}
$$

This implies that

$$
\left| \arg \left(\frac{1}{\binom{m}{k}} \sum_{1 \leq i_1 < \ldots < i_k \leq m} z_{i_1} \cdot \ldots \cdot z_{i_k} \right) \right| < \frac{k\pi}{2m},
\tag{95}
$$

so $|\arg(L_k(z_1, \ldots, z_k))| < \frac{\pi}{2m}$.

$\mathcal{W}$

By Lemma 6, for any fixed $1 \leq j \leq m$, the sequence $\left\{a_n^{(j)}\right\}$ is a family of uniformly bounded analytic functions on any bounded subset of D. Recall the first formulation of Montel's theorem:

Theorem 6 (Montel)**.** *Let $\mathcal{F}$ be a uniformly-bounded family of complex-analytic functions from some open subset D of $\mathbb{C}^n$ into $\mathbb{C}$. Then there exists a subsequence $\{f_n\} \subset \mathcal{F}$ that converges uniformly on compact subsets of D.*

By Montel's theorem, for any compact $D_0 \subset D$, $\left\{a_n^{(j)}\right\}$ contains a sequence that converges uniformly on D_0, and since the limit of uniformly convergent analytic functions on connected open sets is analytic, this subsequence must converge to a complex-analytic function. Since $\left\{a_n^{(j)}\right\}$ is convergent on $\mathbb{R}_+^m$, it follows that on any interval of the form $(0, R)$, $\left\{a_n^{(j)}\right\}$ converges to an analytic function. The theorem follows. $\mathcal{W}$

It follows from Theorem 5 that M_m can be expanded as a locally-convergent power series about any $(b, b, \ldots, b) \in \mathbb{R}_+^m$. Furthermore, the coefficients of the power series of $a_1^{(j)}, a_2^{(j)}, a_3^{(j)}, \ldots$ exhibit exponential stabilization:

Theorem 7. *Let $a_j^{(0)} = b + x_j$ for some $b \in \mathbb{R}_+$, for all $1 \le j \le m$. (For instance, $b = 1$ might be a natural choice.) Then for $x_1, \ldots, x_m$ sufficiently small in modulus, $a_n^{(k)}$ and $M_m\left(a_1^{(0)}, \ldots, a_m^{(0)}\right)$ can be written as power series in $\mathbf{x} \overset{def}{=} (x_1, \ldots, x_m)$, and the first term of disagreement between the power series representations of $a_n^{(k)}$ and $M_m\left(a_1^{(0)}, \ldots, a_m^{(0)}\right)$ is of order at least 2^n.*

Proof. By induction on n. The claim follows trivially in the base case $n = 0$. Assume that the claim holds for some $n \ge 0$, and write

$$\mathrm{M}_m\left(a_1^{(0)}, \ldots, a_m^{(0)}\right) = \sum_{s=0}^{2^{n+1}-1} \sum_{|\sigma|=s} \lambda_\sigma \mathbf{x}^\sigma + O\left(2^{n+1}\right). \tag{96}$$

Then, for all $1 \le j \le k$ and for some coefficients $\kappa_\sigma^{(j)}$,

$$a_n^{(j)} = \sum_{s=0}^{2^n-1} \sum_{|\sigma|=s} \lambda_\sigma^{(j)} \mathbf{x}^\sigma + \sum_{s=2^n}^{2^{n+1}-1} \sum_{|\sigma|=s} \kappa_\sigma^{(j)} \mathbf{x}^\sigma + O\left(2^{n+1}\right). \tag{97}$$

It follows that for all $1 \le k \le m$,

$$\begin{aligned}
a_{n+1}^{(k)} &= L_k\left(a_n^{(1)}, \ldots, a_n^{(m)}\right) \qquad (98)\\
&= \left(\frac{1}{\binom{m}{k}} \sum_{1 \le i_1 < \ldots < i_k \le m} \prod_{t=1}^{k} \left(\sum_{s=0}^{2^n-1} \sum_{|\sigma|=s} \lambda_\sigma^{(i_t)} \mathbf{x}^\sigma + \sum_{s=2^n}^{2^{n+1}-1} \sum_{|\sigma|=s} \kappa_\sigma^{(i_t)} \mathbf{x}^\sigma + O\left(2^{n+1}\right) \right) \right)^{\frac{1}{k}}\\
&= \left(\frac{1}{\binom{m}{k}} \left(\binom{m}{k} \left(\sum_{s=0}^{2^n-1} \sum_{|\sigma|=s} \lambda_\sigma^{(i_t)} \mathbf{x}^\sigma \right)^k + \right.\right.\\
&\qquad \left.\left. + \ \frac{k}{m} \binom{m}{k} \left(\sum_{s=0}^{2^n-1} \sum_{|\sigma|=s} \lambda_\sigma^{(i_t)} \mathbf{x}^\sigma \right)^{k-1} \sum_{t=1}^{m} \sum_{s=2^n}^{2^{n+1}-1} \sum_{|\sigma|=s} \kappa_\sigma^{(t)} \mathbf{x}^\sigma + O\left(2^n(k+1)-k\right) \right) \right)^{\frac{1}{k}}\\
&= \left(\left(\sum_{s=0}^{2^n-1} \sum_{|\sigma|=s} \lambda_\sigma^{(i_t)} \mathbf{x}^\sigma \right)^k + \right.\\
&\qquad \left. + \ \frac{k}{m} \left(\sum_{s=0}^{2^n-1} \sum_{|\sigma|=s} \lambda_\sigma^{(i_t)} \mathbf{x}^\sigma \right)^{k-1} \sum_{t=1}^{m} \sum_{s=2^n}^{2^{n+1}-1} \sum_{|\sigma|=s} \kappa_\sigma^{(t)} \mathbf{x}^\sigma + O\left(2^n(k+1)-k\right) \right)^{\frac{1}{k}}\\
&= \left(\sum_{s=0}^{2^n-1} \sum_{|\sigma|=s} \lambda_\sigma^{(i_t)} \mathbf{x}^\sigma \right) \left(1 + \frac{k}{m} \left(\sum_{s=0}^{2^n-1} \sum_{|\sigma|=s} \lambda_\sigma^{(i_t)} \mathbf{x}^\sigma \right)^{-1} \sum_{t=1}^{m} \sum_{s=2^n}^{2^{n+1}-1} \sum_{|\sigma|=s} \kappa_\sigma^{(t)} \mathbf{x}^\sigma + O\left(2^n+1\right) \right)^{\frac{1}{k}}\\
&= \left(\sum_{s=0}^{2^n-1} \sum_{|\sigma|=s} \lambda_\sigma^{(i_t)} \mathbf{x}^\sigma \right) \left(1 + \frac{1}{m} \left(\sum_{s=0}^{2^n-1} \sum_{|\sigma|=s} \lambda_\sigma^{(i_t)} \mathbf{x}^\sigma \right)^{-1} \sum_{t=1}^{m} \sum_{s=2^n}^{2^{n+1}-1} \sum_{|\sigma|=s} \kappa_\sigma^{(t)} \mathbf{x}^\sigma + O\left(2^n+1\right) \right).\\
&= \sum_{s=0}^{2^n-1} \sum_{|\sigma|=s} \lambda_\sigma^{(i_t)} \mathbf{x}^\sigma + \frac{1}{m} \sum_{t=1}^{m} \sum_{s=2^n}^{2^{n+1}-1} \sum_{|\sigma|=s} \kappa_\sigma^{(t)} \mathbf{x}^\sigma + O\left(2^{n+1}\right)\\
&= \sum_{s=0}^{2^n-1} \sum_{|\sigma|=s} \lambda_\sigma^{(i_t)} \mathbf{x}^\sigma + \sum_{s=2^n}^{2^{n+1}-1} \sum_{|\sigma|=s} \frac{\kappa_\sigma^{(1)} + \ldots + \kappa_\sigma^{(m)}}{m} \mathbf{x}^\sigma + O\left(2^{n+1}\right).
\end{aligned}$$

This calculation shows that the coefficients of the power series representation of $a_{n+1}^{(k)}$, up to order $2^{n+1}-1$, are independent of k. By induction, the claim follows. $\mathcal{W}$

4.3 The univariate gAGM

Whether a closed-form representation for M_m exists, by analogy with (3), is an open question. Gauss was led to the integral representation of the classical AGM by first deriving the formula

$$\frac{1}{\mathrm{M}_2\left(1+x,1-x\right)}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-x^2\sin^2\theta}}, \tag{99}$$

and in fact, though there are a number of proofs of (3), by far the simplest proof uses (99) as a stepping-stone [1]. This suggests that finding a univariate version of the gAGM could lead us to an integral representation analogous to (99), and then to an integral representation analogous to (3). Symbolically extrapolating from the univariate classical AGM $\mathrm{M}_2\left(1+x,1-x\right)$, we could consider

$$\mathrm{M}_m\left(1+x,1+e^{\frac{2\pi\iota}{m}}x,\ldots,1+e^{\frac{2(m-1)\pi\iota}{m}}x\right). \tag{100}$$

This function at first appears to be ill-defined: we have not defined the gAGM for complex inputs! But this is not a problem, for iterating once, we see that

$$\mathrm{M}_m\left(1+x,1+e^{\frac{2\pi\iota}{m}}x,\ldots,1+e^{\frac{2(m-1)\pi\iota}{m}}x\right)=\mathrm{M}_m\left(1,1,\ldots,1,\sqrt{1-(-x)^m}\right). \tag{101}$$

See Figure 4.3 for plots of the univariate generalized AGM.

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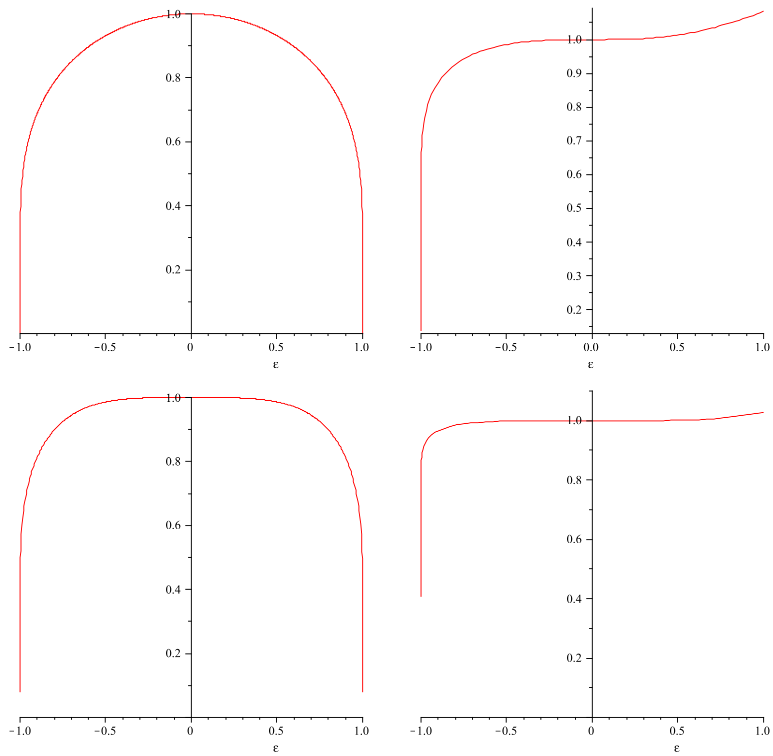

(102)

Figure 1: Clockwise from upper left are the 2-dimensional, 3-dimensional, 5-dimensional, and 4-dimensional univariate arithmetic-geometric means.

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