Grassmann Coordinates

and tableaux

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Autumn 2012
Goals

1. Describe the classical embedding $G(k, n) \hookrightarrow \mathbb{P}^N$.
2. Characterize the image of the embedding
   - quadratic relations.
   - vanishing polynomials.
3. Reinterpret in terms of varieties and ideals.
4. Application: classify representations over $GL_n(\mathbb{C})$. 
What is a Grassmannian?

A **Grassmannian** $G(k, n)$ is the set of all $k$-dimensional subspaces of $\mathbb{C}^n$.

For example,

$$G(1, 3) = \mathbb{P}^2$$

where we identify all lines.

$G(k, n)$ can be given a topology by embedding it as a subspace of $\mathbb{P}^N$. 
The Embedding

- Fix $n, k$ and fix a basis for $\mathbb{C}^n$.
- Let $S_k \in G(k, n)$ be $k$-dimensional subspace.

| Goal: | Map $S_k$ to a point in $\mathbb{P}^{(n)-1}$. |
Let $\alpha_1, \ldots, \alpha_k \in \mathbb{C}^n$ be a basis for $S_k$, and let $A = \begin{bmatrix} \alpha_1 & \cdots \\ \vdots \\ \alpha_k & \cdots \end{bmatrix}$ be the corresponding $k \times n$ matrix.

Let $l = i_1 \ldots i_k$ with each $1 \leq i_j \leq n$ and $i_1 < i_2 < \cdots < i_k$.

Let $A_l$ denote the $k \times k$ submatrix obtained by selecting the columns with suffixes $i_1, \ldots, i_k$. 

$S_k \mapsto p_I \subseteq \mathbb{P}^\left(\binom{n}{k}\right)^{-1}$
Let $\alpha_1, \ldots, \alpha_k \in \mathbb{C}^n$ be a basis for $S_k$, and let $A = \begin{bmatrix} \alpha_1 & \cdots \\ \vdots \\ \alpha_k & \cdots \end{bmatrix}$ be the corresponding $k \times n$ matrix.

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We define coordinate functions $\Phi_l(A_l) = \det A_l := p_l$.

This gives a map $\Phi : G(n, k) \to \mathbb{P}(\binom{n}{k})^{-1}$

$$S_k \mapsto (\ldots, p_l, \ldots), \quad \forall l.$$
Details About Embedding

Proposition

$\Phi$ is injective.

Messy argument with coordinates.
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Messy argument with coordinates.

Proposition
\( \Phi \) is not surjective.
The Plücker Relations

\[ G(k, n) \hookrightarrow \mathbb{P}(n)^{-1}. \]

**Goal:** Characterize the image of \( G(k, n) \). Let \( X = \Phi(G(k, n)) \).

The points in \( X \) satisfy certain quadratic relations.
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The points in $X$ satisfy certain quadratic relations.

Proposition

The points in $X$ do not satisfy any linear relations.
The Plücker Relations

**Theorem (Plücker Relations)**

Fix \( p \in X \). For all \( 1 \leq s \leq n \) and any coordinates \( p_I, p_J \) with \( I = i_1 \ldots i_k \) and \( J = j_1 \ldots j_k \) it holds that

\[
p_I p_J = \sum_{\lambda=1}^{k} p_{i_1 \ldots i_{s-1} \lambda i_{s+1} \ldots i_k} p_{j_1 \ldots j_{\lambda-1} i_s j_{\lambda+1} \ldots j_k}.
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p_I p_J = \sum_{\lambda=1}^{k} p_{i_1 \ldots i_{s-1} j_\lambda i_{s+1} \ldots i_k} p_{j_1 \ldots j_{\lambda-1} i_s j_{\lambda+1} \ldots j_k}.
\]

Theorem (Surjectivity Theorem)

If \( p \in \mathbb{P}^N \) satisfies the Plücker relations then there is a \( k \)-space \( S_k \subseteq \mathbb{P}^n \) with coordinate \( p \).
Definition

Let $1 \leq i_1, \ldots, i_{k-1} \leq n$ and let $1 \leq j_1, \ldots, j_{k+1} \leq n$ be distinct numbers. Denote these two choices by $I$ and $J$. We define a **quadratic basis polynomial**

\[
F_{IJ}(P) = \sum_{\lambda=1}^{k+1} (-1)^\lambda P_{i_1 \ldots i_{k-1} j_\lambda} P_{j_1 \ldots j_\lambda-1 j_{\lambda+1} \ldots j_{k+1}}
\]

with the $P_L$ indeterminates.
Proposition

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But what if $G(p) = 0$? For arbitrary homogeneous $G$. 
Theorem (Basis Theorem I)

If \( G(P) \) is a homogeneous polynomial in the indeterminates \( \ldots, P_L, \ldots \) with \( L = l_1 \ldots l_k \) such that

\[
G(p) = 0, \quad \forall p \in X
\]

then

\[
G(P) = \sum_{I,J} A_{IJ}(P)F_{IJ}(P), \quad I = i_1 \ldots i_{k-1}, J = j_1 \ldots j_{k+1}
\]

(1)

with the \( F_{IJ} \) quadratic basis polynomials and \( A_{IJ} \) homogeneous polynomials in the \( P_L \).
We can embed $G(k, n)$ into $\mathbb{P}(k)^{n-1}$.

The image consists of points satisfying certain quadratic (Plücker) relations.

The set of polynomials which vanish on the image is generated by a set of quadratic polynomials.
We can embed $G(k, n)$ into $\mathbb{P}(n)^{-1}$.

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**Up Next:** This can all be reformulated and proven in terms of varieties and ideals in a coordinate free way.
Let $E$ be a $\mathbb{C}$-vector space, recall that

$$\bigwedge^d E = \left( \bigotimes_1^d E \right) / T$$

with $T = \{v_1 \otimes \cdots \otimes v_d - \text{sign } (\sigma)v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}\}$.

1. $\bigwedge^d E$ is multilinear.
2. $\bigwedge^d E$ is anticommutative.
Fix $S_{n-d} \in G(n - d, n)$. Will map $G(n - d, n) \to \mathbb{P}^* \left( \wedge^d E \right)$ via

$$S_{n-d} \mapsto H_{S_{n-d}}.$$
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The kernel of the map $\bigwedge^d E \to \bigwedge^d (E/S_{n-d})$ is a hyperplane

$$H_{S_{n-d}} \subseteq \bigwedge^d E.$$ 

Recall that $\mathbb{P}^* (E)$ is the quotient of $E$ in which we identify all hyperplanes.
Polynomials on $\mathbb{P}^*(\bigwedge^d E)$

For any $v_1, \ldots, v_d \in E$ we can define a linear form on $H \in \mathbb{P}^*(\bigwedge^d E)$.

$$\bigwedge^d E \xrightarrow{\pi} \left(\bigwedge^d E\right) / H := L$$

For $f \in L^*$.

$$\left(\bigwedge^d E\right)^* \xleftarrow{\pi^*} L^*$$

$$\left( v_1 \wedge \cdots \wedge v_d \right)(H) := \left( v_1 \wedge \cdots \wedge v_d \right)(L^*) \sim \left( \pi^* f \right)(v_1 \wedge \cdots \wedge v_d)$$

\begin{align*}
&\begin{cases}
0 \\
\neq 0
\end{cases}
\end{align*}

Products of the $v_1 \wedge \cdots \wedge v_d$ live in $\text{Sym}^* \left( \bigwedge^d E \right)$. 
Plücker Relations

**Theorem (The Plücker Relations/Surjectivity)**

The Plücker embedding is a bijection from $G(n - d, n)$ to the subvariety of $\mathbb{P}^\ast(\wedge^d E)$ defined by the quadratic equations

\[
(v_1 \wedge \cdots \wedge v_d) \cdot (w_1 \wedge \cdots \wedge w_d) = \sum_{i_1 < i_2 < \cdots < i_k} (v_1 \wedge \cdots \wedge v_{i_1} \wedge \cdots \wedge v_{i_k} \wedge w_{k+1} \wedge \cdots \wedge w_d)
\]
The Basis Theorem II

**Theorem (The Basis Theorem II)**

Let $\tilde{Q}$ be the ideal generated by the Plücker Relations. It holds that

$$\mathcal{I}(\mathcal{Z}(\tilde{Q})) = \tilde{Q}.$$

**Proof.**

- We will prove that $\tilde{Q}$ is prime.
- The Nullstellensatz immediately implies the result.
Proving Primality of $\tilde{Q}$.

Short-Story:

- Goal is to show that $\text{Sym}^* \left( \wedge^d E \right) / \tilde{Q}$ is an integral domain.
- Will prove it embeds as a subring of a polynomial ring.
- Obtain a classification of polynomial representations over $GL_n(\mathbb{C})$. 

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First we need to introduce the tableaux:
Let $E$ be a $\mathbb{C}$-module. For fixed $n$, we let $\lambda$ denote a weakly decreasing partition of $n$, i.e. for $n = 16$ a partition $\lambda$ could be $\lambda = (6, 4, 4, 2)$

$$6 + 4 + 4 + 2 = 16.$$  

The associated tableau (also denoted $\lambda$) is
From each $\lambda$ we can construct a particular $\mathbb{C}$-module $E^\lambda$.

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If $n = 5$ and $\lambda = (2, 2, 1)$ we have an element $v \in E^{\times \lambda}$ is written

\[
\begin{array}{c|c}
V_1 & V_4 \\
V_2 & V_5 \\
V_3 & \\
\end{array}
\]
Let $\lambda$ have $s$ columns and let $d_i, i = 1, \ldots, s$ denote the length of the $i^{th}$ column.

$$E \times \lambda \rightarrow \bigotimes_{i=1}^{s} d_i \bigwedge E : v \mapsto \wedge v$$

For example,

\[
\begin{array}{cc}
V_1 & V_4 \\
V_2 & V_5 \\
V_3 & \mapsto (V_1 \wedge V_2 \wedge V_3) \otimes (V_4 \wedge V_5)
\end{array}
\]
Let $Q^\lambda$ be the submodule generated by

$$\wedge v - \sum \wedge w$$

The sum is over all $w$ obtained from $v$ with an exchange between two given columns with a given subset of boxes in the right chosen column.
The Schur Module: Step 4/4

\[ E^\lambda := \left( \bigotimes_{i=1}^{s} d_i \bigwedge E \right) / Q^\lambda. \]

1. \[ \lambda = \underbrace{\begin{array}{c} \hline \hline \hline \end{array}}_{n \text{ times}} \] then \( E^\lambda = \text{Sym}^n(E) \).

2. \[ \lambda = \begin{array}{c} \hline \hline \hline \end{array} \] then \( E^\lambda = \bigwedge^n E \).
Let $e_1, \ldots, e_n$ be a basis for $E$.

Fill $\lambda$ with the $e_i$.
- Weakly increasing across rows.
- Strictly increasing down columns.

Each such arrangement, $T$, is called a **standard filling**.

The image of this element in $E^\lambda$ will be denoted by $e_T$.

\[
\begin{array}{ccc}
  e_1 & e_2 & e_2 \\
  e_3 & e_4 & e_5 \\
  e_5 & e_5 & \\
\end{array}
\longrightarrow e_T \in E^\lambda
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\quad \rightarrow \quad e_T \in E^\lambda
\]

**Theorem**

$E^\lambda$ is free on the $e_T$. 
A New Polynomial Ring

\[ \mathbb{C}[Z] := \mathbb{C}[\ldots, Z_{i,j}, \ldots], \quad i = 1, \ldots, m \quad j = 1, \ldots, n \]

For \( d \leq m \) choose \( 0 \leq i_1 \leq \cdots \leq i_d \leq n \). Define the polynomial

\[ D_{i_1 \ldots i_d} = \det \begin{bmatrix} Z_{1,i_1} & \cdots & Z_{1,i_d} \\ \vdots & \ddots & \vdots \\ Z_{d,i_1} & \cdots & Z_{d,i_d} \end{bmatrix} \]
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For an arbitrary filling \( T \) of \( \lambda \) with the numbers \( \{1, \ldots, n\} \),

\[ D_T = \prod_{i=1}^{s} D_{T(1,i), T(2,i), \ldots, T(d,i)} \]

**Corollary**

The map \( e_T \mapsto D_T \) is an injective homomorphism \( E^\lambda \to \mathbb{C}[Z] \) and its image \( D^\lambda \) is free on the polynomials \( D_T \).
\( \lambda \) with \( s \) columns with lengths \( d_i \) each occurring with multiplicity \( a_i \).

\[
E^\lambda \simeq \text{Sym}^{a_1}(\bigwedge^{d_1} E) \otimes \cdots \otimes \text{Sym}^{a_s}(\bigwedge^{d_s} E)/Q^\lambda.
\]

Define

\[
S^*(E; d_1, \ldots, d_s) := \bigoplus_{(a_1, \ldots, a_s)} E^\lambda, \quad Q := \bigoplus_{(a_1, \ldots, a_s)} Q^\lambda.
\]

\[
R := \bigoplus_{(a_1, \ldots, a_s)} \text{Sym}^{a_1}(\bigwedge^{d_1} E) \otimes \cdots \otimes \text{Sym}^{a_s}(\bigwedge^{d_s} E)
\]

\[
R/Q = \bigoplus_{(a_1, \ldots, a_s)} E^\lambda
\]
Putting it Together.

We now have

1. \( R/Q = \bigoplus_{(a_1, \ldots, a_s)} E^\lambda. \)
2. \( E^\lambda \simeq D^\lambda \subseteq \mathbb{C}[Z] \) under the map \( e_T \mapsto D_T \)

Proposition

\( Q \) is a prime ideal.

Proof.

- \( R/Q \simeq \bigoplus D^\lambda \subseteq \mathbb{C}[Z] \) via \( e_T \mapsto D_T \).
- \( \bigoplus D^\lambda \) remains direct (requires proof) and thus is a subring.
- A subring of a polynomial ring is an integral domain.

\( \therefore \) \( Q \) is prime.
What about $\tilde{Q}$?

Back to $G(n - d, n)$ and $\tilde{Q}$. Corresponds to $\lambda$ has columns of length $d$.

$$\bigoplus_a E^\lambda = \bigoplus_a \text{Sym}^a \left( \bigwedge^d E \right) / Q^\lambda = \text{Sym}^\cdot \left( \bigwedge^d E \right) / \tilde{Q}.$$ 

Which we just proved embeds as a subring of a polynomial ring.

Hence $\tilde{Q}$ is prime as a special case.

Last item of business (time pending): Why is $\bigoplus D^\lambda$ direct?
A representation of $GL(n, \mathbb{C})$ on $\mathbb{C}$ is a homomorphism $V : GL(n, \mathbb{C}) \to GL(m, \mathbb{C})$ for some $m$.

Let $X_{i,j} : GL(n, \mathbb{C}) \to \mathbb{C}$ be the coordinate function with $1 \leq i, j \leq n$.

We say that a representation, $V$, is polynomial if there is a basis $v_1, \ldots, v_m$ of $V$ such that for $g \in GL(n, \mathbb{C})$ we have

$$gv_b = \sum_a f_{ab}(g)v_a, \quad 1 \leq a, b \leq n.$$  

With $f_{ab} \in \mathbb{C}[X_{ij}]$ (i.e. $f_{ab}$ is a polynomial).
$E^\lambda$ as a Polynomial Representation

Let $|\lambda| = n$, $e_T \in E^\lambda$ acts on a matrix $g \in GL(m, \mathbb{C})$ via the formula

$$g \cdot e_T = \sum g_{i_1,j_1} \cdots g_{i_m,j_m} e_{T'}$$

where the sum is taken over the $n^m$ fillings of $T'$ of obtained from $T$ by replacing the entries $(j_1, \ldots, j_m)$ by $(i_1, \ldots, i_m)$.
$E^\lambda$ as a Polynomial Representation

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**Theorem**

As $\lambda$ varies over all tableaux the $E^\lambda$ classify uniquely all irreducible polynomial representations of $GL(n, \mathbb{C})$. 

**Proposition**

Any sum of irreducible pairwise distinct representations is direct.

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**Proposition**

Any sum of irreducible pairwise distinct representations is direct.

\[ \bigoplus D^\lambda \text{ remains direct.} \]
1. $G(k, n) \hookrightarrow \mathbb{P}^N$ in coordinates and $G(n - d, n) \hookrightarrow \mathbb{P}^* (\wedge^d E)$ via a coordinate free way.

2. Can classify the vanishing polynomials on the respective images.

3. All polynomial representations of $GL(m, k)$ have the form $E^\lambda$. 
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ANY QUESTIONS?