Discrete and continuous inverse boundary problems on a disc

by

David V Ingerman

A dissertation submitted in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

University of Washington

1997

Approved by ______________________________

(Chairperson of Supervisory Committee)

____

____

Program Authorized
to Offer Degree ______________________________

Date ______________________________

University of Washington

Abstract

Discrete and continuous inverse boundary problems on a disc

by David V Ingerman

Chairperson of Supervisory Committee: *Professor James A. Morrow*
Department of Mathematics

A positive function γ (conductivity) on the closed unit disk $\bar{\mathbb{D}}$ or on edges of a circular planar graph Γ (discrete analog of $\bar{\mathbb{D}}$) induces the *Dirichlet-to-Neumann map* Λ_γ on functions on $\partial\bar{\mathbb{D}}$ (on $\partial\Gamma$). The main inverse problems are to give a characterization of the maps Λ_γ and to find out if/when Λ_γ uniquely determines γ. The main results of this thesis are:

It was shown in our joint work with E. Curtis and J.Morrow that a linear map $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ is a Dirichlet-to-Neumann map of a circular planar graph Γ if and only if it is self-adjoint and has the *alternating sign* property. We show that continuous Dirichlet-to-Neumann maps Λ have the continuous analog of the alternating sign property, and that this property is equivalent to the fact that the kernel, K, of Λ satisfies a set of inequalities of the form $\det K(x_i, y_j) > 0$. This implies that restrictions of continuous Dirichlet-to-Neumann maps are discrete Dirichlet-to-Neumann maps.

We constructively show exactly what information about the shape of a circular planar graph can be obtained from a Λ_γ.

We give a geometric characterization of Γ's for which Λ_γ uniquely determines γ, and give an algorithm for finding γ.

In the *layered* case we characterize the set of the discrete and continuous Dirichlet-to-Neumann maps in terms of their kernels and spectra. The characterization in terms of spectra shows that continuous Dirichlet-to-Neumann maps can be viewed as limits of the discrete ones. The characterization in terms of kernels supports the conjecture that the alternating sign property essentially characterizes continuous Dirichlet-to-Neumann maps. The characterizations above give a new interpretation of connections between positive measures, positive definite functions and analytic funtions that map the right half-plane into itself.

We give a probabilistic interpretation to Dirichlet-to-Neumann maps on graphs and use it to give a simple parametrization of the set of *totally positive* matrices (matrices in which determinants of all minors are positive). This characterization leads to an elementary proof of Hadamard's inequality for such matrices.

TABLE OF CONTENTS

ACKNOWLEDGMENTS

I would like to express my deepest gratitude to Professor James Morrow and Professor Edward Curtis who were my first teachers and who introduced me the subject of inverse problems. I would also like to thank Professor James Morrow for his great assistance in every aspect of my career as a student.

I would like to thank Professor Gunther Uhlmann for his inspiring lectures and questions and Professor Donald Marshall and Professor John Sylvester for their help in writing Chapter 5 of this thesis. I would also like to thank Professor Krzysztof Burdzy with whom I coauthored my first paper and Professor Pietro Poggi-Corradini and Professor Mark Lawrence for many fruitful mathematical conversations.

Special considerations go to Alfred P. Sloan Foundation for funding part of my studies, and to the whole Department of Mathematics at the University of Washington for its help and financial assistance.

DEDICATION

to my parents Nellia and Vladimir and my wife Sofia

Chapter 1

INTRODUCTION

A positive function γ (conductivity) on the closed unit disk $\bar{\mathbb{D}}$ or on edges of a circular planar graph Γ (discrete analog of $\bar{\mathbb{D}}$) induces the *Dirichlet-to-Neumann map* Λ_γ on functions on $\partial\bar{\mathbb{D}}$ (on $\partial\Gamma$).

There are two main problems connected with the maps Λ_γ:

- to give a characterization of the Dirichlet-to-Neumann maps. (It was shown in our joint work with E. Curtis ad J. Morrow that alternating sign property essentially caharcterizes the discrete Dirichlet-to-Neumann maps).

- the inverse problem, is to recover γ from Λ_γ. (A sufficient condition for uniqueness is known: It was recently proved in [19] that Λ_γ uniquely determines γ for $\gamma \in W^{2,p}(\overline{\mathbb{D}}) \subset C^3(\overline{\mathbb{D}})$, $p > 1$.)

One of the main motivations of this thesis is to show a strong connection between properties of the continuous and discrete Dirichlet-to-Neumann maps.

In Chapter 3 we show that continuous Dirichlet-to-Neumann maps Λ have the continuous analog of the alternating sign property, and that this property is equivalent to the fact that the kernel, K, of Λ satisfies a set of inequalities of the form $\det K(x_i, y_j) > 0$. This implies that restrictions of continuous Dirichlet-to-Neumann maps are discrete Dirichlet-to-Neumann maps.

In Chapter 4 we constructively show exactly what information about the shape of a circular planar graph can be obtained from a Λ_γ. We also give a geometric characterization of Γ's for which Λ_γ uniquely determines γ, and give an algorithm for finding γ.

In Chapter 5 we consider the case of conductivities that are constant on circles centered at the origin. We also consider a discrete analog of this, so called, layered case. We obtain a clear picture of the sets of both discrete and continuous Dirichlet-to-Neumann maps in this case. We characterize them in terms of their kernels and spectra.

The characterization in terms of the spectra shows that continuous Dirichlet-to-Neumann maps can be viewed as limits of the discrete Dirichlet-to-Neumann maps.

The characterization in terms of the kernels supports the conjecture in [14] that the alternating property essentially characterizes continuous Dirichlet-to-Neumann maps.

We also give sharp conditions on γ for the uniqueness in the continuous inverse problem. For the discrete case, we give a new algorithm, based on the Pick-Nevalinna Interpolation theorem, for the recovery of γ.

The characterizations above give a physical interpretation of the connection between positive measures, positive definite functions and analytic functions from the right half-plane to itself: these objects describe, respectively, spectral measures, kernels and spectra of Dirichlet-to-Neumann maps in the layered case.

In Chapter 6 we give a probabilistic interpretation to Dirichlet-to-Neumann maps on graphs and use it to give a simple parametrization of the set of *totally positive* matrices (matrices in which determinants of all minors are positive). This characterization leads to an elementary proof of Hadamard's inequality for such matrices.

Chapter 2

BACKGROUND AND MAIN RESULTS

2.1 γ-harmonic functions and Dirichlet-to-Neumann maps

We first give the definition of Dirichlet-to-Neumann maps as it is usually done, see [25] for details. For the layered case, which we will consider, the restrictions on γ will be weakened, and the domain of Λ_γ will be shrunk.

Let $\gamma \in C^{1,1}(\overline{\mathbb{D}})$. A function $u \in H^1(\mathbb{D})$ is called γ-harmonic function or potential if

$$\operatorname{div}(\gamma \nabla u) = 0 \text{ in } \mathbb{D}. \tag{2.1}$$

A potential in $\mathbb{D}$ satisfies this equation if there are no sources or sinks of current in $\mathbb{D}$.

For each $f \in H^{1/2}(\partial\mathbb{D})$ there exists a unique γ-harmonic function u such that $u|_{\partial\mathbb{D}} = f$. The Dirichlet-to-Neumann corresponding to γ maps the boundary values of a γ-harmonic function (Dirichlet data) to the current flux $\gamma \frac{\partial u}{\partial r}|_{r=1}$ at the boundary (Neumann data). In symbols

$$\Lambda_\gamma = \gamma \frac{\partial u}{\partial r}|_{r=1},$$

where u is γ-harmonic and $u|_{\partial\mathbb{D}} = f$. The operator $\Lambda_\gamma : H^{1/2}(\partial\mathbb{D}) \to H^{-1/2}(\partial\mathbb{D})$ is a self-adjoint pseudodifferential operator of order 1.

A discrete analog of the disk $\mathbb{D}$ is a circular planar graph. It is a finite graph $\Gamma = (V, E, \partial\Gamma)$ imbedded into $\overline{\mathbb{D}}$, where the set V is the set of nodes of the graph, the set E is the set of edges of Γ, and $\partial\Gamma = V \cap \partial\mathbb{D}$ is the non-empty set of boundary nodes of Γ. The set $V - \partial\Gamma$ is the set of interior nodes of Γ. A conductivity γ is a positive function on the edges of Γ.

A function u on the nodes of Γ is γ-harmonic if at every interior node p it satisfies Kirchhoff's law, that is: *the total current $I_u(p)$ out of p is zero*:

$$I_u(p) = \sum_{pq \in E} \gamma(pq)(u(p) - u(q)) = 0. \tag{2.2}$$

This tells that the value of u at p is the weighted average of the values of u at the neighbors of p (neighbors are the nodes q of the graph for which $pq \in E$). It follows that γ-harmonic functions satisfy the maximum and the minimum principles. From now on we will only consider the graphs in which every interior node is topologically connected to at least one boundary node. On such graphs (and only on them) each γ-harmonic function u is uniquely determined by its values $u|_{\partial\Gamma}$ on the boundary of Γ. The discrete Dirichlet-to-Neumann map Λ_γ is the linear map that sends the boundary values f of a γ-harmonic function u to the corresponding total current out of nodes at the boundary $I_u|_{\partial\Gamma}$. Or algebraically,

$$\Lambda_\gamma f(b) = I_u(b) = \sum_{bq \in E} \gamma(bq)(u(b) - u(q)), b \in \partial\Gamma, \tag{2.3}$$

where u is γ-harmonic and $u|_{\partial\Gamma} = f$.

2.2 Properties of Dirichlet-to-Neumann maps

2.2.1 Alternating sign property

One of the main motivations of this thesis is to show a strong connection between properties of discrete and continuous Dirichlet-to-Neumann maps. An important step in this direction has been made in [14], [6] and [8], where it was shown that both discrete and continuous Dirichlet-to-Neumann maps have the alternating property.

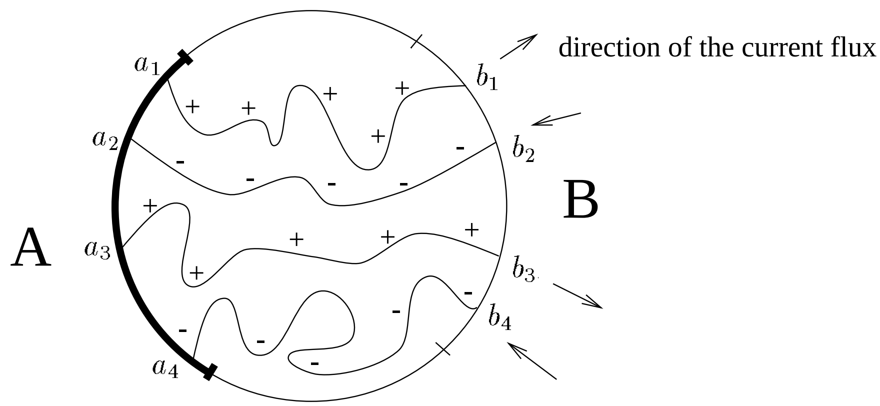
Theorem 2.2.1 (see [14] and [8]). *Let Λ_γ be a Dirichlet-to-Neumann map for $\gamma \in C^2(\overline{\mathbb{D}})$. Then Λ_γ has the alternating sign property. That is:*

Let A *and* B *be a pair of disjoint intervals on* $\partial\mathbb{D}$ *and* $f \in C^\infty(\partial\mathbb{D})$*, such that* $\mathrm{supp} f \subset A$*. Then for any* m *distinct points* $b_1, b_2, \ldots, b_m \in B$*, numbered* *clockwise,* *such that*

$$(-1)^i \Lambda f(b_i) > 0$$

there exist m *distinct points* $a_1, a_2, \ldots, a_m \in A$ *numbered* *counterclockwise,* *such that*

$$(-1)^i f(a_i) < 0.$$



This picture shows the main idea of the proof: the pattern of the directions of the current flux $\Lambda_\gamma f$ on B together with the maximum and minimum principles garantee the existence of non-intersecting curves from b's on which potential alternates in sign. Since $\mathrm{supp} f \subset A$ these curves have to terminate at A. See Section 3.1 for a detailed proof.

The same argument can be applied to the discrete case. In fact the discrete version of the alternating property, which we define next, essentially characterizes the discrete Dirichlet-to-Neumann maps.

We identify the space of real functions on $\partial\Gamma$ with $\mathbb{R}^n$, where n is the number of points in $\partial\Gamma$.

Theorem 2.2.2 ([6]). *A self-adjoint linear map* $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ *is a Dirichlet-to-Neumann map of a circular planar graph if and only if* $\Lambda 1 = 0$ *and* Λ *has the discrete alternating sign property, that is:*

Let A, B be a pair of disjoint intervals on $\partial\mathbb{D}$ and f a function on $\partial\Gamma$ with $\mathrm{supp} f \subset A$. Then for any m points ($2m \leq n$) $b_1, b_2, \ldots, b_m \in B \cap \partial\Gamma$, numbered clockwise and such that

$$(-1)^i \Lambda f(b_i) > 0$$

there exist m distinct points $a_1, a_2, \ldots, a_m \in A \cap \partial\Gamma$, numbered counterclockwise such that

$$(-1)^i f(a_i) < 0.$$

2.2.2 Right sign property of kernels

The following algebraic description of the alternating property turned out to be very useful. To state it we consider the kernel of Λ_γ.

For a discrete Dirichlet-to-Neumann map Λ_γ its kernel is the matrix that represents the linear operator Λ_γ.

The kernel of a continuous Dirichlet-to-Neumann map is the distribution $K(\phi, \theta)$ on $\partial\mathbb{D} \times \partial\mathbb{D}$ such that

$$\Lambda_\gamma f(\phi) = \int_0^{2\pi} K(\phi, \theta) f(\theta) d\theta. \tag{2.4}$$

The existence of K is garanteed by the fact that Λ_γ is a pseudodifferential operator. In fact for $\gamma \in C^2(\overline{\mathbb{D}})$ K is a continuous function off the diagonal of $\partial\mathbb{D} \times \partial\mathbb{D}$, and the singularity at the diagonal is of order 2, (see [14]). The following theorem shows the equivalence of the alternating sign property of an operator Λ and the algebraic property of the kernel of Λ.

Theorem 2.2.3 ([6]). *A symmetric matrix Λ is the kernel of a linear operator $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ that has the alternating property if and only if the kernel of Λ has the discrete right sign property. That is:*

for any two disjoint intervals $A, B \subset \partial\mathbb{D}$ and any $2m$ ($2m \leq n$) distinct points $a_1, a_2, \ldots, a_m \in A \cap \partial\Gamma$, $b_1, b_2, \ldots, b_m \in B \cap \partial\Gamma$, (as before a's are numbered counter-

clockwise and b's are numbered clockwise)

$$det\{-\Lambda(b_i, a_j)\}_{i,j=1}^m \geq 0.$$

A continuous analog of this theorem is proved in Section 3.2 (see also [14]).

Theorem 2.2.4. *Let K be a distribution on $\partial\mathbb{D} \times \partial\mathbb{D}$ such that K is continuous off the diagonal and has a singularity of order 2 on the diagonal. Then the operator*

$$\Lambda f = \int K f$$

has the alternating property if and only if K has the continuous right sign property. That is:

for any two disjoint intervals $A, B \subset \partial\mathbb{D}$ and any $2m$ distinct points $a_1, a_2, \ldots, a_m \in A$, $b_1, b_2, \ldots, b_m \in B$, (a's and b's are numbered as above)

$$det\{-K(a_i, b_j)\}_{i,j=1}^m > 0. \tag{2.5}$$

2.2.3 Main conjecture

We would like to single out the following conjecture on characterizing the kernel of a Dirichlet-to-Neumann map.

Conjecture 2.2.5. *Let $K(x, y) = \frac{k(x,y)}{|x-y|^2}$, where $(x, y) \in \partial\mathbb{D} \times \partial\mathbb{D} - \Delta$, k is continuous on $\partial\mathbb{D} \times \partial\mathbb{D}$, $k(x, x) \neq 0$, and K satisfies (2.4). Then there is a distribution $D(x, y)$ on $\partial\mathbb{D} \times \partial\mathbb{D}$, supported on the diagonal, Δ, and a regularization of K as a distribution on $\partial\mathbb{D} \times \partial\mathbb{D}$, so that $L = K + D$ is the kernel of the Dirichlet-to-Neumann map for some conductivity, γ, on $\mathbb{D}$. The distribution D is determined by the property that*

$$\int_{\partial\mathbb{D}} L(x, y) dy = 0. \tag{2.6}$$

Equation (2.6) is analogous to the fact the the Dirichlet-to-Neumann matrix for an electrical network has row sums equal to zero. This implies that the diagonal is determined by the off-diagonal terms. This is true as well in the continuous case.

2.3 *Discrete inverse problems*

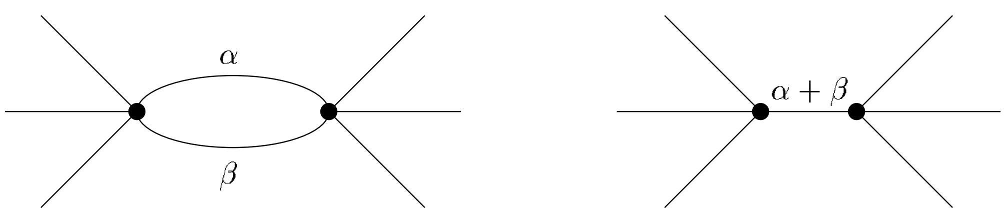
Given the Dirichlet-to-Neumann map $\Lambda(\Gamma_\gamma)$, what can be said about Γ_γ? It is well-known that the following tranformaions i-v of Γ_γ do not change its Dirichlet-to-Neumann map.

Let $p \in int\Gamma$.

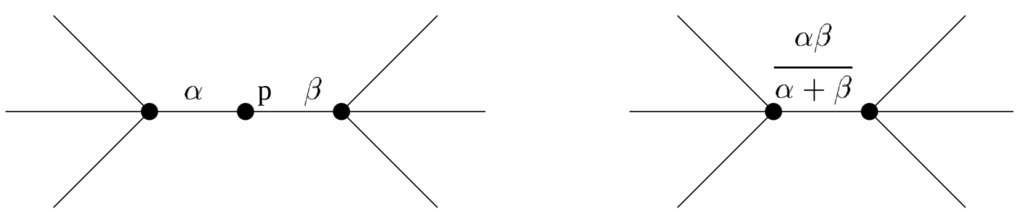
i

ii

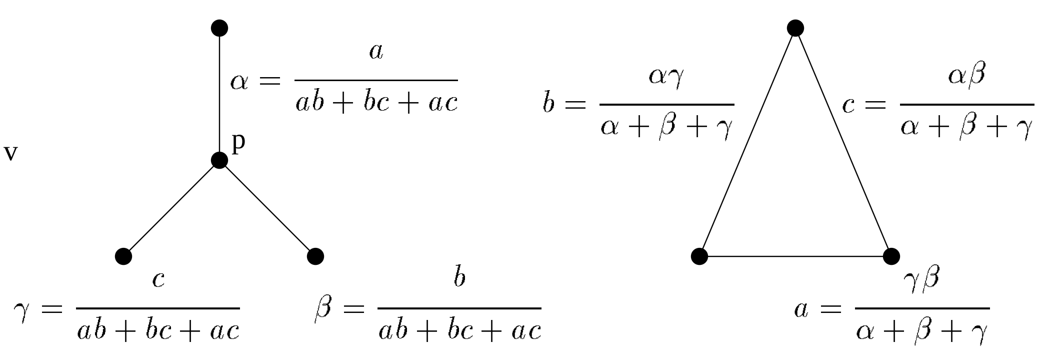
iii



iv



v



In Chapter 4 we will show that if Γ and $\tilde{\Gamma}$ are circular planar graphs and if $\Lambda(\Gamma_\gamma) = \Lambda(\tilde{\Gamma}_{\tilde{\gamma}})$ then Γ_γ can be transformed by i-v into $\tilde{\Gamma}_{\tilde{\gamma}}$ in a finite number of steps. This theorem follows from results in [6]. The proof presented in Chapter 4 is much simpler due to the new Key identity. This identity also gives an efficient algorithm for finding Γ_γ (up to transformations i-v) from $\Lambda(\Gamma_\gamma)$.

Another discrete inverse problem is the following: For what circular planar graphs Γ its Dirichlet-to-Neumann map $\Lambda(\Gamma_\gamma)$ uniquely determines γ ?

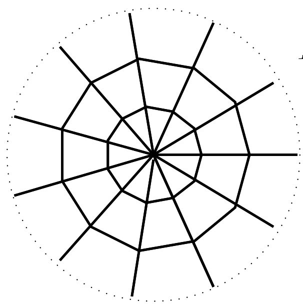
We will give a simple geometric characerization of such graphs in terms of their *medial graphs* in Chapter 4.

2.4 Layered case

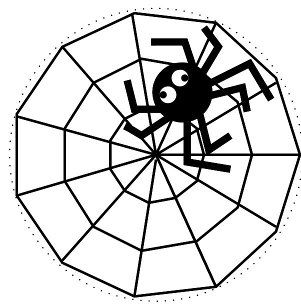
2.4.1 The layered case and the admittance function

In Chapter 5 we consider the case of conductivities that are constant on circles centered at the origin.

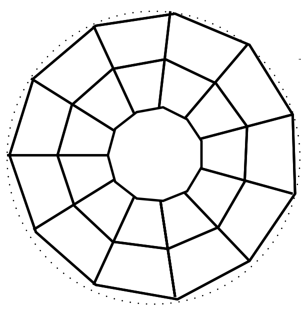
We now introduce a discrete analog of the continuous layered situation. The discrete disks are connected circular planar graphs $D(n,l)$ and $D^*(n,l)$ of the following shapes:



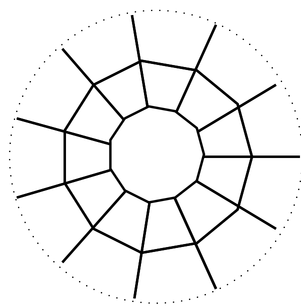
$D(11,5)$



$D(11,6)$



$D^*(11,5)$



$D^*(11,4)$

where n is the number of radial lines and l is the number of layers. The layers of the graphs $D(n,l)$ and $D^*(n,l)$ are minimal subsets of edges invariant under rotations of the graph by the angle $\frac{2\pi}{n}$. Each layer consists of n edges. We assume that the conductivity γ is constant on layers. Therefore, the layered conductivity is determined by l positive numbers.

We first describe the effect that the assumed form of γ has on Λ_γ. In both discrete and continuous situations the γ-harmonic functions are still γ-harmonic after rotations and reflections with respect to the origin. Therefore, the discrete and continuous Dirichlet-to-Neumann maps, corresponding to the layered conductivities γ, commute with rotations and the reflections of functions on the boundaries. For the continuous case it immediately follows that Dirichlet-to-Neumann maps commute with the Laplacian on the boundary of the disk $\frac{d^2}{d\theta^2}$. With a little more effort one gets that $\Lambda_\gamma 1 = 0$ and

$$\Lambda_\gamma e^{\pm ik\theta} = R(k)e^{\pm ik\theta}, k \in \mathbb{N}. \tag{2.7}$$

We call the function R the admittance function. Its values at positive integers uniquely determine Λ_γ. This reduction of layered Dirichlet-to-Neumann maps to one-dimensional objects gives us a way to interpret the notion of γ-harmonic functions when γ is a positive measure, (see Section 5.1). The corresponding Λ_γ's will map trigonometric polynomials to trigonometric polynomials.

We now will make sense of the admittance function for the discrete Dirichlet-to-Neumann maps. The discrete version of the Laplacian on the boundary of a discrete disk $D(n,l)$ or $D^*(n,l)$ is given by the $n \times n$ matrix of the form

$$[\frac{d^2}{d\theta^2}] = -\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

It makes the calculations cleaner if we assume that n is odd. Throughout this section we let $n = 2m+1, m \in \mathbf{N}$. We define

$$\partial_n = \{\frac{2\pi j}{n}, j = -m, \dots, 0, \dots, m\} \tag{2.8}$$

Direct calculation shows that $[\frac{d^2}{d\theta^2}]$ is diagonal in the orthogonal basis

$$e^{ik\theta}|_{\partial_n}, k = -m, \dots, 0, \dots, m$$

with the eigenvalues

$$-|e^{i\frac{2\pi k}{n}} - 1|^2, k = -m, \dots, 0, \dots, m$$

We define

$$\omega_k^{(n)} = \omega_{-k}^{(n)} = |e^{i\frac{2\pi k}{n}} - 1|, k = -m, \dots, 0, \dots, m. \tag{2.9}$$

Note that

$$\lim_{n\to\infty} \frac{n}{2\pi}\omega_k^{(n)} = |k|.$$

The discrete Λ_γ commute with $[\frac{d^2}{d\theta^2}]$ and we get that the eigenvectors of the discrete Dirichlet-to-Neumann maps are the restrictions of the eigenfunctions of the continuous Dirichlet-to-Neumann maps to the boundaries of the discrete disks. In symbols:

$$\Lambda_\gamma e^{\pm ik\theta}|_{\partial_n} = R(\omega_k^{(n)})e^{\pm ik\theta}|_{\partial_n}, k = 1, \ldots, m. \tag{2.10}$$

2.4.2 Characterization of admittance functions

Now, to see how "close" the discrete and continuous Dirichlet-to-Neumann maps are we need to describe their possible eigenvalues. We will give the descriptions by characterizing the discrete and continuous admittance functions.

The discrete admittance functions will turn out to be of the form of the Stieltjes' continued fractions:

$$R(\lambda) = \cfrac{1}{\cfrac{1}{\gamma_l} + \cfrac{1}{\gamma_{l-1}\lambda^2 + \cdots + \cfrac{1}{\cfrac{1}{\gamma_3} + \cfrac{1}{\gamma_2\lambda^2 + \cfrac{1}{\cfrac{1}{\gamma_1}}}}}} \tag{2.11}$$

B where γ_i's are the conductivities on the layers of the discrete disks. This explicit formula will allow us to show a one-to-one correspondence between the admittance functions of discrete disks with l layers and the Blaschke products of degree l. (This correspondence together with the Pick-Nevanlinna interpolation theorem is a key to the discrete inverse problem, see Section 4.3).

It follows from (2.11) that

$$\beta(\lambda) = \frac{R(\lambda)}{\lambda} \tag{2.12}$$

has a natural extension to an analytic function from the right half-plane $\mathbb{C}^+$ to itself. We define

$$\mathfrak{B} = \{\beta : \mathbb{C}^+ \to \mathbb{C}^+ : \beta \text{ is analytic}, \beta(\lambda) > 0 \text{ for } \lambda > 0\}.$$

Applying the Pick-Nevanlinna Interpolation theorem, (see [18],[23]), we will prove

Theorem 2.4.1. *A linear map $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ is the Dirichlet-to-Neumann map of a discrete disk if and only if Λ is diagonal in the orthogonal basis*

$$e^{ik\theta}|_{\partial_n}, k = -m, \ldots, 0, \ldots, m,$$

$\Lambda 1 = 0$ *and there is a function β in $\mathfrak{B}$ such that*

$$\Lambda e^{\pm ik\theta}|_{\partial_n} = \omega_k^{(n)} \beta(\omega_k^{(n)}) e^{\pm ik\theta}|_{\partial_n}, k = 1, \ldots, m.$$

In other words the set of the Dirichlet-to-Neumann maps is equal to

$$\{\sqrt{-[\frac{d^2}{d\theta^2}]}\beta(\sqrt{-[\frac{d^2}{d\theta^2}]}) : \beta \in \mathfrak{B}\}. \tag{2.13}$$

It turns out that a continuous analog of this theorem is true. (Our proof heavily uses the characterization of spectral measures of inhomogenuous strings done by Krein and Kac [15], see also [9].)

Theorem 2.4.2. *A linear map $\Lambda : \partial\mathbb{D} \to \partial\mathbb{D}$ is the Dirichlet-to-Neumann map of the unit disk with a layered conductivity if and only if Λ is diagonal in the orthogonal basis*

$$e^{ik\theta}, k \in \mathbb{Z},$$

$\Lambda 1 = 0$ *and there is a function β in $\mathfrak{B}$ such that*

$$\Lambda e^{\pm ik\theta} = k\beta(k) e^{\pm ik\theta}, k \in \mathbb{N}.$$

In other words the set of the Dirichlet-to-Neumann maps is equal to

$$\{\sqrt{-\frac{d^2}{d\theta^2}}\beta(\sqrt{-\frac{d^2}{d\theta^2}}) : \beta \in \mathfrak{B}\}. \tag{2.14}$$

2.4.3 The "$\gamma \leftrightarrow \frac{1}{\gamma}$" duality

We note, without a proof, that the following identity is true, (see [15] or [9])

$$\Lambda_\gamma \Lambda_{\frac{1}{\gamma}} = \Lambda_{\frac{1}{\gamma}} \Lambda_\gamma = -\frac{d^2}{d\theta^2}.$$

Definition 2.4.3. *Two disks $D_\gamma(n,l)$ and $D^*_{\frac{1}{\gamma}}(n,l)$ with conductivities on layers respectively $\{\delta_1, \xi_1, \delta_2, \xi_2, \delta_3, \dots\}$ and $\{\frac{1}{\delta_1}, \frac{1}{\xi_1}, \frac{1}{\delta_2}, \frac{1}{\xi_2}, \frac{1}{\delta_3}, \dots\}$ are called* <u>*dual*</u>.

We will show that

$$\Lambda(D^*_{\frac{1}{\gamma}})\Lambda(D_\gamma) = \Lambda(D_\gamma)\Lambda(D^*_{\frac{1}{\gamma}}) = -[\frac{d^2}{d\theta^2}].$$

Remark We find the following result amusing. It was motivated by a question of G. Uhlmann. In fact, this question together with the article [23] have stimulated our investigation of the layered case. If γ is identically 1 on $\overline{\mathbb{D}}$ then the corresponding Dirichlet-to-Neumann map as an operator is the positive square root of the minus Laplacian on $\partial\mathbb{D}$. In symbols:

$$\Lambda_1 = \sqrt{-\frac{d^2}{d\theta^2}}.$$

The question: *Is*

$$\sqrt{-[\frac{d^2}{d\theta^2}]}$$

the Dirichlet-to-Neumann map of a circular planar graph? The answer is yes. It is an easy corollary of the Theorem 2.4.1.

2.4.4 Approximation of continuous Dirichlet-to-Neumann maps by discrete ones

Remark We note that from the results in [14] it follows that the discrete disks give all possible Dirichlet-to-Neumann maps of circular planar graphs. In particular, the discrete disks with layered conductivity give all possible Dirichlet-to-Neumann maps Λ of circular planar graphs with the eigenvectors $e^{\pm ik\theta}|_{\partial_n}$. Therefore, for the purposes of the approximations of continuous Dirichlet-to-Neumann maps of the disk with layered conductivity by the discrete ones, one loses nothing essential considering only the discrete disks with layered conductivity and not all circular planar graphs.

The Pick-Nevalinna Interpolation theorem (see [18]) lets us formulate the following "continuous is the limit of discrete" theorem.

Theorem 2.4.4. *In the layered case, the eigenvectors of discrete Dirichlet-to-Neumann maps are restrictions of the eigenfunctions of the continuous Dirichlet-to-Neumann maps.*

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the first (corresponding to $e^{i\theta}, e^{2i\theta}, \ldots, e^{ik\theta}$) eigenvalues of a continuous Dirichlet-to-Neumann map. Then there exist a sequence of discrete disks $\{D_n\}$ with the first eigenvalues $\lambda_1^n, \lambda_2^n, \ldots, \lambda_k^n$ such that

$$\lambda_j = \lim_{n\to\infty} \lambda_j^n, 1 \le j \le k.$$

Conversely, let $\{D_n\}$ be a sequence of discrete disks, such that the limits above exist. Then there exists a continuous Dirichlet-to-Neumann map with the first k eigenvalues being equal to the limits.

2.4.5 Characterization of kernels. Positive definite functions

We will now show the existence and characterize the kernels of the Dirichlet-to-Neumann maps in the layered case. We show the existence by an explicit calculation of the kernel of Λ_γ in terms of the corresponding admittance function $R(\lambda)$. Recall that the layered Dirichlet-to-Neumann maps commute with rotations and reflections (with respect to the origin) of functions. It follows that the kernel of Λ_γ has to be of the convolution type:

$$K(\phi, \theta) = h(\phi - \theta)$$

where h is a distribution on $\mathbb{R}$ such that

$$h(s) = h(s + 2\pi) = h(2\pi - s), s \in \mathbb{R}$$

and

$$\int_0^{2\pi} h(s)\cos\lambda s ds = R(\lambda), \lambda > 0.$$

To proceed we need the following representation of analytic functions from the right half-plane to the right half-plane.

Theorem 2.4.5 (Herglotz). *A function β is in $\mathfrak{B}$ if and only if for some $c, C \geq 0$*

$$\beta(\lambda) = C\lambda + \frac{c}{\lambda} + \int_0^\infty \frac{\lambda(1+t^2)d\sigma(t)}{\lambda^2+t^2}$$

where σ is a positive measure of bounded variation on $(0,\infty)$.

A straightforward calculation gives us

Lemma 2.4.6. *A distribution K on $\partial\mathbb{D}\times\partial\mathbb{D}$ is the kernel of a layered Dirichlet-to-Neumann map Λ_γ if and only if*

$$K(\phi,\theta) = h(\phi-\theta)$$

where h is a distribution on $\mathbb{R}$ such that $h(s) = h(s+2\pi) = h(2\pi - s), s \in \mathbb{R}$,

$$\int_0^{2\pi} h(s)ds = 0,$$

and for $s \in [0, 2\pi)$

$$h(s) = c\delta(0) - C\delta''(0) + \int_0^\infty t\frac{e^{-st}+e^{(s-2\pi)t}}{1-e^{-2\pi t}}(1+t^2)d\sigma(t), \tag{2.15}$$

where $c, C \geq 0$ and σ is a positive measure of bounded variation on $(0,\infty)$.

Corollary 2.4.7. *The kernel of a layered Dirichlet-to-Neumann map is C^∞ off the diagonal.*

We will now explain a connection of the characterization in lemma 2.10 with the alternating property.

Definition 2.4.8. *A continuous function f on a possibly infinite interval (a, b) is* *positive definite* *if*

$$\det\{f(x_i + y_j)\}_1^m \geq 0$$

for all $m \in \mathbb{N}$, $x_i + y_j \in (a, b)$.

It follows that f is positive definite on $(0, 2\pi)$ if and only if the kernel $K(\phi, \theta) = f(\phi - \theta)$ satisfies the right sign property. We are now one step from restating the characterization of the kernels in terms of their right sign property. We need the following classical characterization of the positive definite functions. (see [16])

Theorem 2.4.9 (Bochner). *A continuous function f is positive definite on a possibly infinite interval (a, b) if and only if there exists a positive σ-finite measure ν on $\mathbb{R}$ such that*

$$f(x) = \int_{-\infty}^{+\infty} e^{xt} d\nu(t).$$

We now state one of the main resuls of Chapter 5.

Theorem 2.4.10. *A distribution K on $\partial\mathbb{D} \times \partial\mathbb{D}$ is the kernel of a layered Dirichlet-to-Neumann map Λ_γ if and only if*

$$K(\phi, \theta) = h(\phi - \theta)$$

where h is a distribution on $\mathbb{R}$ such that $h(s) = h(s + 2\pi) = h(2\pi - s), s \in \mathbb{R}$,

$$\int_0^{2\pi} h(s) ds = 0, \qquad \int_0^{2\pi} h(s)(\cos s - 1) ds < \infty$$

and h is positive definite on $(0, 2\pi)$.

2.4.6 The inverse problems

The continuous inverse problem will be reduced to an inverse Sturm-Liouville problem studied by Krein, (see [15]). We obtain the following result.

Theorem 2.4.11. *If $\gamma(r)$ is a measurable function on $[0,1]$ such that for all $\epsilon > 0$*

$$\int_\epsilon^1 (\gamma + \frac{1}{\gamma})dr < \infty$$

then γ is uniquely determined by Λ_γ a.e.. If for some $\epsilon > 0$ the integral is infinite, then no information about γ on $[0,\epsilon)$ can be obtained from Λ_γ.

Our main result on the discrete inverse problem can be roughly stated as

Theorem 2.4.12. *A layered conductivity on $D(2m+1,l)$ or $D^*(2m+1,l)$ can be recovered from the corresponding Dirichlet-to-Neumann map if and only if $l \leq m$.*

(See Section 5.2.3 for a refined version.) Theorem 2.4.12 follows from the general theory in [6] and [8]. In this paper the proof of the uniqueness and the conductivity recovery algorithm are much simpler due to the assumed form of the conductivity.

Our algorithm shows an intimate connection between the discrete inverse problem and the Pick-Nevalinna interpolation problem.

2.4.7 The case of a half plane

One often considers the Dirichlet-to-Neumann maps of the lower half plane with a conductivity that is constant on horizontal lines. For that layered case the following results hold.

Theorem 2.4.13. *The set of Dirichlet-to-Neumann maps of the half planes with layered conductivity is equal to*

$$\{\sqrt{-\frac{d^2}{dx^2}}\beta(\sqrt{-\frac{d^2}{dx^2}}) : \beta \in \mathfrak{B}\}.$$

Theorem 2.4.14. *A distribution K on $\mathbb{R} \times \mathbb{R}$ is the kernel of the Dirichlet-to-Neumann map Λ_γ of the half plane with a layered conductivity if and only if*

$$K(\phi,\theta) = h(\phi - \theta)$$

where h is a distribution on $\mathbb{R}$ such that $h(s) = h(-s)$,

$$\int_0^\infty h(s)ds = 0, \qquad \int_0^\infty h(s)(\cos s - 1)ds < \infty$$

and h is positive definite on $(0, \infty)$.

2.5 Probabilistic interpretation of Dirichlet-to-Neumann maps

Let us consider *a directed graph* $\Gamma = (V, E, \partial\Gamma)$ where V is the finite set of *nodes* of the graph, E is the set of *directed edges* and $\partial\Gamma$ is a subset of V, called *boundary* of Γ. The elements of $\partial\Gamma$ are called *boundary nodes* of Γ. The subset $int\Gamma = V - \partial\Gamma$ of V is called *interior* of Γ. The elements of $int\Gamma$ are called *interior nodes* of Γ. A node q is called a *neighbor* of a node p if there is a directed edge e from p to q.

A weighted directed (WD) graph Γ_γ is a directed graph Γ together with a positive function γ on the edges E of the graph.

We consider the following random walk on a WD graph Γ_γ. A particle starts its motion at a node of Γ. Suppose at the moment $t = n$ it occupies a node p then at the moment $t = n + 1$ it will be at a neighbor q of p. The probability of going to a particular neighbor q is proportional to the weight $\gamma(pq)$.

Given a WD graph Γ_γ let $b_1, b_2, \ldots, b_N$ denote its boundary nodes.

The *percolation matrix* of Γ_γ is the $N \times N$ matrix $X(\Gamma_\gamma) = \{x_{ij}\}$ such that

- N = number of boundary nodes of Γ

- x_{ij} = probability that the next boundary node that a particle, starting its random walk at the boundary node b_i, hits is the boundary node b_j. If b_i does not have neighbors, $x_{ij} = 0$ for all j.

One can think of a circular planar graph Γ with conductivity γ as of a WD graph $\Gamma_\gamma = \{(V, E, \partial\Gamma), \gamma\}$, such that

$$pq \in E \Leftrightarrow qp \in E,$$

and

$$\gamma(pq) = \gamma(qp).$$

Let Λ be the Dirichlet-to-Neumann map of Γ_γ. Let X be the percolation matrix of Γ_γ. Let $D = \{d_{ij}\}$ be the $N \times N$ diagonal matrix where

$$d_{ii} = \sum_{b_i p \in E} \gamma(b_i p),$$

then

$$X = I - D^{-1}\Lambda.$$

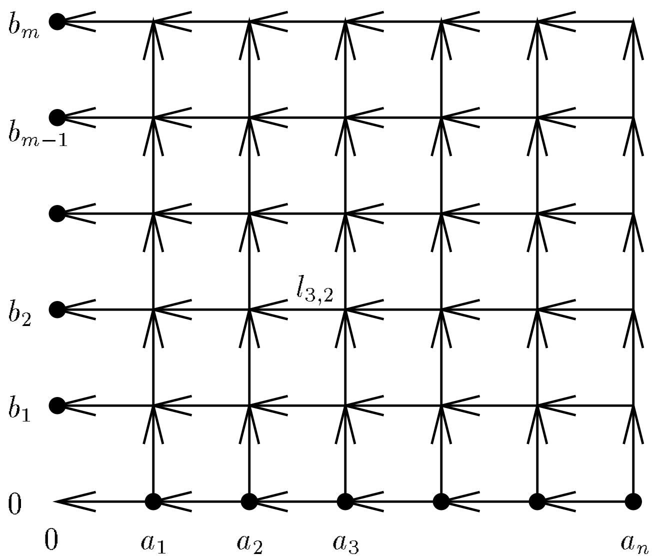
This shows that the percolation matrix of a circular planar graph with conductivity is the renormalized Dirichlet-to-Neumann map of the graph.

2.5.1 Parametrization and properties of totally positive matrices

A matrix Z is called *totally positive* if determinants of all its minors are positive. These matrices arise as restrictions of kernels of planar Dirichlet-to-Neumann maps, see section 2.2.2. Let $\mathbb{S}$ be the set of totally positive matrices with row sums strictly between 0 and 1.

We obtain a parametrization of the set $\mathbb{S}$ that gives a simple explanation to some important properties of totally positive matrices, (e.g. Hadamard's inequality).

For a pair of natural numbers n, m let $R(n, m)$ be the following graph.



$y(a_i, b_j) =$ probability that particle starting at a_i exits at b_j

WD graph R(n,m)

We will refer to the nodes of $R(n,m)$ by its coordinates. There are $n+m$ boundary nodes in $R(n,m)$ which are

$$\{a_i = (i,0)\} \cup \{b_j = (0,j)\}, \text{ where } i = 1,\dots,n, \qquad j = 1,\dots,m.$$

For $1 \le i \le n$ and $0 \le j \le m-1$ let

$l_{i,j} \in (0,1) =$ probability that a particle at (i,j) will make the next move to the left.

Let $Y(R_l) = \{y_{ij}\}$ be the $n \times m$ matrix where

$$y_{ij} = \text{ probability that a particle starting random walk at } a_i \text{ will go to } b_j.$$

Theorem 2.5.1. *Let C and D be two subsets of size k of the boundary nodes of a_i's and b_i's respectively.*

Then, $\det\{y(c,d)\}_{c\in C, d\in D} =$ *probability that k particles starting random walk at C will follow disjoint paths and exit at D.*

Theorem 2.5.2. *(Parametrization of $\mathbb{S}$)*

$$\textit{The map } \Psi = \begin{cases} (0,1)^{nm} \to \mathbb{S}, \\ l(i,j) \to Y(R_l) \end{cases} \quad \textit{is diffeomorphism onto.}$$

The proof of this theorem is in section 6.6.4.

Chapter 3

PROPERTIES OF PLANAR DIRICHLET-TO-NEUMANN MAPS

3.1 Alternating sign property

We first restate and prove a result of [6]. Suppose that $\partial\mathbb{D} = A \cup B$, where A and B are disjoint connected arcs. Then we have the following theorem.

Theorem 3.1.1. *Let f be a smooth function on $\partial\mathbb{D}$ such that $f = 0$ on A. Suppose there is a sequence of points $\{p_1, \ldots, p_n\} \subset A$ in circular order such that*

$$(-1)^{i+1}\Lambda f(p_i) > 0. \tag{3.1}$$

Then there is a sequence of points $\{q_1, \ldots, q_n\} \subset B$ in circular order such that

$$(-1)^{n}\Lambda f(p_i) f(q_i) > 0. \tag{3.2}$$

Proof. Equation (3.2) is equivalent to

$$\Lambda f(p_i) f(q_{n+1-i}) < 0. \tag{3.3}$$

We first describe how to pick the point q_n. Let u be the potential such that $u = f$ on $\partial\mathbb{D}$. By (3.1) $\frac{\partial u}{\partial n}(p_1) > 0$. Hence there is a small open line segment, α, such that $\alpha \subset \mathbb{D}$, p_1 is one end of α and $u < 0$ on α. Let W be the connected component of $\{z \in \mathbb{D} \,:\, u(z) < 0\}$ that contains α. Suppose that $\overline{W} \cap B = \emptyset$. Then $u = 0$ on ∂W. But this contradicts the maximum principle since $u < 0$ in W and $W \neq \emptyset$. Thus $\overline{W} \cap B \neq \emptyset$. Now $u = 0$ at every point of ∂W that is in $\mathbb{D}$. Using the maximum principle again we see that there is a $q_n \in \overline{W} \cap B$ such that $f(q_n) < 0$ and there is an open line segment $\beta \subset W$ such that q_n is an end point of β. Now we can connect the ends of α and β that are inside W by a smooth curve in W. Hence there is a smooth curve C_1 such that C_1 is diffeomorphic to a line segment, has end points points p_1 and q_n, and $C_1 - p_1 - q_n \subset W$. Then $u(z) < 0$ for all $z \in C_1 - p_1$. We can repeat this argument to produce curves C_j such that C_j joins p_j to a point $q_{n+1-j} \in B$, $C_j - p_j - q_{n+1-j} \subset \mathbb{D}$, and $(-1)^j u(z) < 0$ for all $z \in C_j - p_j$. These curves cannot intersect and by the Jordan curve theorem the points $p_1, \ldots, p_n, q_1, \ldots, q_n$ must be in circular order on $\partial\mathbb{D}$. It is easy to see that these points satisfy (3.3). ■

3.2 Right sign property of kernels

3.2.1 The Weak Inequality

The domain of Λ may be taken to be $H^{\frac{1}{2}}(\partial\mathbb{D})$ and the image is in $H^{-\frac{1}{2}}(\partial\mathbb{D})$. Λ is a pseudo-differential operator of order 1 and as such has a kernel, $K(x, y)$, defined as a distribution on $\partial\mathbb{D} \times \partial\mathbb{D}$. The kernel gives a representation of Λ by the formula

$$\Lambda f(x) = \int_{\partial\mathbb{D}} K(x, y) f(y) dy, \tag{3.4}$$

where x and y are arc length coordinates on $\partial\mathbb{D}$. For the pseudo-differential operator Λ, the kernel K is a symmetric function, $K(x, y) = K(y, x)$, and for a fixed $x \in \partial\mathbb{D}$, $\lim_{y \to x} |K(x, y)| = \infty$. More precisely,

$$K(x, y) = \frac{k(x, y)}{|x - y|^2} + D(x, y), \tag{3.5}$$

where k is continuous on $\partial\mathbb{D} \times \partial\mathbb{D}$, $k(x, y) = k(y, x)$, $k(x, x) \neq 0$, and D is a distribution supported on $\Delta = \{(x, x) : x \in \partial\mathbb{D}\}$. (In this formula, $|x - y|$ is the separation in arc length of points with arc length coordinates x and y and the continuous term in this expansion has been incorportated into the term $\frac{k(x,y)}{|x-y|^2}$.) If $x \notin \operatorname{supp}(f)$, then the integral is an ordinary integral and there are no convergence questions. Since we will be interested in the behaviour of $K(x, y)$ for $x \neq y$ we will ignore D and will pretend that $K(x,y) = \frac{k(x,y)}{|x-y|^2}$. The expansion (3.5) follows from Lemma 3.7 of [24] or Theorem 0.1 in [26].

We first prove the weaker statement:

Theorem 3.2.1. *Let $(x_1, \ldots, x_n; y_1, \cdots, y_n)$ be a circular pair on $\partial\mathbb{D}$. Let $L = (l_{ij})$ be the $n \times n$ matrix with entries defined by $l_{ij} = K(x_i, y_j)$. Then*

$$(-1)^{\frac{n(n+1)}{2}} \det(L) \geq 0 \tag{3.6}$$

Proof. The proof is by induction on n. We first consider $n = 1$. The proof goes by contradiction. Suppose that there are points $p, q \in \partial\mathbb{D}$ with $p \neq q$ and $K(p,q) > 0$. Then there is an $\epsilon > 0$ such that $p \notin D_\epsilon = \{y : |y-q| < \epsilon\}$ and $K(p,y) > 0$ for $y \in D_\epsilon$. Let $f(y)$ be a continuous function on $\partial\mathbb{D}$ such that $\text{supp}(f) \subset D_\epsilon = \{y : |y-q| < \epsilon\}$, $f(q) > 0$, and $f(s) \geq 0$ for all $s \in \partial\mathbb{D}$. Then

$$\gamma(p)\frac{\partial u}{\partial n}(p) = \Lambda f(p) = \int_{D_\epsilon} K(p,y)f(y)dy > 0,$$

where u is γ-harmonic and $u(s) = f(s),\ s \in \partial\mathbb{D}$ But then there must be a point z near p in $\mathbb{D}$ such that $u(z) < 0$. This contradicts the maximum principle.

Next we assume that the result is true for all $(n-1)\times(n-1)$ matrices and prove that it is true for $n \times n$ matrices. If the result is not true, then we have a circular pair $(x_1, \ldots, x_n; y_1, \cdots, y_n)$ such that

$$(-1)^{\frac{n(n+1)}{2}} \det(L) < 0. \tag{3.7}$$

Consider the matrix L^{-1} with entries (h_{ij}). Then

$$h_{ij} = (-1)^{i+j}\frac{\det(L_{ij})}{\det(L)}, \tag{3.8}$$

where L_{ij} is the (i,j) minor of L. By induction, (3.7), and (3.8)

$$(-1)^{i+j+\frac{n(n-1)}{2}+\frac{n(n+1)}{2}+1}h_{ij} = (-1)^{i+j+n+1}h_{ij} \geq 0. \tag{3.9}$$

Since L is nonsingular, for fixed i there must be some j for which

$$(-1)^{i+j+n+1} h_{ij} > 0. \tag{3.10}$$

Now let $w = [1, -1, 1, \ldots, (-1)^{n+1}]^T$ be an n-vector with alternating signs. Let $z = L^{-1}w$. Then using (3.9) and (3.10) it is easy to verify that

$$(-1)^{i+n} z_i > 0. \tag{3.11}$$

To summarize, we have a vector z such that

$$(-1)^{i+1} = w_i = \sum_{j=1}^{n} K(x_i, y_j) z_j \tag{3.12}$$

and

$$(-1)^{n+1} z_i w_i > 0. \tag{3.13}$$

Now, choose small intervals D_j around the points y_j such that the D_j are disjoint and do not contain any of the points x_i. Choose the D_j so small that

$$|K(x_i, y) - K(x_i, y_j)| < \epsilon, \; y \in D_j, \; i = 1, \ldots, n. \tag{3.14}$$

Also choose functions f_j such that

$$\operatorname{supp}(f_j) \subset D_j, \; z_j f_j(y) \geq 0, \text{ and } \int_{D_j} f_j = z_j. \tag{3.15}$$

Let $f = \sum f_j$. Then

$$\begin{aligned} |\Lambda f(x_i) - w_i| &= \left| \int_{\partial\mathbb{D}} K(x_i, y) f(y) dy - \sum_{j=1}^{n} K(x_i, y_j) z_j \right| \\ &= \left| \int_{\partial\mathbb{D}} (K(x_i, y) - K(x_i, y_j)) f(y) dy \right| \\ &\leq \epsilon \sum_{j=1}^{n} |z_i|. \end{aligned} \tag{3.16}$$

Thus we conclude that for ϵ small enough $\Lambda f(x_i)$ has the same sign as w_i. By the alternating property, there would have to be a set of n points t_i in circular order such that

$$(-1)^n w_i f(t_i) > 0. \tag{3.17}$$

For such a set of points we would have to have $t_i \in D_i$ and hence $f(t_i)$ would have the same signs as z_i. This contradicts (3.13).

■

3.2.2 *The Strong Inequality*

We now prove the strong version of Theorem 3.2.1. We consider the cases $n = 1$ and $n > 1$ separately. Let us assume the arc length of $\partial\mathbb{D}$ is S and that points on $\partial\mathbb{D}$ are parametrized by the numbers in the interval $[0, S)$. When $n = 1$, suppose there is a pair of points x_1, y_1 with $0 \leq x_1 < y_1$ and $K(x_1, y_1) = 0$. By (3.5) there is no sequence of points z_j such that $x_1 < z_j < y_1$, $\lim_{j\to\infty} z_j = x_1$, and $\lim_{j\to\infty} K(x_1, z_j) = 0$. Hence there is a point η_2 with $x_1 < \eta_2 < y_1$ such that

$$K(x_1, \eta_2) = 0, \text{ and } K(x_1, \eta) < 0, \text{ for } x_1 < \eta < \eta_2.$$

Let x be any number such that $x_1 < x < \eta_2$ and choose η_1 so that $x < \eta_1 < \eta_2$. Then $(x_1, x; \eta_1, \eta_2)$ is a circular pair and hence

$$\begin{vmatrix} K(x_1, \eta_1) & K(x_1, \eta_2) \\ K(x, \eta_1) & K(x, \eta_2) \end{vmatrix} \leq 0. \tag{3.18}$$

Since

$$K(x_1, \eta_2) = 0, \; K(x, \eta_2) \leq 0, \text{ and } K(x_1, \eta_1) < 0,$$

it follows that

$$K(x, \eta_2) = 0. \tag{3.19}$$

This shows that for *all* x, with $x_1 < x < \eta_2$, $K(x, \eta_2) = 0$. Hence we get the contradiction that $\lim_{x \to \eta_2} K(x, \eta_2) = 0$.

The proof for $n > 1$, makes use of the following result in [6]. It was later pointed out to us that Charles Dodgson (Lewis Carroll) used a version of this identity in [2]. Let $(x_1, \ldots, x_n; y_1, \ldots, y_n)$ be a circular pair. We assume that the coordinates on $\partial\mathbb{D}$ are chosen so that $0 \leq x_1 < \cdots < x_n < y_1 < \cdots < y_n < S$. Let L be the matrix with i, j entry equal to $K(x_i, y_j)$. We will use the notation

$$\kappa(x_1, \ldots, x_n; y_1, \ldots, y_n) = \det(L). \tag{3.20}$$

Lemma 3.2.2. *Let $(a_1, \ldots, a_{n+1}; b_1, \ldots, b_{n+1})$ be a circular pair. Then*

$$\begin{aligned} \kappa(a_1, \ldots, a_{n+1}; b_1, \ldots, b_{n+1})\kappa(a_1, \ldots, a_{n-1}; b_3, \ldots, b_{n+1}) = \\ \kappa(a_1, \ldots, a_n; b_1, b_3, \ldots, b_{n+1})\kappa(a_1, \ldots, a_{n-1}, a_{n+1}; b_2, \ldots, b_{n+1}) \\ - \kappa(a_1, \ldots, a_n; b_2, \ldots, b_{n+1})\kappa(a_1, \ldots, a_{n-1}, a_{n+1}; b_1, b_3, \ldots, b_{n+1}) \end{aligned} \tag{3.21}$$

Assume that

$$\kappa(x_1, \ldots, x_n; y_1, \ldots, y_n) = 0 \tag{3.22}$$

for some circular pair. First we claim that there is no sequence of points z_j such that $x_n < z_j < y_1$, $\lim_{j \to \infty} z_j = x_n$, and $\lim_{j \to \infty} \kappa(x_1, \ldots, x_n; z_j, y_2, \ldots, y_n) = 0$. For this would imply that there are constants c_k (independent of j) so that

$$K(x_n, z_j) = \sum_{k<n} c_k K(x_k, z_j), \tag{3.23}$$

and hence

$$\lim_{j \to \infty} K(x_n, z_j) = \sum_{k<n} c_k K(x_k, x_n), \tag{3.24}$$

contradicting (3.5). Thus there is a number η_1 with $x_n < \eta_1 < y_1$ such that

$$\kappa(x_1, \ldots, x_n; \eta_1, y_2, \ldots, y_n) = 0 \text{ and} \tag{3.25}$$

$$\kappa(x_1, \ldots, x_n; \eta, y_2, \ldots, y_n) \neq 0, \text{ for } x_n < \eta < \eta_1. \tag{3.26}$$

Let x be such that $x_n < x < \eta_1$. Then there is an η such that $x < \eta < \eta_1$ and hence $(x_1, \ldots, x_n, x; \eta, \eta_1, y_2, \ldots, y_n)$ is a circular pair. By (3.21), (3.22), and (3.6)

$$\begin{aligned} 0 \geq \kappa(x_1, \ldots, x_n, x; \eta, \eta_1, y_2, \ldots, y_n)\kappa(x_1, \ldots, x_{n-1}; y_2, \ldots, y_n) = \\ \kappa(x_1, \ldots, x_n; \eta, y_2, \ldots, y_n)\kappa(x_1, \ldots, x_{n-1}, x; \eta_1, y_2, \ldots, y_n) \\ - \kappa(x_1, \ldots, x_n; \eta_1, y_2, \ldots, y_n)\kappa(x_1, \ldots, x_{n-1}, x; \eta, y_2, \ldots, y_n) \\ = \kappa(x_1, \ldots, x_n; \eta, y_2, \ldots, y_n)\kappa(x_1, \ldots, x_{n-1}, x; \eta_1, y_2, \ldots, y_n) \geq 0. \end{aligned} \tag{3.27}$$

Using this and (3.26) we see that

$$\kappa(x_1, \ldots, x_{n-1}, x; \eta_1, y_2, \ldots, y_n) = 0, \text{ for } x_n < x < \eta_1. \tag{3.28}$$

As above, this contradicts (3.5) and proves the theorem.

3.3 The Hopf Lemma

We now show how the fact that $K(x, y) < 0$ for $x \neq y$ implies the Hopf lemma (reference) for the conductivity equation.

Theorem 3.3.1. *Let u be a non constant solution of $\nabla(\gamma\nabla u) = 0$, and let $p \in \partial\mathbb{D}$ be a point where u assumes a minimum. Then*

$$\frac{\partial u}{\partial n}(p) < 0 \tag{3.29}$$

Proof. We may assume that $u(p) = 0$. Let $f = u|_{\partial\mathbb{D}}$. Since u is not constant, $\mathrm{supp}(f)$ is not empty. Thus there is an interval D around p in $\partial\mathbb{D}$ such that $\mathrm{supp}(f) - D$ is not empty. Let ψ be a smooth function on $\partial\mathbb{D}$ such that $\psi = 1$ on $\mathrm{supp}(f) - D$, $\psi = 0$ on an interval around p, and $0 \leq \psi \leq 1$. Let $g = \psi f$ and let v be the solution of $\nabla(\gamma\nabla v) = 0$ with $v|_{\partial\mathbb{D}} = g$. Since $f \geq g$ it follows that $u \geq v$. It is also true that $g \geq 0$. Since $p \notin \mathrm{supp}(g)$ and $K(p, y) < 0$,

$$0 > \int_{\partial\mathbb{D}} K(p, y)g(y)dy = \gamma(p)\frac{\partial v}{\partial n}(p) \geq \gamma(p)\frac{\partial u}{\partial n}(p), \tag{3.30}$$

which proves the theorem. ■

3.4 The Variation Diminishing Property

We will use the following notation. Let $M(x,y)$ be a continuous function on $[c,d] \times [a,b]$. Let $c \le x_1 < x_2 < \cdots < x_n \le d$, $a \le y_1 < y_2 < \cdots < y_n \le b$. Let T be the $n \times n$ matrix with i,j entry equal to $M(x_i, y_j)$. Let

$$\mu(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) = \det(T).$$

The following lemma from [11] is sometimes paraphrased by saying that the kernel M has the *variation diminishing property.* It will be used to show that the strong inequalities of the form 3.6 imply the alternating sign property.

Lemma 3.4.1. *Let f be a continuous, not identically 0, function defined on the interval $[a,b]$, such that f changes its sign on this interval no more than $n-1$ times. Let $M(x,y)$, $x,y \in [c,d] \times [a,b]$, be a continuous kernel with the property that*

$$\mu(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) > 0, \tag{3.31}$$

whenever $c \le x_1 < x_2 < \cdots < x_n \le d$, $a \le y_1 < y_2 < \cdots < y_n \le b$. Then the function

$$g(x) = \int_a^b M(x,y) f(y) dy$$

vanishes in $[c,d]$ no more than $n-1$ times.

By saying that function f changes its sign k times on the interval $[a,b]$ we mean that there are $k+1$ points $x_1 < x_2 < \cdots < x_{k+1}$ in $[a,b]$ such that for $i = 1, 2, \ldots, k$

$$f(x_i) f(x_{i+1}) < 0. \tag{3.32}$$

Proof. By hypothesis there are points $a = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = b$ such that in each interval (s_{i-1}, s_i), $i = 1, 2, \ldots, n$ function f does not change its sign and is not identically 0. For $i = 1, 2, \ldots, n$ let

$$g_i(x) = \int_{s_{i-1}}^{s_i} M(x, y) f(y) dy. \tag{3.33}$$

Then

$$g(x) = \sum_{i=1}^{n} g_i(x). \tag{3.34}$$

For any $c \le x_1 < x_2 < \cdots < x_n \le d$ the determinant

$$\det(\{g_i(x_j\}) = \int_{s_{n-1}}^{s_n} \cdots \int_{s_0}^{s_1} \mu(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) f(y_1) \ldots f(y_n) dy_1 \ldots dy_n \tag{3.35}$$

is not 0 since the integrand is not identically zero and has constant sign. This shows that there is no non-trivial linear combination of g_i's vanishing at n points and hence that $g(x) = \sum_{i=1}^{n} g_i(x)$ cannot vanish at n points. ■

We note that this proof only used the fact that $\mu(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n)$ has constant sign. We need one more lemma before coming to the proof of the alternating principal.

Let $K(x, y)$ be a kernel on $\partial\mathbb{D} \times \partial\mathbb{D}$. We assume that $K(x, y)$ is continuous when $x \neq y$, but we don't assume anything about K on the diagonal of $\partial\mathbb{D} \times \partial\mathbb{D}$. Let $\kappa(x_1, \ldots, x_n; y_1, \ldots, y_n)$ be defined as in section 4.

Lemma 3.4.2. *Suppose that $\kappa(x_1, \ldots, x_n; y_1, \ldots, y_n)$ is never zero and has constant sign for all circular n-pairs $(x_1, \ldots, x_n; y_1, \ldots, y_n)$. Let $\partial\mathbb{D} = I \cup J$ where I and J are disjoint connected arcs. Let f be a continuous function on $\partial\mathbb{D}$ with supp$(f) \subset J$. Let*

$$g(x) = \int_{\partial\mathbb{D}} K(x, y) f(y) dy. \tag{3.36}$$

Then if there is a sequence of $n+1$ points in I in circular order at which g alternates in sign, then there is a sequence of at least $n+1$ points in J in circular order at which f alternates in sign.

Proof. If there is no sequence of $n+1$ points of J at which f alternates in sign, then f can change its sign no more than $n-1$ times in J. By Lemma 3.4.1, g can vanish no more than $n-1$ times in I. But we are assuming that g has $n+1$ alternations of sign in I and hence at least n zeros in I. This contradiction proves the lemma. ■

We now state and prove the theorem.

Theorem 3.4.3. *Using the notation of Lemma 3.4.2, suppose that*

$$(-1)^{\frac{n(n+1)}{2}}\kappa(x_1,\ldots,x_n;y_1,\ldots,y_n)>0 \tag{3.37}$$

for all $n>0$ and all circular n-pairs $(x_1,\ldots,x_n;y_1,\ldots,y_n)$. Let f be a continuous function on $\partial\mathbb{D}$ with supp$(f)\subset J$. Let

$$g(x)=\int_{\partial\mathbb{D}}K(x,y)f(y)dy. \tag{3.38}$$

Suppose there is a sequence of points $\{p_1,\ldots,p_n\}\subset I$ in circular order such that

$$(-1)^{i+1}g(p_i)>0 \tag{3.39}$$

Then there is a sequence of points $\{q_1,\ldots,q_n\}\subset J$ in circular order such that

$$(-1)^{n}g(p_i)f(q_i)>0. \tag{3.40}$$

Proof. By Lemma 3.4.2 there is a sequence of points in J at which f alternates in sign. If there is no sequence with the desired alteration property then J is a disjoint union of subintervals J_i, in circular order, such that

1. f is not identically 0 on J_i, $i=1,\ldots,n$,

2. f does not change its sign on $J_i,\ i = 1, \ldots, n$

3. for some $z_i \in J_i$,

$$(-1)^{n+i} f(z_i) > 0. \tag{3.41}$$

We use the idea of Lemma 3.4.1. For $i = 1, 2, \ldots, n$ let

$$g_i(x) = \int_{J_i} K(x, y) f(y) dy. \tag{3.42}$$

Then

$$g(x) = \sum_{i=1}^{n} g_i(x). \tag{3.43}$$

Let

$$G = \begin{bmatrix} g_1(x_1) & g_2(x_1) & \ldots & g_n(x_1) \\ g_1(x_2) & \ldots & & g_n(x_2) \\ \vdots & & & \vdots \\ g_1(x_n) & \ldots & & g_n(x_n) \end{bmatrix}. \tag{3.44}$$

Let u be the n-vector with $u_i = 1,\ i = 1, \ldots, n$. Then

$$Gu = \begin{bmatrix} g(x_1) \\ g(x_2) \\ \vdots \\ g(x_n) \end{bmatrix}. \tag{3.45}$$

Using (3.41) we will show that the signs of u are all negative. This contradiction will prove the theorem. We need to compute the signs of the entries of G^{-1}. Rather than get lost in a cloud of indices, we will give the proof in the case that $n = 3$ and leave the general proof to the reader. In this case the assumption (3.41) implies that $f(y) \geq 0$ in J_1, $f(y) \leq 0$ in J_2, and $f(y) \geq 0$ in J_3. As in section 3.2.1 we compute the signs of the cofactors of G. First we have

$$\det(G) = \int_{J_1} \int_{J_2} \int_{J_3} \kappa(x_1, x_2, x_3; y_1, y_2, y_3) f(y_1) f(y_2) f(y_3) dy_1 dy_2 dy_3 < 0. \tag{3.46}$$

We find that

$$\begin{vmatrix} g_2(x_2) & g_3(x_2) \\ g_2(x_3) & g_3(x_3) \end{vmatrix} = \int_{J_2}\int_{J_3} \kappa(x_2, x_3; y_2, y_3) f(y_2) f(y_3) dy_2 dy_3 > 0. \tag{3.47}$$

Hence $(G^{-1})_{11} < 0$. Next we compute that

$$(-1)^{1+2} \begin{vmatrix} g_1(x_2) & g_3(x_2) \\ g_1(x_3) & g_3(x_3) \end{vmatrix} = \int_{J_1}\int_{J_3} \kappa(x_2, x_3; y_1, y_3) f(y_1) f(y_3) dy_1 dy_3 > 0, \tag{3.48}$$

and thus $(G^{-1})_{21} < 0$. Continuing the calculation we find that the signs of G^{-1} are as follows

$$G^{-1} = \begin{bmatrix} - & + & - \\ - & + & - \\ - & + & - \end{bmatrix}. \tag{3.49}$$

This yields the contradiction

$$\begin{bmatrix} - & + & - \\ - & + & - \\ - & + & - \end{bmatrix} \begin{bmatrix} + \\ - \\ + \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{3.50}$$

■

Chapter 4

CIRCULAR PLANAR GRAPHS

4.1 Background

We first need to establish some geometric properties of circular planar graphs.

4.1.1 Basic transformations of graphs

Two graphs Γ_1 and Γ_2 are called electrically equivalent if they have the same sets of Dirichlet-to-Neumann maps.

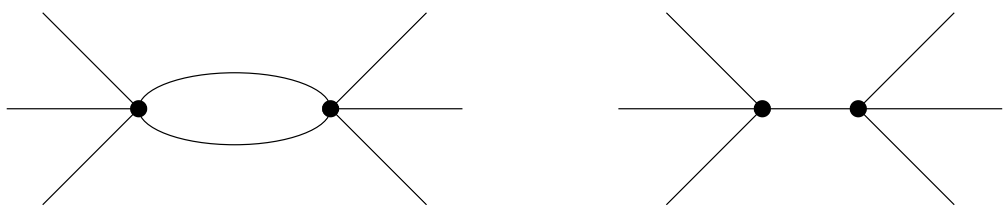
It is well known that if Γ_1 can be transformed to Γ_2 by a sequence of the five following transformations then Γ_1 and Γ_2 are electrically equivalent.

In the following p is an interior node.

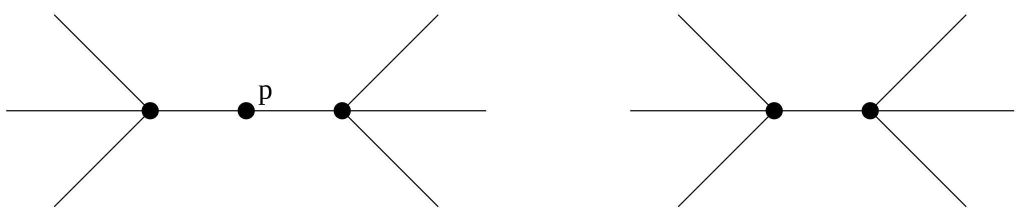
i

ii

iii



iv



v

We will prove that for the circular planar graphs the strong converse is true, i.e. if the sets of Dirichlet-to-Neumann maps of Γ_1 and Γ_2 intersect then Γ_1 can be transformed to Γ_2 by a sequence of the transformations i-v. We will call two graphs *equivalent under transformations i-v* if one of the graphs can be transformed to another by a finite sequence of transformations i-v. We will call this equvalence relation *induced* by the transformations i-v. First, we will describe what is the set of circular planar graphs quotiented by the iequivalence relation induced by the transformations i-v. To do that we consider

4.1.2 Medial graphs

A *medial graph* M is a circular planar graph such that its boundary nodes are 1-valent and its interior nodes a 4-valent.

The name "medial" comes from the following construction that for each circular planar graph Γ produces corresponding medial graph $\mathbb{M}(\Gamma)$:

Suppose $\Gamma = (V, E, \partial\Gamma)$ is a circular planar graph with n boundary nodes. Γ is assumed to be embedded in in the closed unit disk $\bar{\mathbb{D}}$ so that the boundary nodes $v_1, v_2, \ldots, v_n$ occur in clockwise order around a circle $C = \partial\mathbb{D}$ and the rest of Γ is in the interior of $\mathbb{D}$. The construction of the medial graph $\mathbb{M}(\Gamma)$ is similar to that in [12] (p 239). The medial graph $\mathbb{M}(\Gamma)$ depends on the embedding. First, for each edge e of Γ, let m_e be its midpoint. Next, place $2n$ boundary points $t_1, t_2, \ldots t_{2n}$ on C so that

$$t_1 < v_1 < t_2 < t_3 < v_2 < \ldots < t_{2n-1} < v_n < t_{2n} < t_1$$

in the clockwise circular order around C.

(1) The vertices of $\mathbb{M}(\Gamma)$ consist of the points m_e for $e \in E$, and the points t_i for $i = 1, 2, \ldots, 2n$.

(2) The edges in $\mathbb{M}(\Gamma)$ are as follows. Two vertices m_e and m_f are joined by an edge whenever e and f have a common vertex and e and f are incident to the same face in Γ. There is also one edge for each point t_j as follows. The point t_{2i} is joined by an edge to m_e where e is the edge of the form $e = v_i r$ which comes first after arc $v_i t_{2i}$ in clockwise order around v_i. The point t_{2i-1} is joined by an edge to m_f where f

is the edge of the form $f = v_i s$ which comes first after arc $v_i t_{2i-1}$ in counter-clockwise order around v_i.

The vertices of the form m_e of $\mathbb{M}(\Gamma)$ are 4-valent; the vertices of the form t_i are 1-valent.

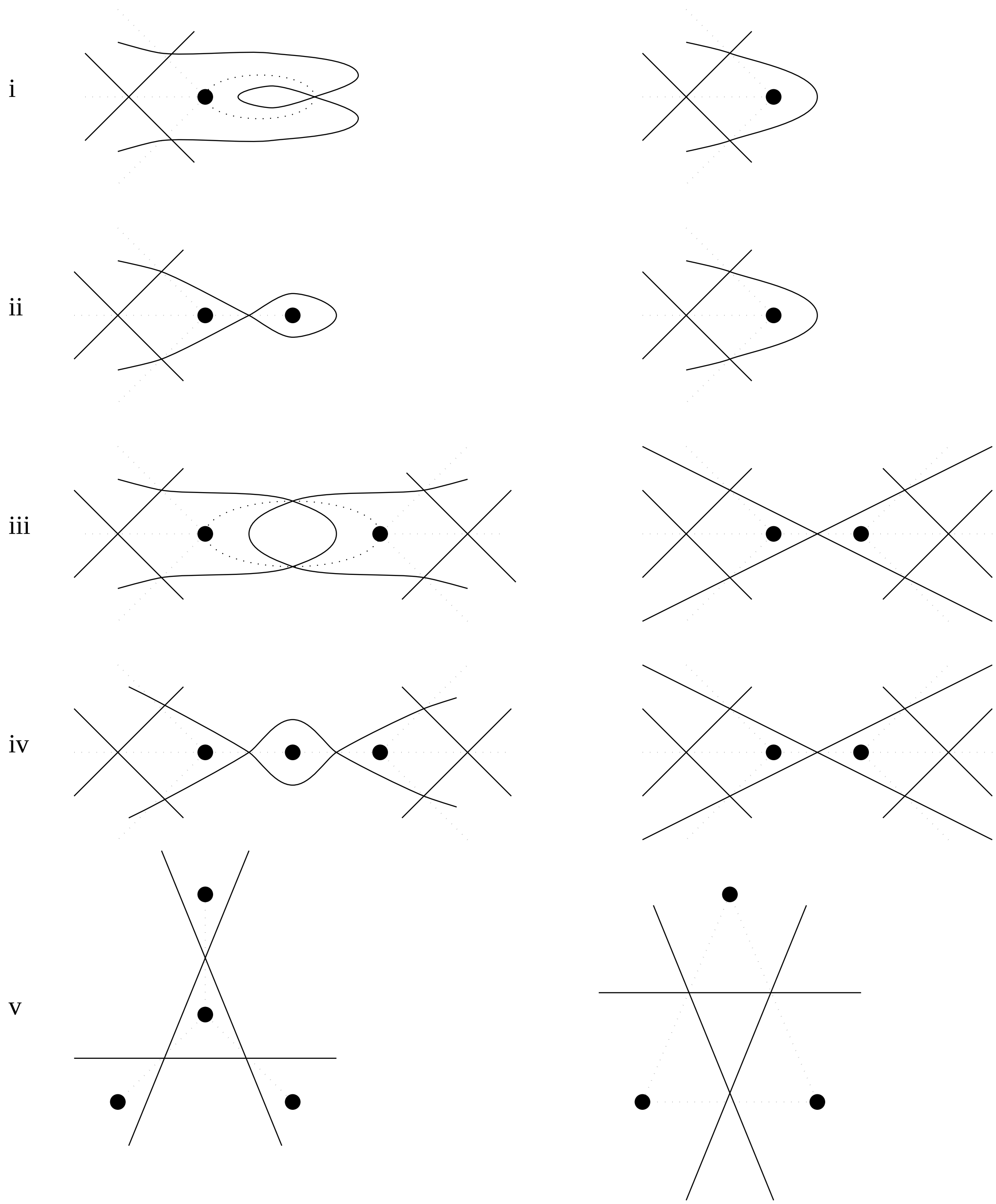
An edge uv of a medial graph M has a direct extension vw if the edges uv and vw separate the other two edges incident to the vertex v. A path $u_0 u_1 \dots u_k$ in M is called a geodesic arc if each edge $u_{i-1} u_i$ has edge $u_i u_{i+1}$ as a direct extension. A geodesic arc $u_0 u_1 \dots u_k$ is called a geodesic if either

(1) u_0 and u_k are points on the circle C.

or

(2) $u_k = u_0$ and $u_{k-1} u_k$ has $u_0 u_1$ as direct extension.

The following picture shows transformations of $\mathbb{M}(\Gamma)$ corresponding to the transformations i-v of Γ.



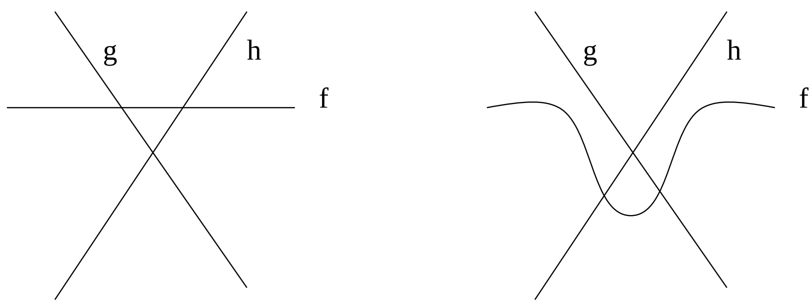
A subgraph $\mathbb{L}$ of M is called a lens provided that:

(1) $\mathbb{L}$ consists of a simple closed path path $u_0 u_1 \ldots u_k v_0 v_1 \ldots v_m u_0$ and all the nodes and edges of M in the bounded connected component of the complement of $\mathbb{L}$ in the plane.

(2) $u_0u_1 \dots u_kv_0$ and $v_0v_1 \dots v_mu_0$ are two geodesic arcs such that no inner edge of $\mathbb{L}$ is incident to u_0 or v_0.

If each geodesic in M begins and ends on C, has no self-intersection, and if M has no lenses, we will say that M is lensless.

A triangle in M is a triple $\{f, g, h\}$ of geodesics which intersect to form a triangle with no other intersections within the configuration, as in the picture below.



Suppose $\{f, g, h\}$ form a triangle as in the picture above. A motion of $\{f, g, h\}$ consists of interchanging the configurations above.

Lemma 4.1.1. *Two circular planar graphs are $Y - \Delta$ equivalent if and only if their medial graphs are equivalent under motions.*

Proof. Each $Y - \Delta$ transformation of Γ corresponds to a motion on $\mathbb{M}(\Gamma)$. Conversely, a motion on $\mathbb{M}(\Gamma)$ corresponds to a $Y - \Delta$ transformation of Γ. ∎

We shall make extensive use of the following Lemma. Our proof is an adaptation of a proof of Steinitz to our situation; see [12]and [22].

Lemma 4.1.2. *Suppose M is lensless medial graph. Suppose g and h intersect at p. Suppose g intersects C at q and h intersects C at r. Assume $\mathbb{F} = \{f_1, \dots, f_m\}$ is a set of geodesics with the property that for each $1 \leq i \leq m$, f_i intersects g between p and q if and only if f_i intersects h between p and r. Then a finite sequence of motions will remove all members of $\mathbb{F}$ from the sector qpr.*

Proof. For each $i = 1, \ldots, m$, let v_i be the point of intersection (if there is one) of f_i with g between p and q. For each f_i which intersects another of the f_j within sector qpr, let d_i be the first point of intersection on f_i after v_i in sector qpr. Let $D = \{d_i\}$ be the set of points obtained in this way. If D is empty, let f_i be the geodesic in $\mathbb{F}$ such that v_i is closest to p, and $\{g, h, f_i\}$ form a triangle. A motion will remove f_i from sector qpr. Otherwise, D is nonempty. Each point $d_i \in D$ is the point of intersection of two of the geodesics, say f_i and f_j. Let d be a point in D for which the number of regions within the configuration formed by f_i and f_j and g is a minimum. This minimum must be one, or there would be another geodesic which intersects f_i between v_i and d or which intersects f_j between v_j and d. Then $\{g, f_i, f_j\}$ form a triangle. A motion will reduce the number of regions within sector qpr. After a finite number of motions, no f_i will cross into the sector. ■

Lemma 4.1.3. *Suppose M is a medial graph that has a lens. Then M is equivalent under motions to a medial graph $\tilde{M}$ that has an empty lens.*

Proof. Suppose g and h are two geodesics which intersect at p_1 and p_2 to form a lens $\mathbb{L}$. WLOG assume that $\mathbb{L}$ is a lens with the fewest number of regions inside $\mathbb{L}$. Each geodesic f which intersects g between p_1 and p_2 also intersects h between p_1 and p_2, or there would be a lens with fewer regions than $\mathbb{L}$. An argument similar to that of 4.1.2 shows that all of these f's may be removed from $\mathbb{L}$. ■

This process will be called "clearing the lens".

Corollary 4.1.4. *Every medial graph can be made lensless by a finite sequence of transformations i-v.*

Proof. If a medial graph has a lens, by the previous lemma we can assume that the lense is empty. Then applying one of the transformations i-iv we can eliminate the lens. Every such elimination decreases the number of interior points of M. Therefore, M can be made lensless by a finite number of steps. ∎

Corollary 4.1.5. *Every circular planar graph* Γ *is electrically equivalent to a circular planar graph* $\tilde{\Gamma}$ *such that* $\mathbb{M}(\tilde{\Gamma})$ *is lensless.*

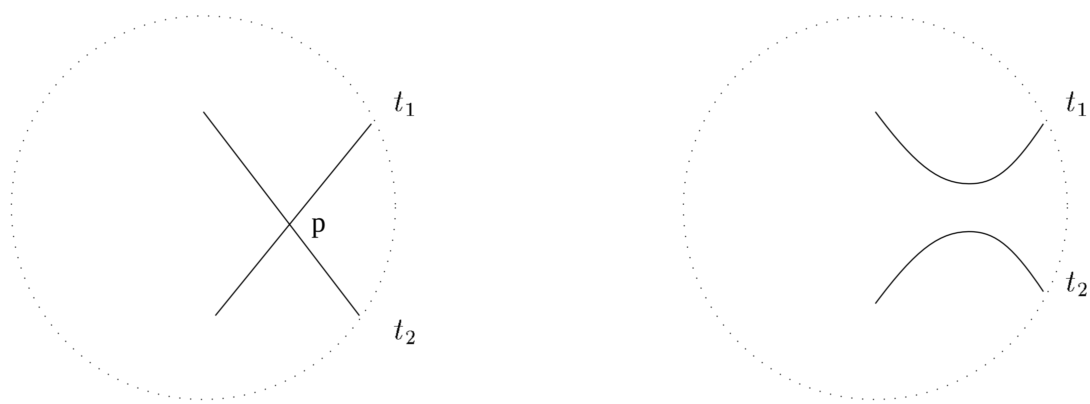
4.1.3 Z-sequences

Let M be a lensless medial graph. Then M will have n geodesics each of which intersects C twice. The n geodesics intersect C in $2n$ distinct boundary points. These $2n$ points are labelled $t_1, \ldots, t_{2n}$, so that

$$t_1 < t_2 < t_3 < \ldots < t_{2n-1} < t_{2n} < t_1$$

are in circular order around C. The geodesics will be labelled as follows. Let g_1 be the geodesic which begins at t_1. The remaining geodesics are labelled g_2, g_3, ..., g_n so that if $i < j$, then the first point of intersection of g_i with C occurs before the first point of intersection of g_j with C in clockwise order starting from t_1. For each $i = 1, 2, \ldots, 2n$, let z_i be the number associated with the geodesic which intersects C at t_i. In this way we obtain a sequence $z(M) = z_1, z_2, \ldots, z_{2n}$, called the z-sequence for M. Each of the numbers from 1 to n occurs in z exactly twice.

We now need to define a new transformation on a lensless medial graph M. Let t_1 and t_2 be two adjacent boundary nodes of M such that the geodesics from t_1 and t_2 intersect at an interior node p (if that is the case then it can always be read from $z(M)$). Then using the idea of "clearing lenses" in lemma 4.1.3 we can transform M by motions so that no other geodesic intersects the triangle $t_1 p t_2$. That is, near the boundary points t_1 and t_2 the transformed graph looks like the picture on the left.

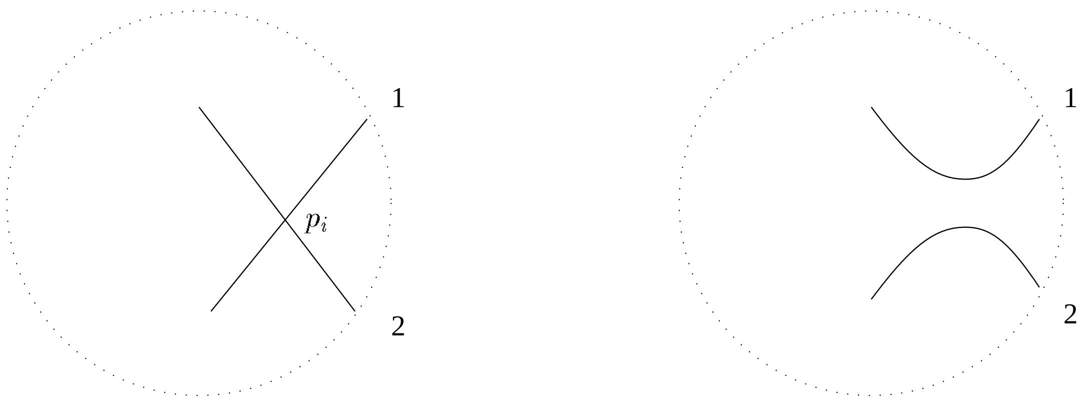


The transformation from left to right will be called *unwinding between* t_1 *and* t_2, the inverse transformation, defined if the geodesics from t_1 and t_2 are different and do not intersect in a lensless graph, will be called *winding between* t_1 *and* t_2. After winding or unwinding the medial graph is still lensless and its z-sequence changes by one transposition.

Lemma 4.1.6. *Two lensless medial graphs* M_1 *and* M_2 *are equivalent under motions if and only if* $z(M_1) = z(M_2)$.

Proof. Obviously, motions of a medial graph do not change its z-sequence.

We show the other direction by an induction on the number of interior nodes of the medial graphs. Clearly, the theorem is true if M_1 or M_2 have no interior points. Now, suppose they have at least one. Then not all geodesics in M_1 or M_2 are parallel. WLOG we can assume that none of the geodesics of M_1 or M_2 terminate at two adjacent boundary nodes, that is there are no two equal adjacent symbols in $z(M_1)$ or $z(M_2)$. Therefore, WLOG we can assume that the geodesics that go through boundary nodes 1 and 2 intersect in an interior node p_i in M_i, $i = 1, 2$. The trick of "clearing lenses" in lemma 4.1.3 shows that by a finite sequence of motions all other geodesics can be moved out of the triangle $12p_i$. Therefore, WLOG the medial graphs look like the following figure near boundary points 1 and 2.



The unwinding transformation above produces two new lensless medial graphs with equal z-sequences. By the inductive statement, since these new medial graphs have fewer interior points, they are equivalent under motions, and therefore, so are the original graphs. ■

4.1.4 Connections and Z-sequences. Key identity

Let Γ be a circular planar graph. A *path* β between boundary nodes a and b of Γ is either an edge (ab) or a sequence of interior nodes $p_1, \ldots, p_m$ such that

$$(ap_1), (p_1p_2), \ldots, (p_{m-1}p_m), (p_m b)$$

are edges of Γ.

A *disjoint connection* α between two disjoint k-tuples of boundary nodes $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ is a set of pairwise disjoint paths α_i between a_i's and b_i's.

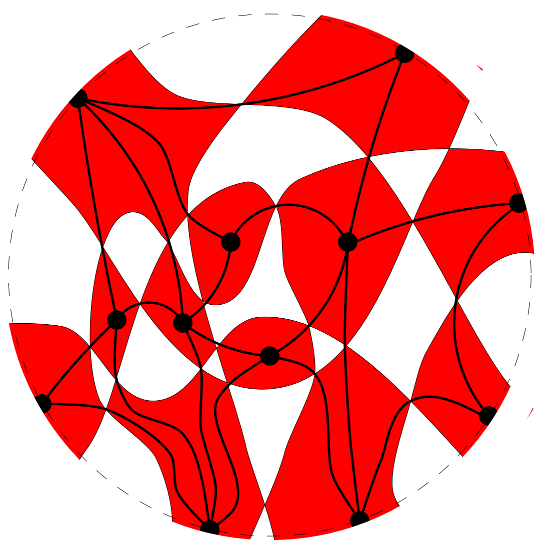
The following theorem, proved in [6], shows that the existense of disjoint connections between non-interlacing k-tuples of boundary nodes of Γ on C can be read directly from a Dirichlet-to-Neumann map $\Lambda(\Gamma_\gamma)$.

Theorem 4.1.7. *(see [6]) Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be a disjoint pair of non-interlacing boundary nodes of Γ. Then there is a disjoint connection between a_i's and b_i's if and only if*

$$\det\{\Lambda(a_i, b_j)\} \neq 0.$$

We now extend the notion of disjoint connection to medial graphs.

A *face* of a medial graph M is a connected component of $\bar{\mathbb{D}} - M$. Due to the valences of the nodes in M one can color the faces of M in black and white so that no two faces with the same edge are of the same color (so called 2-coloring). If transformations i-v are applied to a colored medial graph then the new coloring is always chosen to respect the old one. If $M = \mathbb{M}(\Gamma)$ then one can choose the 2-coloring of M so that a face is black if and only if it contains a node of Γ. Let us call this coloring *induced.*



The boundary nodes of M split C in $2n$ intervals $I_1, \ldots, I_{2n}$. A 2-coloring of M induces a 2-coloring of the intervals.

For the rest of the section let c and d be two points in two distinct intervals I_k

and I_j. Let $C - \{c, d\} = A \cup B$ where A and B are connected disjoint arcs. Let I and J be two black intervals on the boundary such that $I \subset A$ and $J \subset B$. A *path* G between I and J is a sequence of black faces $F_1, \ldots, F_m$ such that $I \in \bar{F}_1$, $J \in \bar{F}_m$, $\bar{F}_i \cap \bar{F}_{i+1} \neq \phi$, $\bar{F}_2, \ldots, \bar{F}_{m-1} \cap C = \phi$ and c and d are not in the closures of F_i's.

Let I_i and J_i be two disjoint k-tuples of the black intervals, such that $I_i \subset A$ and $J_i \subset B$. A *disjoint connection* between I_i's and J_i's is a sequence of pairwise disjoint paths G_i between I_i's and J_i's.

The definitions above are chosen so that the following lemma is true.

Lemma 4.1.8. *Let Γ be a circular planar graph. Suppose $M = \mathbb{M}(\Gamma)$ is its medial graph with the induced coloring. Let $\{a_i\} \in A$ and $\{b_i\} \in B$ be two disjoint k-tuples of boundary nodes of Γ. Let I_i and J_i be corresponding black intervals. Then there is a disjoint connection between a_i's and b_i's if and only if there is a disjoint connection between I_i's and J_i's.*

Proof. From the construction of $\mathbb{M}(\Gamma)$ one has 1-1 correspondence between nodes of Γ and black faces of $\mathbb{M}(\Gamma)$. Moreover, interior nodes of Γ correspond to black faces that do not touch C. This induces the 1-1 correspondence between disjoint connections in Γ and $\mathbb{M}(\Gamma)$. ∎

Lemma 4.1.9. *The existence of disjoint connections is invariant under the transformations i-v.*

Proof. The check of several simple cases is left to the reader. ∎

We have by theorem 4.1.7 that a Dirichlet-to-Neumann map $\Lambda(\Gamma_\gamma)$ gives complete information about disjoint connections between non-interlacing k-tuples of boundary intervals in $\mathbb{M}(\Gamma)$. The following identity provides a link between existence of the disjoint connections and the z-sequence of a medial graph M.

Theorem 4.1.10. *(Key Identity) Let $Max(A)$ be the size of the biggest disjoint connection between a set of black intervals in A and a set of black intervals in $B = C - \bar{A}$. Let $Black(A)$ = the number of black intervals in A. Let $R(A)$ be the number of geodesics that start and terminate at A. let $card(A)$ be the number of boundary nodes of M in A. Then*

$$R(A) = card(A) - Black(A) - Max(A). \tag{4.1}$$

Proof. Let t_1 and t_2 be two adjacent boundary nodes of M so that $\{t_1, t_2\} \subset A$ or B. The following observation is crucial: windings or unwindings between t_1 and t_2 do not change $Max(A, B)$, $Black(A)$ or $R(A)$. By a sequence of such windings and unwindings M can be transformed to a lenless medial graph without interior nodes. For that graph the identity is trivially true. By the observation it is true for a general lensless medial graph. ■

We are now in a position to prove a corollary that is crucial for disrete inverse problems.

Corollary 4.1.11. *Two lensless medial graphs have the same disjoint connections if and only if they have the same z-sequence.*

Proof. By the lemma 4.1.6 two lensless medial graphs with the same z-sequence are equivalent under motions. So, one direction follows from the invariance of disjoint connections under the transformations i-v. Suppose now that two medial graphs have the same disjoint connections. Then by the key identity, for every connected arc A on C both graphs have the same $R(A)$.

Let $t_i < t_j$ be two boundary nodes of M. Consider $\epsilon > 0$ such that there are no other boundary nodes of M in $(t_i - \epsilon, t_i + \epsilon)$ and $(t_j - \epsilon, t_j + \epsilon)$. Then there is a geodesic in M with the endpoints t_i and t_j if and only if

$$R(t_i - \epsilon, t_j + \epsilon) - 1 = R(t_i + \epsilon, t_j + \epsilon) = R(t_i - \epsilon, t_J - \epsilon) = R(t_i + \epsilon, t_j - \epsilon).$$

This implies the equality of the z-sequnces of the graphs. ■

Corollary 4.1.12. *The set of medial graphs quotiented by the equivalence relation induced by the transformations i-v is in 1-1 correspondence with the set of z-sequences.*

4.2 Inverse problems

4.2.1 Finding shape

What can be said about a circular planar graph Γ given $\Lambda(\Gamma_\gamma)$ for some γ? The results of the previous section give a simple answer to this question:

Theorem 4.2.1. *Let Γ and $\tilde{\Gamma}$ be two circular planar graphs. Then*

$$\Lambda(\Gamma_\gamma) = \Lambda(\tilde{\Gamma}_{\tilde{\gamma}})$$

for some γ and $\tilde{\gamma}$ if and only if Γ and $\tilde{\Gamma}$ are equivalent under transformations i-v.

Proof. One direction is obvious. Now, suppose that $\Lambda(\Gamma_\gamma) = \Lambda(\tilde{\Gamma}_{\tilde{\gamma}})$. It follows that Γ and $\tilde{\Gamma}$ have the same disjoint connections, and therefore $\mathbb{M}(\Gamma)$ and $\mathbb{M}(\tilde{\Gamma})$ have the same disjoint connections. From corollary 4.1.11 $\mathbb{M}(\Gamma)$ and $\mathbb{M}(\tilde{\Gamma})$ have the same z-sequence, and, therefore, are i-v equivalent. By the correspondence of transformations we are done. ■

4.2.2 Finding conductivities

Another discrete inverse problem is the following: For what circular planar graphs Γ does its Dirichlet-to-Neumann map $\Lambda(\Gamma_\gamma)$ uniquely determine γ ?

First, it is clear that $\mathbb{M}(\Gamma)$ has to be lensless. We will show in this section that this is a sufficient condition. To do that we need the following calculations from [6]. First,

Lemma 4.2.2. *([6]) Let* Γ *and* $\tilde{\Gamma}$ *be two* $Y-\Delta$ *equivalent graphs, then* $\Lambda : \gamma \to \Lambda(\Gamma_\gamma)$ *is injective if and only if* $\Lambda : \gamma \to \Lambda(\tilde{\Gamma}_\gamma)$ *is injective.*

A *boundary-to-boundary edge* in Γ is an edge between two adjacent boundary nodes of Γ. A *spike* (bp) is an edge between a 1-valent boundary node b and an interior node p.

There are two operations on Γ corresponding to unwindings on $\mathbb{M}(\Gamma)$ (two because of two ways to 2-color $\mathbb{M}(\Gamma)$).

(1) breaking of a boundary-to-boundary edge is removing this boundary-to-boundary edge,

(2) collapsing a spike (bp) is removing the spike and making p boundary node.

Lemma 4.2.3. *(see [6]) Let* Γ_γ *be a circular planar graph. Suppose* $\tilde{\Gamma}_{\tilde{\gamma}}$ *is obtained from* Γ_γ *by breaking of a boundary-to-boundary edge or collapsing a spike of conductivty* β*. Then* $\Lambda(\tilde{\Gamma}_{\tilde{\gamma}})$ *can be explicitly calculated from* $\Lambda(\Gamma_\gamma)$ *and* β*.*

Moreover, if $\tilde{\Gamma}$ *has fewer disjoint connections than* Γ *then* β *can be explicitly calculated from* $\Lambda(\Gamma_\gamma)$*.*

Theorem 4.2.4. *Let* Γ *be a circular planar graph. Then the map* $\Lambda : \gamma \to \Lambda(\Gamma_\gamma)$ *is injective if and only if* $\mathbb{M}(\Gamma)$ *is lensless.*

Proof. We will prove this theorem by induction on the number of conductors in Γ (number of interior nodes in $\mathbb{M}(\Gamma)$). Let M be the medial graph of Γ. Not all geodesic of M are parallel and, therefore, WLOG M can be unwinded between two nodes t_1 and t_2. The new medial graph will have a different z-sequence and therefore fewer connections. The unwinding in M corresponds to breaking of a boundary-to-boundary edge or collapsing a spike in Γ. By the previous lemma its value can be calculated from $\Lambda(\Gamma_\gamma)$ and moreover the new Dirichlet-to-Neumann map can be found. All other conductors can be calculated by the inductive hypothesis. ■

Corollary 4.2.5. *Let Γ_1 and Γ_2 be two circular planar graphs. Then the sets of Dirichlet-to-Neumann maps of Γ_1 and Γ_2 intersect if and only if Γ_1 can be transformed to Γ_2 by a sequence of the transformations i-v.*

Chapter 5

LAYERED CASE

5.1 Continuous problem

In this chapter we consider the case of conductivities that are constant on circles centered at the origin. Our assumption that γ depends only on r makes it possible to reduce our subject of study to a 1-dimensional one.

5.1.1 Reduction to a 1-dimensional problem

Lemma 5.1.1. *The solution to the Dirichlet problem on $\mathbb{D}$ with the boundary data $e^{ik\theta}, k \in \mathbb{Z}$ is of the form*

$$u_k(r,\theta) = a_k(r)e^{ik\theta}$$

Proof. Suppose u is γ-harmonic and $u|_{\partial\mathbb{D}} = e^{ik\theta}$. Since the conductivity is constant on circles, for all $\epsilon > 0$

$$v(r,\theta) = u(r,\theta+\epsilon) - u(r,\theta)$$

is also γ-harmonic and $v|_{\partial\mathbb{D}} = e^{ik\theta}(e^{ik\epsilon} - 1)$. Hence, by the uniqueness of the solution of the Dirichlet problem,

$$u_k(r,\theta)(e^{ik\epsilon} - 1) = u_k(r,\theta+\epsilon) - u_k(r,\theta)$$

$$\Rightarrow u_k(r,\theta)e^{ik\epsilon} = u_k(r,\theta+\epsilon)$$

$$\Rightarrow u_k(r,\theta) = a_k(r)e^{ik\theta}.$$

■

Corollary 5.1.2. *Λ_γ and $\frac{d^2}{d\theta^2}$ have the same eigenfunctions*

$$e^{ik\theta}, k \in \mathbb{Z}.$$

We have that for $k \in \mathbb{N}$

$$\begin{cases} a_k(0) = 0, \\ a_k(1) = 1. \end{cases}$$

Writing (2.1) in polar coordinates gives

$$\frac{d}{dr}\gamma(r)r\frac{d}{dr}a_k(r) - k^2\frac{\gamma(r)}{r}a_k(r) = 0. \tag{5.1}$$

The eigenvalue of Λ_γ corresponding to $e^{ik\theta}$ and $e^{-ik\theta}$ is

$$R(k) = \gamma\frac{da_k}{dr}(1).$$

We make the change of variable

$$x = \int_r^1 \frac{dt}{t\gamma(t)}.$$

Let $x_\infty = \int_0^1 \frac{dt}{t\gamma(t)} \leq \infty$. We have

$$\frac{d^2}{dx^2}a_k(x) = k^2\gamma^2(x)a_k(x), x \in (0, x_\infty), \tag{5.2}$$

$$\begin{cases} a_k(0) = 1, \\ a_k(x_\infty) = 0, \end{cases}$$

and

$$R(k) = -\frac{da_k}{dx}(0).$$

The investigations by Krein and Kac, outlined in the next section, show that the admittance function R is well-defined even if γ^2 in (5.2) is replaced by a positive measure. This fact allows us to extend the map $\gamma \to \Lambda_\gamma$ to conductivities given by positive measures. (See [20] for a similar extension made by geophysicists).

We will view from now on Λ_γ's as linear operators from trigonometric polynomials to trigonometric polynomials, given by

$$\begin{cases} \Lambda_\gamma 1 = 0, \\ \Lambda_\gamma e^{\pm ik\theta} = R(k)e^{\pm ik\theta}, k \in \mathbb{N}. \end{cases}$$

5.1.2 Small Vibrations of Strings

We will give now, without proofs, an outline of some results of Krein and Kac, see also [9].

Definition 5.1.3. *A string is a pair lm, where l is the lenght of the string* $(0 < l \leq \infty)$ *and*

$$m = m(x), x \in [0, l]$$

is a non-decreasing function with

$$0 \leq m(x) < \infty \text{ for } 0 \leq x < l$$

The value of m at z represents the mass of the interval $[0, x]$.

We note that this definition is slightly different from the one in [15] and [9], instead of considering different ways of attaching the right end of the string we allow a weightless interval at that end.

If the right end l of the string is fixed, and a pulsating force

$$F = A \sin \sqrt{\zeta} t, \zeta \notin \mathbb{R}$$

is applied to the left end in the direction perpendicular to the x-axis, the forced oscillation of the left end satisfies the law

$$y = H(\zeta) A \sin \sqrt{\zeta} t.$$

The function H is called the coefficient of dynamic compliance of the string.

The amplitude function of the oscillation satisfies the following integral equation

$$\psi(x,\zeta) = \psi(0,\zeta) + \psi'_{-}(0,\zeta)x - \zeta \int_0^x (x-s)\psi(s,\zeta)dm(s). \tag{5.3}$$

If m has the density $\rho(x) = dm/dx$ this equation has an equivalent differential form

$$\frac{1}{\rho(x)} \frac{d^2}{dx^2}\psi(x,\zeta) = -\zeta\psi(x,\zeta). \tag{5.4}$$

The integral form makes the general characterizations below possible.

Let $\phi(x,\zeta)$ and $\theta(x,\zeta)$ be the solutions of (5.3) with the boundary conditions

$$\begin{cases} \tilde{\phi}(0,\zeta) = 1 \\ \dfrac{d\tilde{\phi}}{dx}(0,\zeta) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\theta}(0,\zeta) = 0 \\ \dfrac{d\tilde{\theta}}{dx}(0,\zeta) = 1. \end{cases}$$

For every $x \in [0, x_\infty)$ the functions $\phi(x,\zeta)$ and $\theta(x,\zeta)$ are entire functions of ζ. The coefficient of dynamic compliance is determined by these fundamental solutions

$$H(\zeta) = \lim_{x\to l} \frac{\theta(x,\zeta)}{\phi(x,\zeta)}.$$

Note that

$$\psi(x,\zeta) = \phi(x,\zeta) - \frac{1}{\mathbb{D}(\zeta)}\theta(x,\zeta)$$

is the solution of (5.3) with

$$\begin{cases} \psi(0,\zeta) = 1, \\ \psi(l,\zeta) = 0. \end{cases}$$

The following fundamental theorem is proved in [15], see also [9].

Theorem 5.1.4. *For every function of the form*

$$H(\zeta) = C - \frac{c}{\zeta} + \int_0^\infty \frac{(1+t^2)d\sigma(t)}{t^2-\zeta}, \zeta \in \mathbb{C} - [0,+\infty),$$

where σ is a positive measure of bounded variation on $(0,\infty)$ there exists a unique string for which H serves as the coefficient of dynamic compliance. And for every string its coefficient of dynamic compliance is of this form.

The measure σ is essentially the spectral measure of the operator $\frac{1}{\rho(x)}\frac{d^2}{x^2}$. In particular for any $x, y \in [0, x_\infty)$

$$\int_0^\infty \phi(x,\zeta)\phi(y,\zeta)(1+\zeta^2)d\sigma(\zeta) = \delta(x-y).$$

Corollary 5.1.5. *The formula*

$$\beta(\zeta) = \zeta H(-\zeta^2)$$

gives a one-to-one correspondence between coefficients of dynamic compliance of strings and analytic functions

$$\beta : \mathbb{C}^+ \to \mathbb{C}^+$$

with $\beta(\zeta) > 0$ *for* $\zeta > 0$.

Proof. Herglotz's theorem in Introduction. ■

5.1.3 Corollaries of the results of Krein and Kac

The definition of the Dirichlet-to-Neumann maps in section 5.1 and corollary 5.1.5 immediately imply Theorem 2.4.2.

We will now show that the uniqueness in Theorem 5.1.4 implies Theorem 2.4.11.

For a conductivity γ satisfying the hypothesis of Theorem 2.4.11 we put in correspondence the string lm with

$$l = \int_0^1 \frac{dr}{r\gamma(r)} \leq \infty,$$

and

$$\rho(\int_r^1 \frac{dt}{t\gamma(t)}) = \gamma^2(r).$$

The admittance function corresponding to γ and the coefficient of dynamic compliance of the string are connected by

$$R(\lambda) = \frac{1}{H(-\lambda^2)}. \tag{5.5}$$

By Corollary 5.1.5 we have that

$$\frac{R(\lambda)}{\lambda} : \mathbb{C}^{+} \to \mathbb{C}^{+}$$

is analytic in $\mathbb{C}^{+}$ and, therefore, is determined by its values at integers (see [21]). Therefore, the Dirichlet-to-Neumann map Λ_{γ} determines the coefficient of dynamic compliance $\mathbb{D}(\lambda)$ of the correspondent string. Theorem 5.1.4 garantees a unique corresponding string with density ρ. The conductivity can be found then by

$$\gamma(e^{-\int_0^x \sqrt{\rho(t)}dt}) = \sqrt{\rho(x)}. \tag{5.6}$$

5.1.4 Characterization of kernels. Positive definite functions

We will now explore the role of positive definite functions.

Lemma 5.1.6. *Let*

$$f(s) = -\int_0^{\infty} t\frac{e^{-st} + e^{(s-2\pi)t}}{1 - e^{-2\pi t}}(1+t^2)d\sigma(t), \qquad s \in (0, \pi), \tag{5.7}$$

where σ is a positive measure of bounded variation on $(0, \infty)$, then

$$\int_0^{\pi} f(s)(\cos \lambda s - 1)ds = \int_0^{\infty} \frac{\lambda^2(1+t^2)d\sigma(t)}{\lambda^2 + t^2}$$

for $\lambda \in \mathbb{C}^{+}$.

Proof. Tonelli's theorem. ■

We are now in a position to prove Theorem 2.4.11.

Proof. The fact that the kernel of a layered Dirichlet-to-Neumann map has the properties of the Theorem 2.4.11 follows directly from the Herglotz's theorem and the lemma 5.1.6. We now show the other direction.

By the Bochner's theorem in Section 2.4, if $-h$ is positive definite on $(0, 2\pi)$, there exists a σ-finite measure ν on $\mathbb{R}$ such that

$$h(s) = -\int_{-\infty}^{+\infty} e^{-st} d\nu(t), s \in (0, 2\pi).$$

Since

$$h(s) = h(2\pi - s),$$

there exists a σ-finite measure τ on $(0, \infty)$ such that

$$h(s) = -\frac{1}{2}\int_{-\infty}^{+\infty} e^{-st} + e^{-(2\pi - s)t} d\nu(t) = -\int_0^{\infty} e^{-st} + e^{-(2\pi-s)t} d\tau(t).$$

Since

$$\int_0^{\pi} f(s)(\cos s - 1) ds < \infty,$$

by lemma 3.6 with $\lambda = 1$ and $d\tau(t) = \frac{t(1+t^2)}{1-e^{-2\pi t}} d\sigma(t)$

$$\int_0^{\infty} \frac{1 - e^{-2\pi t}}{t(1+t^2)} d\tau(t) < \infty.$$

$$\Rightarrow f(s) = -\int_0^{\infty} t\frac{e^{-st} + e^{(s-2\pi)t}}{1 - e^{-2\pi t}}(1+t^2) d\sigma(t), \qquad s \in (0, \pi)$$

for a positive measure of bounded total variation σ. Invoking of lemma 5.1.6 and Theorem 5.1.4 finishes the proof. ■

The arguments above can be easily transformed to give theorems 2.4.13 and 2.4.14 for the half-plane. Lemma 5.1.6 should be replaced by

Lemma 5.1.7. *Let*

$$g(s) = -\int_0^{\infty} te^{-st}(1+t^2) d\sigma(t), \qquad s \in (0, \infty), \tag{5.8}$$

where σ is a positive measure of bounded variation on $[0, \infty)$, then

$$\int_0^{\infty} g(s)(\cos \lambda s - 1) ds = \int_0^{\infty} \frac{\lambda^2(1+t^2) d\sigma(t)}{\lambda^2 + t^2}$$

for $\lambda \geq 0$.

5.2 Discrete problem

We will proceed in a manner similar to the continuous case.

5.2.1 Reduction to a 1-dimensional problem

Lemma 5.2.1. *The solution to the Dirichlet problem on D_n with the boundary data $e^{ik\theta}|_{\partial_n}$ is of the form*

$$u_k(r,\theta) = a_k(r)e^{ik\theta}$$

Proof. Suppose u is γ-harmonic and $u|_{\partial_n} = e^{ik\theta}$. Since the conductivity is constant on layers

$$v(r,\theta) = u(r,\theta + \frac{2\pi}{n}) - u(r,\theta)$$

is also γ-harmonic and $v|_{\partial_n} = e^{ik\theta}(e^{i\frac{2\pi k}{n}} - 1)$. Hence, by the uniqueness of the solution of the Dirichlet problem,

$$u_k(r,\theta)(e^{i\frac{2\pi k}{n}} - 1) = u_k(r,\theta + \frac{2\pi}{n}) - u_k(r,\theta)$$

$$\Rightarrow u_k(r,\theta)e^{i\frac{2\pi}{n}} = u_k(r,\theta + \frac{2\pi}{n})$$

$$\Rightarrow u_k(r,\theta) = a_k(r)e^{ik\theta}.$$

■

Corollary 5.2.2. *Λ_γ and $[\frac{d^2}{d\theta^2}]$ have the same eigenvectors.*

We will now derive an explicit formula for the eigenvalues of Λ_γ in terms of γ. We will do it in a way that emphasizes the relevance of the Sturm-Liouville and beads-on-a-string inverse problems ([11]) to discrete impedance tomography. Let us first consider the case of $D(n,l)$ with an odd l.

Let $\{\delta_1, \xi_1, \delta_2, \xi_2, \ldots, \delta_{\frac{l+1}{2}}\}$ denote the conductivities on layers of $D(n,l)$ starting from the origin. For $k \neq 0$

$$\begin{cases} a_k(0) = 0, \\ a_k(1) = 1, \\ \delta_j(a_k(r_j) - a_k(r_{j-1})) + \delta_{j+1}(a_k(r_j) - a_k(r_{j+1})) + \xi_j a_k(\omega_k^{(n)})^2 = 0. \end{cases}$$

Let $P(\lambda, r_j)$ be the unique solution of the following problem:

$$\begin{cases} P(\lambda, 0) = 0, \\ P(\lambda, r_1) = 1, \\ \delta_j(P(\lambda, r_j) - P(\lambda, r_{j-1})) + \delta_{j+1}(P(\lambda, r_j) - P(\lambda, r_{j+1})) + \lambda^2 \xi_j P(\lambda, r_j) = 0. \end{cases}$$

Let

$$Q(\lambda, r_j) = \delta_j(P(\lambda, r_j) - P(\lambda, r_{j-1})).$$

Then

$$\lambda_k^{(n)} = \frac{Q(\omega_k^{(n)}, 1)}{P(\omega_k^{(n)}, 1)}.$$

We have also

$$\begin{cases} P(\lambda, r_j) = P(\lambda, r_{j-1}) + \dfrac{1}{\delta_j} Q(\lambda, r_j), \\ Q(\lambda, r_j) = Q(\lambda, r_{j-1}) + \xi_{j-1}\lambda^2 P(\lambda, r_{j-1}). \end{cases} \tag{5.9}$$

Therefore,

$$\frac{Q(\lambda, r_j)}{P(\lambda, r_j)} = \frac{Q(\lambda, r_j)}{P(\lambda, r_{j-1}) + \dfrac{1}{\delta_j} Q(\lambda, r_j)}$$

$$= \frac{1}{\dfrac{1}{\delta_j} + \dfrac{P(\lambda, r_{j-1})}{Q(\lambda, r_j)}} = \frac{1}{\dfrac{1}{\delta_j} + \dfrac{P(\lambda, r_{j-1})}{Q(\lambda, r_{j-1}) + \xi \lambda^2 P(\lambda, r_{j-1})}}$$

$$= \cfrac{1}{\cfrac{1}{\delta_j} + \cfrac{1}{\xi_{j-1}\lambda^2 + \cfrac{Q(\lambda, r_{j-1})}{P(\lambda, r_{j-1})}}}. \tag{5.10}$$

Let

$$R(\lambda) = \cfrac{1}{\cfrac{1}{\delta_{\frac{l+1}{2}}} + \cfrac{1}{\xi_{\frac{l-1}{2}}\lambda^2 + \cdots + \cfrac{1}{\cfrac{1}{\delta_3} + \cfrac{1}{\xi_2\lambda^2 + \cfrac{1}{\cfrac{1}{\delta_2} + \cfrac{1}{\xi_1\lambda^2 + \delta_1}}}}}} \tag{5.11}$$

Then the eigenvalues $\lambda_k^{(n)}$ of Λ_γ are

$$\lambda_k^{(n)} = R(\omega_k^{(n)}).$$

To get the similar formula for other disks one should make corresponding δ_1 or $\dfrac{1}{\delta_{\frac{l+1}{2}}}$ or both zero.

5.2.2 Characterizations of the Dirichlet-to-Neumann maps

We consider the function

$$\beta(\lambda) = \frac{1}{\lambda}R(\lambda) = \cfrac{1}{\cfrac{1}{\delta_{\frac{l+1}{2}}}\lambda + \cfrac{1}{\xi_{\frac{l-1}{2}}\lambda + \cdots + \cfrac{1}{\cfrac{1}{\delta_3}\lambda + \cfrac{1}{\xi_2\lambda + \cfrac{1}{\cfrac{1}{\delta_2}\lambda + \cfrac{1}{\xi_1\lambda + \cfrac{1}{\cfrac{1}{\delta_1}\lambda}}}}}}}. \tag{5.12}$$

The function β has the following properties:

1. β is rational,
2. $\beta(\lambda) : \mathbb{C}^+ \to \mathbb{C}^+$,
3. $\beta(\lambda) > 0$ for $\lambda > 0$,
4. $\beta(-\bar{\lambda}) = -\bar{\beta(\lambda)}$.

It turns out that these four properties characterize the continued fractions of the form (5.2.4), see [17].

Corollary 5.2.3. *The set of the discrete Dirichlet-to-Neumann maps belongs to*

$$\{\sqrt{-[\frac{d^2}{d\theta^2}]}\beta(\sqrt{-[\frac{d^2}{d\theta^2}]}) : \beta \in \mathfrak{B}\}.$$

Let

$$\tau(z) = \frac{1-z}{1+z} : \mathbb{C}^+ \xrightarrow{1-1} \mathbb{D} \tag{5.13}$$

The characterization in [17] can be restated as

Theorem 5.2.4. *Let*

$$B : \mathbb{D} \to \mathbb{D}$$

be a real Blaschke product. Then

$$\tau \circ B \circ \tau : \mathbb{C}^+ \to \mathbb{C}^+ \tag{5.14}$$

can be written in unique way as a continued fraction of the form (5.2.4) with positive δ_k, ξ_k*. The number of coefficients in this continued fraction is equal to the number of terms in the product.*
Conversely, every continued fraction of the form (5.2.4) with positive δ_k, ξ_k *can be written in the form (5.2.6) for some real Blaschke product* B*.*

The following theorem is a consequence of the Pick-Nevanlinna interpolation algorithm. (see [10], see also [18] and [23]).

Theorem 5.2.5. *Consider*

$$\{z_i\}, \{w_i\} \subset \mathbb{R}^+, i = 1, \ldots, m$$

There exists an analytic function

$$F : \mathbb{C}^+ \to \mathbb{C}^+$$

$$F(z_i) = w_i, i = 1, \ldots, m$$

if and only if the matrix

$$W = \left(\frac{w_i + w_j}{z_i + z_j} \right)_{i,j=1}^{m}$$

is positive semidefinite,
if and only if there exists a real Blaschke product B *such that*

$$\tau \circ B \circ \tau(z_i) = w_i, i = 1, \ldots, m,$$

which may not be equal to F*, e.g.* $F \equiv 1$.

If W is singular, the Blaschke product is unique, and the number of terms in it is equal to the size of the largest non-singular principal minor of W.

If W is not singular there are exactly two desired Blaschke products with the number of terms m, and an infinite family of the Blaschke products with the number of terms $> m$.

Corollary 5.2.6. *The set of the discrete Dirichlet-to-Neumann maps contains*

$$\{\sqrt{-[\frac{d^2}{d\theta^2}]}\beta(\sqrt{-[\frac{d^2}{d\theta^2}]}) : \beta \in \mathfrak{B}\}.$$

This finishes the proof of the Theorem 2.4.5.

5.2.3 Solution of the discrete inverse problem

Let Λ be an $n \times n$, $n = 2m + 1$ discrete layered Dirichlet-to-Neumann map with the non-zero eigenvalues

$$\lambda_k^{(n)}, k = 1, 2, \ldots, m.$$

Consider

$$W = \left(\frac{\lambda_i^{(n)}/\omega_i^{(n)} + \lambda_j^{(n)}/\omega_j^{(n)}}{\omega_i^{(n)} + \omega_j^{(n)}} \right)_{i,j=1}^{m}.$$

If W is singular, there is unique discrete disk $D = D(n, l)$ or $D^*(n, l)$ with unique radially symmetric conductivity γ on it, such that

$$\Lambda(D_\gamma) = \Lambda.$$

Theorems (5.2.3) and (5.2.4) give an explicit construction of $D_\gamma(n, l)$. Also, it follows that l is equal to the size of the largest non-singular principal minor of W, in particular $l < m$.

If W is non-singular, there are unique conductivities γ, γ' on the disks $D(n, m)$, $D^*(n, m)$, with

$$\Lambda(D_\gamma(n, m)) = \Lambda(D^*_{\gamma'}(n, m)) = \Lambda,$$

And for every $D = D(n,l)$ or $D^*(n,l)$ with $l > m$ there are infinitely many condutivities γ with

$$\Lambda(D_\gamma) = \Lambda.$$

Chapter 6

RANDOM WALKS ON WEIGHTED GRAPHS

6.1 *Percolation matrix of a weighted graph*

Let us consider *a directed graph* $\Gamma = (V, E, \partial\Gamma)$ where V is the finite set of *nodes* of the graph, E is the set of *directed edges* and $\partial\Gamma$ is a subset of V, called *boundary* of Γ. The elements of $\partial\Gamma$ are called *boundary nodes* of Γ. The subset $int\Gamma = V - \partial\Gamma$ of V is called *interior* of Γ. The elements of $int\Gamma$ are called *interior nodes* of Γ. A node q is called a *neighbor* of a node p if there is a directed edge e from p to q.

A weighted directed (WD) graph Γ_γ is a directed graph Γ together with a positive function γ on the edges E of the graph.

We consider the following random walk on a WD graph Γ_γ. A particle starts its motion at a node of Γ. Suppose at the moment $t = n$ it occupies a node p then at the moment $t = n + 1$ it will be at a neighbor q of p. The probability of going to a particular neighbor q is proportional to the weight $\gamma(pq)$.

Given a WD graph Γ_γ let us number its boundary nodes.

The *percolation matrix* of Γ_γ is the $N \times N$ matrix $X(\Gamma_\gamma) = \{x_{ij}\}$ such that

- N = number of boundary nodes of Γ

- x_{ij} = probability that the next boundary node that a particle, starting its random walk at the boundary node b_i, hits is the boundary node b_j. If b_i does not have neighbors, $x_{ij} = 0$ for all j.

A motivation for the definition above is the fact that the percolation matrix of a resistor network (a WD graph in which $pq \in E \Leftrightarrow qp \in E$ and $\gamma(pq) = \gamma(qp)$) is essentially the Dirichlet-to-Neumann map of the network. The calculation in the section 6.2 makes the statement precise.

Two basic properties of any percolation matrix $X = \{x_{ij}\}$ are:

- $x_{ij} \in [0, 1]$
- sum of entries in each row is less or equal to 1

A *circular planar weighted directed graph* is a WD graph that can be embedded in a closed disc in such a way that the boundary of the graph belong to the boundary of the disc.

6.2 A probabilistic interpretation of Dirichlet-to-Neumann maps

One can think of a graph Γ with conductivity γ as of a WD graph $\Gamma_\gamma = \{(V, E, \partial\Gamma), \gamma\}$, such that

$$pq \in E \Leftrightarrow qp \in E,$$

and

$$\gamma(pq) = \gamma(qp).$$

Given a real-valued function u on the nodes of Γ_γ the corresponding *current out of the nodes* of Γ_γ is the function I_u on the nodes of Γ_γ defined by

$$I_u(p) = \sum_{pq \in E} \gamma(pq)(u(p) - u(q)). \tag{6.1}$$

A Kirchhoff's matrix K of Γ_γ is a matrix that represents the linear map from the functions on the nodes of Γ_γ to the corresponding currents. Numbering the nodes of Γ_γ so that boundary goes first, we can write K in the block form

$$K = \left(\begin{array}{c|c} \mathrm{A} & B^T \\ \hline \mathrm{B} & \mathrm{C} \end{array} \right). \tag{6.2}$$

A is $N \times N$, C is $n \times n$, where N is the number of boundary nodes of Γ_γ and n is the number of interior nodes of Γ_γ.

From the formula (6.1) we get that

- $k_{ij} = 0$ if $i \neq j$ and $p_i p_j \notin E$
- $k_{ij} = -\gamma(p_i p_j)$ if $i \neq j$ and $p_i p_j \in E$
- $k_{ii} = \sum_{p_i q \in E} \gamma(p_i q)$

A function u on the nodes of Γ_γ is called γ-*harmonic* if it satisfies the Kirchhoff's Law:

$$I_u|_{int\Gamma} = Ku|_{int\Gamma} = Bu|_{\partial\Gamma} + Cu|_{int\Gamma} = 0. \tag{6.3}$$

Or in other words the value of u at an interior node p is the weighted average of the values of u at the neighbors of p. It follows that γ-harmonic functions satisfy maximum and minimum principle; and therefore, the values of a γ-harmonic function u at the interior nodes of Γ_γ are determined by the values of u at the boundary nodes of Γ_γ. From the equation (2) we get that:

$$u|_{int\Gamma} = -C^{-1}Bu|_{\partial\Gamma}. \tag{6.4}$$

The Dirichlet-to-Neumann map Λ of the resistor network Γ_γ is the map that sends a function f on the boundary of Γ_γ to the current out of the boundary nodes of the γ-harmonic continuation of f to the interior of Γ_γ. In other words given a function f on the boundary of Γ_γ,

$$\Lambda f = I_u|_{\partial\Gamma}, \qquad u|_{\partial\Gamma} = f, \qquad I_u|_{int\Gamma} = 0.$$

The explicit formula is:

$$\Lambda f = Af + B^T(-C^{-1}Bf) = (A - B^T C^{-1} B)f. \tag{6.5}$$

For more detailed discussion of Dirichlet-to-Neumann maps see [6].

Let us calculate the percolation matrix X of Γ_γ. Let u_j be the function on the nodes of Γ such that for an interior node p

$u_j(p)$ = probability that a particle starting its random walk on Γ_γ goes to the boundary node b_j before hitting any other boundary node of Γ_γ;

$u_j(b_i) = \delta^i_j$.

From the definition of the random walk we get that u_j is γ -harmonic. And, therefore, from the equation (6.3)

$$u_j = -C^{-1}B\delta^i_j.$$

We have now that, if $i \neq j$

$$x_{ij} = \frac{1}{\sum_{p_i q\in E}\gamma(p_i q)}(\gamma(b_i b_j) + \sum_{p_l\in int\Gamma}\gamma(b_i p_l)u_j(p_l)).$$

and

$$x_{ii} = \frac{1}{\sum_{p_i q\in E}\gamma(p_i q)}\sum_{p_l\in int\Gamma}\gamma(b_i p_l)u_i(p_l).$$

Let I_Σ be a diagonal $N \times N$ matrix such that $I_\Sigma(ii) = k_{ii} = \sum_{p_i q\in E}\gamma(p_i q)$.

Then,

$$X = I_\Sigma^{-1}(-A + I_\Sigma - B^T(-C^{-1}B)) = I - I_\Sigma^{-1}(A - B^T C^{-1}B).$$

Therefore,

$$X = I - I_\Sigma^{-1}\Lambda. \tag{6.6}$$

This shows that the percolation matrix of a resistor network is its renormalized Dirichlet-to-Neumann map.

6.3 *Graphs without cycles*

In this section we will explore a probabilistic interpretation of determinants of minors of percolation matrices of WD graphs without cycles.

A *cycle* in a WD graph Γ is a sequence of edges

$$(p_1p_2), (p_2p_3), \dots, (p_{m-1}p_m), (p_mp_1)$$

B where p_i's are interior nodes of Γ.

Throughout this section let $A = \{a_i\}$ and $B = \{b_i\}$, $i = 1, \dots, n$ be two subsets of boundary nodes of Γ. Let X be the percolation matrix of Γ, and let S_n be the group of permutations on n elements. From the expansion formula for determinants one has that

$$\det\{X(a_i, b_j)\} = \sum_{\sigma \in S_n} (-1)^{sign \sigma} \text{probability that } n \text{ particles starting} \quad (6.7)$$

$$\text{the random walk at } a_1, \dots, a_n \text{ exit respectively at } b_{\sigma(1)}, \dots, b_{\sigma(n)}.$$

A *path* β from a boundary node a to a boundary node b is either the edge (ab) or a sequence of edges

$$(ap_1), (p_1p_2), \dots, (p_{m-1}p_m), (p_mb)$$

where p_i's are interior nodes.

A *connection* $\alpha(\sigma)$ from $a_1, \dots, a_n$ to $b_1, \dots, b_n$ is a set of paths $\alpha_i(\sigma)$ from a_i's to $b_{\sigma(i)}$'s.

Let $\omega(\alpha)$ be the probability that n particles starting the random walk at $a_1, \dots, a_n$ follow α.

With these definitions:

$$\det\{X(a_i, b_j)\} = \sum_{\sigma \in S_n} (-1)^{sign \sigma} \sum_{\text{connection } \alpha(\sigma) \text{ from } a_i\text{'s to } b_i\text{'s}} \omega(\alpha(\sigma)). \quad (6.8)$$

A connection $\alpha(\sigma)$ from a_i's to b_i's is called *disjoint* if none of the pairs of paths in it go through the same interior node.

Theorem 6.3.1. *If Γ has no cycles then*

$$\det\{X(a_i, b_j)\} = \sum_{\sigma \in S_n} (-1)^{sign \sigma} \sum_{disjoint\ connection \alpha(\sigma)\ from\ a_i\text{'s to } b_i\text{'s}} \omega(\alpha(\sigma)). \quad (6.9)$$

Proof. In view of 6.8 it suffices to obtain a map ϕ of the set of non-disjoint connections $\alpha(\sigma)$ from a_i's to b_i's to itself such that

$$\phi^2 = identity,$$

and if

$$\phi(\alpha(\sigma)) = \tilde{\alpha}(\tilde{\sigma})$$

then

$$\omega(\tilde{\alpha}(\tilde{\sigma})) = \omega(\alpha(\sigma))$$

and

$$sign\tilde{\sigma} = -sign\sigma.$$

For the existence of such ϕ the fact that Γ does not have cycles is crucial. One of the constructions of ϕ is the following:

Take an ordering on $\{1,\dots,n\}\times\{1,\dots,n\}$. Now, given a non-disjoint

$$\alpha(\sigma) = \{\alpha_1(\sigma),\dots,\alpha_2(\sigma)\}$$

consider the non-disjoint pair $(\alpha_j(\sigma),\alpha_k(\sigma))$ with the maximum (j,k). Since Γ does not have cycles there is a well defined <u>last</u> interior point p where $\alpha_j(\sigma)$ intersects $\alpha_k(\sigma)$, that is the interior point p such that

$$\alpha_j(\sigma) = (a_jp_1)(p_1p_2)\dots(p_sp)(pp_{s+1})\dots(p_tb_{\sigma(j)}),$$

$$\alpha_k(\sigma) = (a_kq_1)(q_1q_2)\dots(q_{\tilde{s}}p)(pq_{\tilde{s}+1})\dots(q_{\tilde{t}}b_{\sigma(k)}),$$

and

$$\{p_{s+1},\dots,p_t\}\cap\{q_{\tilde{s}+1},\dots,q_{\tilde{t}}\} = \phi.$$

We now define $\tilde{\alpha}(\tilde{\sigma}) = \phi(\alpha(\sigma))$. Let

$$\tilde{\sigma}(i) = \sigma(i) \text{ for } i \neq j,k$$

$$\tilde{\sigma}(j) = \sigma(k)$$

$$\tilde{\sigma}(k) = \sigma(j)$$

$$\tilde{\alpha}_i(\tilde{\sigma}) = \alpha_i(\sigma) \text{ for } i \neq j, k$$

$$\tilde{\alpha_j}(\tilde{\sigma}) = (a_j p_1)(p_1 p_2) \ldots (p_s p)(p q_{\tilde{s}+1}) \ldots (q_{\tilde{t}} b_{\sigma(k)})$$

and

$$\tilde{\alpha_k}(\tilde{\sigma}) = (a_k q_1)(q_1 q_2) \ldots (q_{\tilde{s}} p)(p p_{s+1}) \ldots (p_t b_{\sigma(j)}).$$

It follows from the construction that

$$\phi^2 = identity.$$

The check that ϕ has the desired properties is left to the reader. ■

Corollary 6.3.2. *Let Γ be a circular planar WD graph without cycles. Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ be a pair of disjoint non-interlacing sets of boundary nodes. Then*

$$|\det X\{(a_i, b_j)\}| = \textit{probability that } n \textit{ particles starting the random walk at } a_i\textit{'s} \tag{6.10}$$

$$\textit{will follow disjoint paths and exit at } b_i\textit{'s.}$$

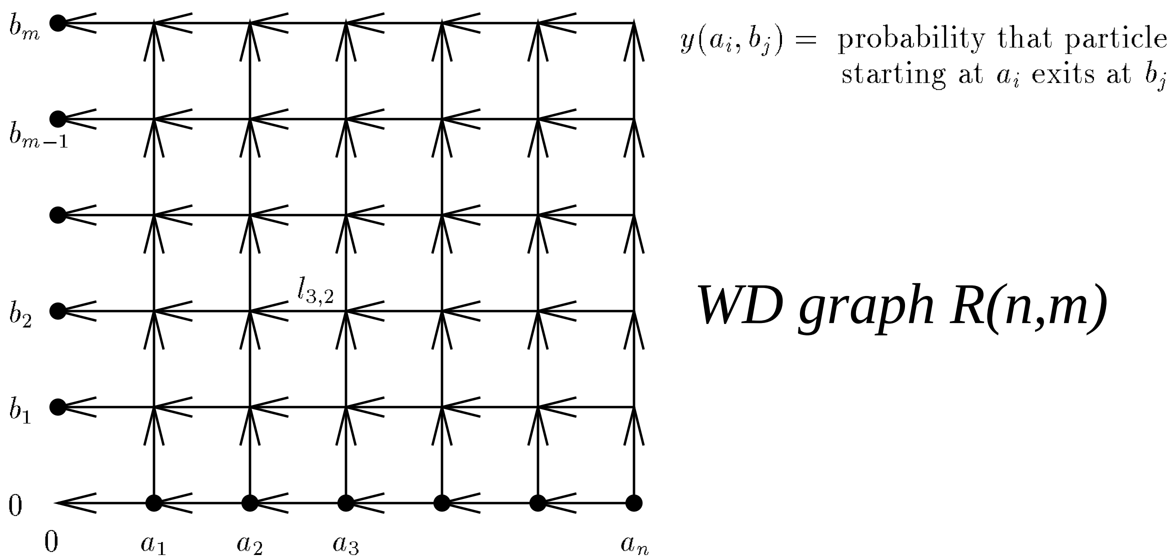
Proof. It follows from the hypothesis of the theorem that there may be at most one $\sigma \in S_n$ for which there exists a disjoint connection $\alpha(\sigma)$. Invoking the theorem 6.3.1 finishes the proof. ■

6.4 Characterization of totally positive matrices

An $n \times m$ matrix is called *totally non-negative* (*totally positive*) if determinants of all its minors are non-negative (positive). It is not hard to show ([11]) that the set of $n \times m$ totally non-negative matrices is in the closure of the set of totally positive matrices in the usual topology of $\mathbb{R}^{nm}$. Let $\mathbb{S}$ be the set of totally positive matrices with the row sums strictly between 0 and 1. Then $\bar{\mathbb{S}}$ is the set of totally non-negative matrices with the row sums in $[0,1]$.

In this section we will obtain a characterization of the set $\bar{\mathbb{S}}$ as essentially the set of percolation matrices of circular planar WD graphs, and give a simple parametrization to the set $\mathbb{S}$.

For a pair of natural numbers n, m let $R(n,m)$ be the following graph.



We will refer to the nodes of $R(n,m)$ by its coordinates. There are $n+m$ boundary nodes in $R(n,m)$:

$$\{a_i = (i,0)\} \cup \{b_j = (0,j)\}, \text{ where } i = 1,\ldots,n, \qquad j = 1,\ldots,m.$$

Let n and m be fixed. For $1 \le i \le n$ and $0 \le j \le m-1$ let

$l_{i,j} \in [0,1] =$ probability that a particle at (i,j) will make the next move to the left.

The probability that a particle at (i,j) will make the next move up is $1 - l_{i,j} \in [0,1]$. We will view l a point in the cube $[0,1]^{nm}$.

Let $Y(l) = \{y_{ij}\}$ be the $n \times m$ matrix where

$y_{ij} =$ probability that a particle starting a random walk at a_i will go to b_j.

Note that the matrix Y is a minor of the percolation matrix of $R(n,m)$. The results of the previous section give us the following properties of $Y(l)$.

Lemma 6.4.1. *For all $l \in [0,1]^{nm}$ the corresponding $Y(l)$ is in $\bar{\mathbb{S}}$. Moreover, $Y(R_l)$ is in $\mathbb{S}$ if and only if $0 < l(i,j) < 1$, i.e if and only if*

$$l \in (0,1)^{nm}$$

Proof. The first statement follows from corollary 6.3.2. This corollary also implies that $Y(\{(0,1)^{nm}\}) \subset \mathbb{S}$ since all k-tuples from a's to b's in R are disjointly connected. It is now left to show that if for some (i,j) the probability $l(i,j) = 0$ or 1 then $Y(l) \neq \mathbb{S}$. There are three simple cases to consider:

(1) if $l(i,0) = 1$ then the i'th row of $Y(l)$ is the zero row, (2) if $l(i,0) = 0$ then a particle starting at i has to exit at b's, and, therefore, the sum of the i'th row in $Y(l)$ is 1, (3) $l(i,j) = 0$ or 1 and $j \neq 0$, then at least one disjoint connection in R is broken (the inspection of the graph is left to the reader), and by corollary 6.3.2 at least one determinant in $Y(l)$ is 0. ■

Lemma 6.4.2.

$$\text{The map } \Psi = \begin{cases} (0,1)^{nm} \to \mathbb{S}, \\ l(i,j) \to Y(R_l) \end{cases} \quad \text{is injective}$$

and the inverse is smooth on the range of Ψ.

Proof. Note that y_{ij} does not depend on $l_{\tilde{i},\tilde{j}}$ if $\tilde{i} > i$. This observation leads to a direct calculation of l from $Y(l)$. The map from $Y(l)$ to l is obviously rational. ■

Theorem 6.4.3. *(Parametrization of $\mathbb{S}$ and Characterization of $\bar{\mathbb{S}}$)*

$$\text{The map } \Psi = \begin{cases} [0,1]^{nm} \to \bar{\mathbb{S}}, \\ l(i,j) \to Y(R_l) \end{cases} \quad \text{is onto.}$$

and

$$\Psi = \begin{cases} (0,1)^{nm} \to \mathbb{S}, \\ l(i,j) \to Y(R_l) \end{cases} \quad \textit{is diffeomorphism onto.}$$

Proof. Obviously, $\mathbb{S}$ is open. It is not hard to see that $\mathbb{S}$ is connected ([11]). From the results above we have that $\Psi|_{(0,1)^{nm}}$ is continuous, open and injective. Also $\Psi(\{\partial(0,1)^{nm}\}) \in \bar{\mathbb{S}} - \mathbb{S}$. The rest is basic topology. ■

6.4.1 Hadamard's Inequality

The probablistic characterization of the set $\bar{\mathbb{S}}$ leads to a simple proof of Hadamard's Inequality for totally non-negative matrices:

Lemma 6.4.4. *Let $Z = \{z_{ij}\}$ be an $n \times n$ totally non-negative matrix. Consider a family of disjoint intervals I_k of natural numbers such that*

$$\cup I_k = \{1, \dots, n\}.$$

Let Z_k be principal minors of Z corresponding to I_k's, that is

$$Z_k = \{z_{ij}\}_{i,j \in I_k}.$$

Then

$$\det Z \leq \prod \det Z_k \tag{6.11}$$

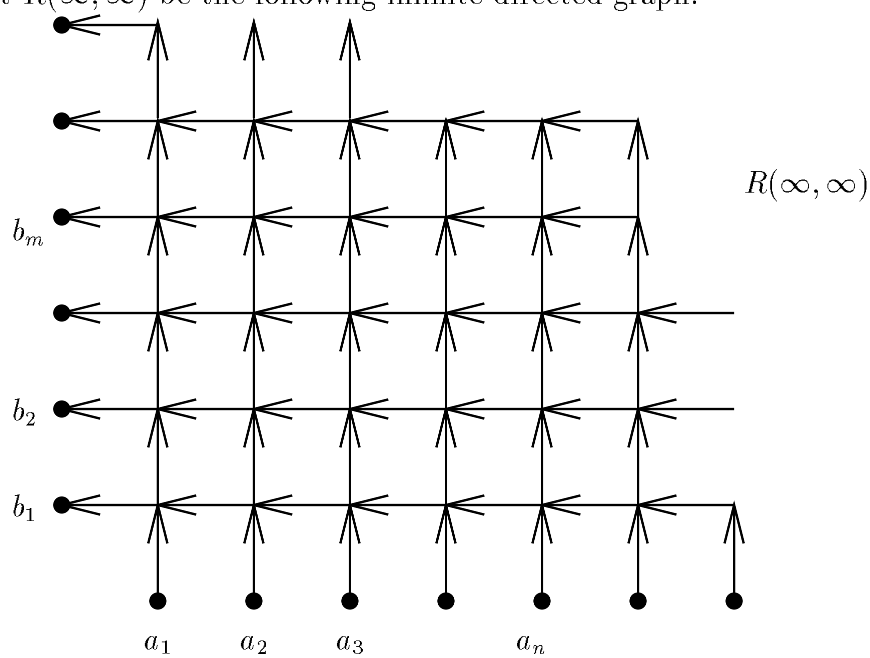
Proof. WLOG, by normalizing the matrix, $Z \in \bar{\mathbb{S}}$ and, therefore, by theorem 6.4.3 $Z = Y(R_l(n,n))$ for some l. Therefore, by corollary 6.3.2, one can think of $\det Z_k$ as probability that k particles follow disjoint trajectories from a_i's to b_i's, where $i \in I_k$. The inequality follows from the rule for calculating conditional probabilities. ■

6.4.2 Some properties of Pascal's triangle

The results of two previous sections can be applied to derive some interesting properties of Pascal's triangle:

$$\Delta_P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \\ 1 & 3 & 6 & 10 & 15 & 21 & \ddots & \ddots \\ 1 & 4 & 10 & 20 & 35 & 56 & \ddots & \ddots \\ 1 & 5 & 15 & 35 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Let $R(\infty, \infty)$ be the following infinite directed graph.



It is not hard to see that $\Delta_P(i, j) =$ the number of different paths from a_i to b_j in $R(\infty, \infty)$. In fact the following identity follows from the proof of theorem 6.3.1.

Lemma 6.4.5. *Let A and B be two k-tuples of natural numbers. Then*

$$\det\{\Delta_P(i, j)\}_{i \in A, j \in B} = \textit{ number of disjoint connections} \qquad (6.12)$$

$$\textit{from } a_i\textit{'s to } b_j\textit{'s in } R(\infty, \infty)$$

Corollary 6.4.6. $\Delta_P(i,j)$ *is totally positive.*

Corollary 6.4.7. *Let l and k be two natural numbers. Then*

$$\det\{\Delta_P(i,j)\}_{1\leq i<l, k\leq j<k+l} = \det\{\Delta_P(i,j)\}_{k\leq i<k+l, 1\leq j<l} = 1.$$

It is not hard to see that the last property completely characterizes Pascal's triangle.

BIBLIOGRAPHY

[1] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, New York, 1961-63.

[2] C. L. Dodgson, *Condensation of Determinants*, Royal Society of London, Proceedings **15** (1866), 150-155.

[3] Y. Colin de Verdiere, *Reseaux Electriques Planaires*, Publ. de l'Institut Fourier, no. 225, (1992), pp = 1–20.

[4] Y. Colin de Verdiere, *Reseaux Electriques Planaires I*, preprint, (1993), pp = 1–20.

[5] Y. Colin de Verdiere, I. Gitler, D. Vertigan, *Planar Electric Networks II*, preprint, (1994).

[6] E. Curtis, D. Ingerman, J. Morrow, *Circular planar graphs and resistor networks*, submitted.

[7] E. B. Curtis and J. A. Morrow, *Determining the resistors in a network*, SIAM J. of Applied Math., 50 (1990), pp. 918–930.

[8] E. Curtis, E. Mooers, J. Morrow, *Finding conductors in circular networks from boundary measurments*, Math. Modelling and Numerical Mathematical Modelling and Numerical Analysis 28 (1994), 781-813.

[9] H. Dym and H. P. McKean, *Gaussian Processes, Function Theory, and the Inverse Spectral Problem*, Academic Press, 1976.

[10] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, Inc., 1981.

[11] F. R. Gantmakher and M. G. Krein, *Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems*, Moscow, 1950.

[12] B. Grunbaum, *Convex Polytopes*, Interscience, (1967).

[13] D. Ingerman, *Discrete and continuous Dirichlet-to-Neumann maps in the layered case*, submitted.

[14] D. Ingerman and J. Morrow, *On a Characterization of the kernel of the Dirichlet-to-Neumann map for a planar region*, to appear in SIAM J. of Math. Analysis.

[15] I. S. Kac, M. G. Krein, *On the spectral function of the string*, AMS Translations, S.2, Vol 103, 1974, 19-102.

[16] S. Karlin, *Total Positivity*, Stanford University Press, 1968.

[17] N. Levinson, R. M. Redheffer, *Complex Variables*, Holden-Day, Inc., 1970.

[18] D. Marshall, *An Elementary Proof of the Pick-Nevalinna Interpolation Theorem*, Michigan Math. J. 21 (1974), 219-233.

[19] A. I. Nachman ,*Global Uniqueness for a Two-Dimentional Inverse Boundary Value Problema*, Annals of Mathematics(2), 143 (1996), no. 1, 71-96.

[20] R. Parker, *The Inverse Problem of Electromagnetic Induction: Existence and Construction of Solutions Based on Incomplete Data*, J. of Geophysical Research, Vol. 85, No. B8, August 10, (1980), 4421-4428.

[21] Rudin, *Complex and Real Analysis*, McGraw-Hill Book Company, 1987.

[22] E. Steinitz and H. Rademacher, *Vorlesungen uber die Theorie der Polyhedra*, Berlin, (1914).

[23] J. Sylvester, *A Convergent Layer Stripping algorithm for the Radially Symmetric Impedance Tomography Problem*, Commun. in Partial Differential Equations, 17(11&12), (1992), 1955-1994.

[24] J. Sylvester, *An Anisotropic Inverse Boundary Value Problem*, Communications on Pure and Applied Mathematics XLIII (1990), 201-232.

[25] J. Sylvester, G. Uhlmann, *Inverse Boundary Value Problems at the Boundary - Continuous Dependence*, Comm. Pure Appl. Math. XLI, (1988), 197-219.

[26] J. Sylvester and G. Uhlmann, *Inverse Boundary Value Problems at the Boundary – Continuous Dependence*, Communications on Pure and Applied Mathematics XLI (1988), 197-219.