Grassmann Coordinates

and tableaux

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Goals

1. Describe the classical embedding $G(k, n) \hookrightarrow \mathbb{P}^N$.
2. Characterize the image of the embedding
   - quadratic relations.
   - vanishing polynomials.
3. Reinterpret in terms of varieties and ideals.
4. Application: classify representations over $GL_n(\mathbb{C})$.

What is a Grassmannian?

A **Grassmannian** $G(k, n)$ is the set of all k-dimensional subspaces of $\mathbb{C}^n$.

For example,

$$G(1, 3) = \mathbb{P}^2$$

where we identify all lines.

$G(k, n)$ can be given a topology by embedding it as a subspace of $\mathbb{P}^N$.

The Embedding

- Fix n, k and fix a basis for $\mathbb{C}^n$.
- Let $S_k \in G(k, n)$ be k-dimensional subspace.

Goal: Map S_k to a point in $\mathbb{P}^{\binom{n}{k}-1}$.

$S_k \mapsto p_I \subseteq \mathbb{P}^{\binom{n}{k}-1}$

- Let $\alpha_1, \ldots, \alpha_k \in \mathbb{C}^n$ be a basis for S_k, and let $A = \begin{bmatrix} \alpha_1 & \cdots & \\ \vdots & & \\ \alpha_k & \cdots & \end{bmatrix}$ be the corresponding $k \times n$ matrix.
- Let $I = i_1 \ldots i_k$ with each $1 \leq i_j \leq n$ and $i_1 < i_2 < \cdots < i_k$.
- Let A_I denote the $k \times k$ submatrix obtained by selecting the columns with suffixes $i_1, \ldots, i_k$.

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- Let A_I denote the $k \times k$ submatrix obtained by selecting the columns with suffixes $i_1, \ldots, i_k$.
- We define coordinate functions $\Phi_I(A_I) = \det A_I := p_I$.
- This gives a map $\Phi : G(n, k) \to \mathbb{P}^{\binom{n}{k}-1}$

$$S_k \mapsto \underbrace{(\ldots, p_I, \ldots)}_{\binom{n}{k}\text{-tuple}}, \qquad \forall I.$$

Details About Embedding

Proposition

Φ *is injective.*

Messy argument with coordinates.

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Messy argument with coordinates.

Proposition

Φ *is not surjective.*

The Plücker Relations

$$G(k,n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}.$$

Goal: Characterize the image of $G(k,n)$. Let $X = \Phi(G(k,n))$.

The points in X satisfy certain quadratic relations.

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Goal: Characterize the image of $G(k,n)$. Let $X = \Phi(G(k,n))$.

The points in X satisfy certain quadratic relations.

Proposition

The points in X do not satisfy any linear relations.

The Plücker Relations

Theorem (Plücker Relations)

Fix $\boldsymbol{p} \in X$. *For all* $1 \leq s \leq n$ *and any coordinates* p_I, p_J *with* $I = i_1 \ldots i_k$ *and* $J = j_1 \ldots j_k$ *it holds that*

$$p_I p_J = \sum_{\lambda=1}^{k} p_{i_1 \ldots i_{s-1} j_\lambda i_{s+1} \ldots i_k} p_{j_1 \cdots j_{\lambda-1} i_s j_{\lambda+1} \cdots j_k}.$$

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Theorem (Surjectivity Theorem)

If $\boldsymbol{p} \in \mathbb{P}^N$ *satisfies the Plücker relations then there is a* k*-space* $S_k \subseteq \mathbb{P}^n$ *with coordinate* $\boldsymbol{p}$*.*

Basis Theorem I

Definition

Let $1 \leq i_1, \ldots, i_{k-1} \leq n$ and let $1 \leq j_1, \ldots, j_{k+1} \leq n$ be distinct numbers. Denote these two choices by I and J. We define a ***quadratic basis polynomial***

$$F_{IJ}(P) = \sum_{\lambda=1}^{k+1} (-1)^{\lambda} P_{i_1 \ldots i_{k-1} j_\lambda} P_{j_1 \ldots j_{\lambda-1} j_{\lambda+1} \cdots j_{k+1}}$$

with the P_L inderminates.

Basis Theorem I

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But what if $G(p) = 0$? For arbitrary homogeneuos G.

Basis Theorem

Theorem (Basis Theorem I)

If $G(P)$ is a homogeneous polynomial in the indeterminates $\ldots, P_L, \ldots$ with $L = l_1 \ldots l_k$ such that

$$G(\boldsymbol{p}) = 0, \qquad \forall \boldsymbol{p} \in X$$

then

$$G(P) = \sum_{I,J} A_{IJ}(P) F_{IJ}(P), \qquad I = i_1 \ldots i_{k-1}, J = j_1 \ldots j_{k+1} \tag{1}$$

with the F_{IJ} quadratic basis polynomials and A_{IJ} homogeneous polynomials in the P_L.

Summary

- We can embed $G(k,n)$ into $\mathbb{P}^{\binom{n}{k}-1}$.
- The image consists of points satisfying certain quadratic (Plücker) relations.
- The set of polynomials which vanish on the image is generated by a set of quadratic polynomials.

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Up Next: This can all be reformulated and proven in terms of varieties and ideals in a coordinate free way.

Coordinate-Free Version

Let E be a $\mathbb{C}$-vector space, recall that

$$\bigwedge^d E = \left(\bigotimes_1^d E \right) / T$$

with $T = \{v_1 \otimes \cdots \otimes v_d - \text{sign}\,(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}\}$.

1. $\bigwedge^d E$ is multilinear.
2. $\bigwedge^d E$ is anticommutative.

Coordinate-Free Embedding

Fix $S_{n-d} \in G(n-d, n)$. Will map $G(n-d, n) \to \mathbb{P}^* \left(\bigwedge^d E\right)$ via

$$S_{n-d} \mapsto H_{S_{n-d}}.$$

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The kernel of the map $\bigwedge^d E \to \bigwedge^d (E/S_{n-d})$ is a hyperplane

$$H_{S_{n-d}} \subseteq \bigwedge^d E.$$

Recall that $\mathbb{P}^*(E)$ is the quotient of E in which we identify all hyperplanes.

Polynomials on $\mathbb{P}^*(\bigwedge^d E)$

For any $v_1, \ldots v_d \in E$ we can define a linear form on $H \in \mathbb{P}^*(\bigwedge^d E)$.

$$\bigwedge^d E \xrightarrow{\pi} \left(\bigwedge^d E\right)/H := L$$

$$\left(\bigwedge^d E\right)^* \xleftarrow{\pi^*} L^*$$

For $f \in L^*$.

$$\begin{aligned}(v_1 \wedge \cdots \wedge v_d)(H) &:= (v_1 \wedge \cdots \wedge v_d)(L^*) \\ &\sim (\pi^* f)(v_1 \wedge \cdots \wedge v_d) \\ &\begin{cases} \equiv 0 \\ \not\equiv 0 \end{cases}\end{aligned}$$

Products of the $v_1 \wedge \cdots \wedge v_d$ live in $\mathrm{Sym}^\bullet\left(\bigwedge^d E\right)$.

Plücker Relations

Theorem (The Plücker Relations/Surjectivity)

The Plücker embedding is a bijection from $G(n-d,n)$ to the subvariety of $\mathbb{P}^*(\bigwedge^d E)$ defined by the quadratic equations

$$(v_1 \wedge \cdots \wedge v_d) \cdot (w_1 \wedge \cdots \wedge w_d) \;=\; \sum_{i_1 < i_2 < \cdots < i_k} (v_1 \wedge \cdots \wedge w_1 \wedge \cdots \wedge w_k \wedge \cdots \wedge v_d) \cdot (v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge w_{k+1} \wedge \cdots \wedge w_d)$$

The Basis Theorem II

Theorem (The Basis Theorem II)

Let $\tilde{Q}$ be the ideal generated by the Plücker Relations. It holds that

$$\mathcal{I}(\mathcal{Z}(\tilde{Q})) = \tilde{Q}.$$

Proof.

- We will prove that $\tilde{Q}$ is prime.
- The Nullstellensatz immediately implies the result.

□

Proving Primality of $\tilde{Q}$.

Short-Story:

- Goal is to show that $\mathrm{Sym}^\bullet\left(\bigwedge^d E\right)/\tilde{Q}$ is an integral domain.
- Will prove it embeds as a subring of a polynomial ring.
- Obtain a classification of polynomial representations over $GL_n(\mathbb{C})$.

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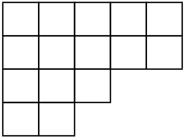
First we need to introduce the tableaux:

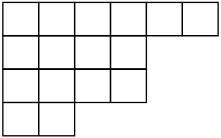


Tableaux

Let E be a $\mathbb{C}$-module. For fixed n, we let λ denote a weakly decreasing **partition of** n, i.e. for $n = 16$ a partition λ could be $\lambda = (6, 4, 4, 2)$

$$6 + 4 + 4 + 2 = 16.$$

The associated **tableau** (also denoted λ) is



Constructing the Schur Module: Step 1/4

From each λ we can construct a particular $\mathbb{C}$-module E^{λ}.

Start with cartesian product $E^{\times\lambda}$

Instead of *n*-tuples - put elements in boxes.

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If $n = 5$ and $\lambda = (2, 2, 1)$ we have an element $\mathbf{v} \in E^{\times\lambda}$ is written

v_1	v_4
v_2	v_5
v_3	

Constructing the Schur Module: Step 2/4

Let λ have s columns and let d_i, $i = 1, \ldots, s$ denote the length of the i^{th} column.

$$E^{\times\lambda} \to \bigotimes_{i=1}^{s} \bigwedge_{1}^{d_i} E : \mathbf{v} \mapsto \wedge\mathbf{v}$$

For example,

$$\begin{array}{|c|c|}\hline v_1 & v_4 \\ \hline v_2 & v_5 \\ \hline v_3 & \\ \cline{1-1} \end{array} \mapsto (v_1 \wedge v_2 \wedge v_3) \otimes (v_4 \wedge v_5)$$

The Quadratic Relations: Step 3/4

Let Q^λ be the submodule generated by

$$\wedge \mathbf{v} - \sum \wedge \mathbf{w}$$

The sum is over all **w** obtained from **v** with an exchange between two given columns with a given subset of boxes in the right chosen column.

1	6	11	15
2	7	12	6
3	8	13	
4	9	14	
5	10		

$\wedge \mathbf{v}$

11	6	1	15
2	7	12	6
13	8	3	
4	9	5	
14	10		

$\wedge \mathbf{w}$

The Schur Module: Step 4/4

$$E^\lambda := \left(\bigotimes_{i=1}^{s} \bigwedge_{1}^{d_i} E \right) / Q^\lambda.$$

1. $\lambda = \underbrace{\square\square\cdots\square}_{n \text{ times}}$ then $E^\lambda = \mathrm{Sym}^n(E)$.

2. $\lambda = \begin{array}{c}\square\\ \square\\ \square\end{array}$ then $E^\lambda = \bigwedge^n E$.

E^λ in Coordinates

- Let $e_1, \ldots, e_n$ be a basis for E.
- Fill λ with the e_i.
  - Weakly increasing across rows.
  - Strictly increasing down columns.
- Each such arrangement, T, is called a **standard filling.**

The image of this element in E^λ will be denoted by e_T.

$$\begin{array}{|c|c|c|}\hline e_1 & e_2 & e_2 \\ \hline e_3 & e_4 & e_5 \\ \hline e_5 & e_5 \\ \cline{1-2}\end{array} \longmapsto e_T \in E^\lambda$$

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Theorem

E^λ is free on the e_T.

A New Polynomial Ring

$$\mathbb{C}[Z] := \mathbb{C}[\ldots, Z_{i,j}, \ldots], \qquad i = 1, \ldots, m \quad j = 1, \ldots, n$$

For $d \leq m$ choose $0 \leq i_1 \leq \cdots \leq i_d \leq n$. Define the polynomial

$$D_{i_1 \ldots i_d} = \det \begin{bmatrix} Z_{1,i_1} & \cdot & Z_{1,i_d} \\ \cdot & \cdot & \cdot \\ Z_{d,i_1} & \cdot & Z_{d,i_d} \end{bmatrix}$$

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For an arbitrary filling T of λ with the numbers $\{1, \dots, n\}$,

$$D_T = \prod_{i=1}^{s} D_{T(1,i),T(2,i),\dots,T(d_i,i)},$$

Corollary

The map $e_T \mapsto D_T$ is an injective homomorphism $E^\lambda \to \mathbb{C}[Z]$ and its image D^λ is free on the polynomials D_T.

Tying it Together

λ with s columns with lengths d_i each occurring with multiplicity a_i.

$$E^\lambda \simeq \mathrm{Sym}^{a_1}(\bigwedge^{d_1} E) \otimes \cdots \otimes \mathrm{Sym}^{a_s}(\bigwedge^{d_s} E)/Q^\lambda.$$

Define

$$S^\bullet(E; d_1, \ldots, d_s) := \bigoplus_{(a_1,\ldots,a_s)} E^\lambda, \qquad Q := \bigoplus_{(a_1,\ldots,a_s)} Q^\lambda.$$

$$R := \bigoplus_{(a_1,\ldots,a_s)} \mathrm{Sym}^{a_1}(\bigwedge^{d_1} E) \otimes \cdots \otimes \mathrm{Sym}^{a_s}(\bigwedge^{d_s} E)$$

$$R/Q = \bigoplus_{(a_1,\ldots,a_s)} E^\lambda$$

Putting it Together.

We now have

1. $R/Q = \bigoplus_{(a_1,\ldots,a_s)} E^\lambda$.
2. $E^\lambda \simeq D^\lambda \subseteq \mathbb{C}[Z]$ under the map $e_T \mapsto D_T$

Proposition

Q is a prime ideal.

Proof.

- $R/Q \simeq \bigoplus D^\lambda \subseteq \mathbb{C}[Z]$ via $e_T \mapsto D_T$.
- $\bigoplus D^\lambda$ remains direct (requires proof) and thus is a subring.
- A subring of a polynomial ring is an integral domain.

$\therefore$ Q is prime. □

What about $\tilde{Q}$?

Back to $G(n-d,n)$ and $\tilde{Q}$. Corresponds to λ has columns of length d.

$$\bigoplus_a E^\lambda = \bigoplus_a \mathrm{Sym}^a \left(\bigwedge^d E \right) / Q^\lambda = \mathrm{Sym}^\bullet \left(\bigwedge^d E \right) / \tilde{Q}.$$

Which we just proved embeds as a subring of a polynomial ring.

Hence $\tilde{Q}$ is prime as a special case.

Last item of business (time pending): Why is $\bigoplus D^\lambda$ direct?

Some Representation Theory

A **representation** of $GL(n, \mathbb{C})$ on $\mathbb{C}$ is a homomorphism $V : GL(n, \mathbb{C}) \to GL(m, \mathbb{C})$ for some m.

Let $X_{i,j} : GL(n, \mathbb{C}) \to \mathbb{C}$ be the coordinate function with $1 \leq i, j \leq n$.

We say that a representation, V, is **polynomial** if there is a basis $v_1, \ldots, v_m$ of V such that for $g \in GL(n, \mathbb{C})$ we have

$$gv_b = \sum_a f_{ab}(g) v_a, \qquad 1 \leq a, b \leq n.$$

With $f_{ab} \in \mathbb{C}[X_{ij}]$ (i.e. f_{ab} is a polynomial).

E^λ as a Polynomial Representation

Let $|\lambda| = n$, $e_T \in E^\lambda$ acts on a matrix $g \in GL(m, \mathbb{C})$ via the formula

$$g \cdot e_T = \sum g_{i_1,j_1} \cdots g_{i_m,j_m} e_{T'}$$

where the sum is taken over the n^m fillings of T' of obtained from T by replacing the entries $(j_1, \ldots, j_m)$ by $(i_1, \ldots, i_m)$.

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Theorem

As λ varies over all tableaux the E^λ classify uniquely all irreducible polynomial representations of $GL(n, \mathbb{C})$.

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Theorem

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Proposition

Any sum of irreducible pairwise distinct representations is direct.

$\bigoplus D^\lambda$ remains direct.

Conclusion

1. $G(k, n) \hookrightarrow \mathbb{P}^N$ in coordinates and $G(n-d, n) \hookrightarrow \mathbb{P}^*(\bigwedge^d E)$ via a coordinate free way.
2. Can classify the vanishing polynomials on the respective images.
3. All polynomial representations of $GL(m, k)$ have the form E^λ.

Conclusion

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