ON A CHARACTERIZATION OF THE KERNEL OF THE DIRICHLET-TO-NEUMANN MAP FOR A PLANAR REGION

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ABSTRACT. We will show that the Dirichlet-to-Neumann map Λ for the electrical conductivity equation on a simply connected plane region has a *Variation Diminishing Property*, which may be considered as a generalized maximum principle. Using this property, we will prove that the kernel, K, of Λ satisfies a set of inequalities of the form $(-1)^{\frac{n(n+1)}{2}} \det K(x_i, y_j) > 0$. We will show that these inequalities imply Hopf's lemma for the conductivity equation. We will also show that these inequalities imply the Variation Diminishing Property of a kernel.

1. INTRODUCTION

In this paper we will derive some properties of the Dirichlet-to-Neumann map for the electrical conductivity equation in $\mathbf{R}^2$. We recall some definitions. Let Ω be a relatively compact, smoothly bounded, simply connected open set in $\mathbf{R}^2$. Let $\gamma(p) > 0$ be a smooth function on $\overline{\Omega}$. Let f be a function defined on $\partial\Omega$. Then there is a unique function u, defined on $\overline{\Omega}$, such that

$$\nabla(\gamma\nabla u) = 0 \tag{1}$$

and $u(p) = f(p)$, for $p \in \partial\Omega$. (Equation (**??**) is the electrical conductivity equation and a function, u, that satisfies (**??**) is called a γ-harmonic function.) Let $\frac{\partial u}{\partial n}(p)$ be the directional derivative of u in the direction of the outward pointing unit normal n at the point $p \in \partial\Omega$. Then the Dirichlet-to-Neumann map, Λ, is defined by the formula

$$\Lambda f(p) = \gamma(p)\frac{\partial u}{\partial n}(p). \tag{2}$$

The domain of Λ may be taken to be $H^{\frac{1}{2}}(\partial\Omega)$ and the image is in $H^{-\frac{1}{2}}(\partial\Omega)$. Λ is a pseudo-differential operator of order 1 and as such has a kernel, $K(x, y)$, defined as a distribution on $\partial\Omega \times \partial\Omega$. The kernel gives a representation of Λ by the formula

$$\Lambda f(x) = \int_{\partial\Omega} K(x, y)f(y)dy, \tag{3}$$

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where x and y are arc length coordinates on $\partial\Omega$. For the pseudo-differential operator Λ, the kernel K is a symmetric function, $K(x,y) = K(y,x)$, and for a fixed $x \in \partial\Omega$, $\lim_{y\to x} |K(x,y)| = \infty$. In fact,

$$K(x,y) = \frac{k(x,y)}{|x-y|^2}, \tag{4}$$

where k is continuous on $\partial\Omega \times \partial\Omega$ and $k(x,x) \neq 0$. (In this formula, $|x-y|$ is the separation in arc length of points with arc length coordinates x and y.) If $x \notin \mathrm{supp}(f)$, then there the integral is an ordinary integral and there are no convergence questions.

The boundary, $\partial\Omega$, is a Jordan curve and hence is homeomorphic to a circle. Pick an orientation on $\partial\Omega$. We say that $(x_1,\dots,x_n;y_1,\cdots,y_n)$ is a circular pair if there are points $p,q \in \partial\Omega$ which divide $\partial\Omega$ into two connected components, A,B such that $\{x_1,\dots,x_n\} \subset A$, $\{y_1,\dots,y_n\} \subset B$, and $x_1,\dots,x_n,y_1,\dots,y_n$ are in circular order on $\partial\Omega$. (Note that this definition is modified from the definition in CIM.) The main theorem of this paper is the following.

Theorem 1.1. *Let $(x_1,\dots,x_n;y_1,\cdots,y_n)$ be in circular order on $\partial\Omega$. Let $L = (l_{ij})$ be the $n \times n$ matrix with entries defined by $l_{ij} = K(x_i,y_j)$. Then*

$$(-1)^{\frac{n(n+1)}{2}} \det(L) > 0 \tag{5}$$

We consider this to be a generalization of a result in CIM. We will see how it implies the classical Hopf lemma for the conductivity equation in dimension 2.

2. The Variation Diminishing Property

3. The Weak Inequality

We first prove the weaker statement:

Theorem 3.1. *Let $(x_1,\dots,x_n;y_1,\cdots,y_n)$ be in circular order on $\partial\Omega$. Let $L = (l_{ij})$ be the $n \times n$ matrix with entries defined by $l_{ij} = K(x_i,y_j)$. Then*

$$(-1)^{\frac{n(n+1)}{2}} \det(L) \geq 0 \tag{6}$$

We will need the a modification of a theorem from CMM, which we restate and prove here. Proof supplied later – this is the variation diminishing property. Maybe it deserves its own section.

Proof. The proof is by induction on n. We first consider $n = 1$. The proof goes by contradiction. Suppose that there are points $p,q \in \partial\Omega$ with $p \neq q$ and $K(p,q) > 0$. Then there is an $\epsilon > 0$ such that $p \notin D_\epsilon = \{y : |y-q| < \epsilon\}$ and $K(p,y) > 0$ for $y \in D_\epsilon$. Let $f(y)$ be a continuous function on $\partial\Omega$ such that $\mathrm{supp}(f) \subset D_\epsilon = \{y : |y-q| < \epsilon\}$, $f(q) > 0$, and $f(s) \geq 0$ for all $s \in \partial\Omega$. Then

$$\gamma(p)\frac{\partial u}{\partial n}(p) = \Lambda f(p) = \int_{D_\epsilon} K(p,y)f(y)dy > 0,$$

where u satisfies (**??**) and $u(s) = f(s)$, $s \in \partial\Omega$ But then there must be a point z near p in Ω such that $u(z) < 0$. This contradicts the maximum principle.

Next we assume that the result is true for all $(n-1) \times (n-1)$ matrices and prove that it is true for $n \times n$ matrices. If the result is not true, then we have a circular pair $(x_1, \ldots, x_n; y_1, \cdots, y_n)$ such that

$$(-1)^{\frac{n(n+1)}{2}} \det(L) < 0. \tag{7}$$

Consider the matrix $L^{-1} = (h_{ij})$. Then

$$h_{ij} = (-1)^{i+j} \frac{\det(L_{ij})}{\det(L)}, \tag{8}$$

where L_{ij} is the (i, j) minor of L. By induction, (**??**), and (**??**)

$$(-1)^{i+j+\frac{n(n-1)}{2}+\frac{n(n+1)}{2}+1} h_{ij} = (-1)^{i+j+n+1} h_{ij} \geq 0. \tag{9}$$

Since L is nonsingular, for fixed i there must be some j for which

$$(-1)^{i+j+n+1} h_{ij} > 0. \tag{10}$$

Now let $w = [1, -1, 1, \ldots, (-1)^{n+1}]^T$ be an $n \times n$ unit vector with alternating signs. Let $z = L^{-1}w$. Then using (**??**) and (**??**) it is easy to verify that

$$(-1)^{i+n} z_i > 0. \tag{11}$$

To summarize, we have a vector z such that

$$(-1)^{i+1} = w_i = \sum_{j=1}^{n} K(x_i, y_j) z_j \tag{12}$$

and

$$(-1)^{n+1} z_i w_i > 0. \tag{13}$$

Now, choose small intervals D_j around the points y_j such that the D_j are disjoint and do not contain any of the points x_i. Choose the D_j so small that

$$|K(x_i, y) - K(x_i, y_j)| < \epsilon, \; y \in D_j, \; i = 1, \ldots, n. \tag{14}$$

Also choose functions f_j such that

$$\mathrm{supp}(f_j) \subset D_j, \; z_j f_j(y) \geq 0, \text{ and } \int_{D_j} f_j = z_j. \tag{15}$$

Let $f = \sum f_j$. Then

$$
\begin{aligned}
|\Lambda f(x_i) - w_i| &= |\int_{\partial\Omega} K(x_i, y) f(y) dy - \sum_{j=1}^{n} K(x_i, y_j) z_j| \\
&= |\int_{\partial\Omega} (K(x_i, y) - K(x_i, y_j)) f(y) dy| \qquad (16)\\
&\leq \epsilon \sum_{j=1}^{n} |z_i|.
\end{aligned}
$$

Thus we conclude that for ϵ small enough $\Lambda f(x_i)$ has the same sign as w_i. By CMM, there would have to be a set of points t_i in circular order such that

$$(-1)^n w_i f(t_i) > 0. \qquad (17)$$

For such a set of points we would have to have $t_i \in D_i$ and hence $f(t_i)$ would have the same signs as z_i. This contradicts (**??**).

□

4. The Strong Inequality

We now prove Theorem (**??**). This proof is also by induction on n. Let us assume the the arc length of $\partial\Omega$ is S and that points on $\partial\Omega$ are parametrized by the numbers in the interval $[0, S)$. For the case $n = 1$, suppose there is a pair of points x_1, y_1 with $0 \leq x_1 < y_1$ and $K(x_1, y_1) = 0$. By (**??**) there is no sequence of points z_j such that $x_1 < z_j < y_1$, $\lim_{j\to\infty} z_j = x_1$, and $\lim_{j\to\infty} K(x_1, z_j) = 0$. Hence there is a point η_2 with $x_1 < \eta_2 < y_1$ such that

$$K(x_1, \eta_2) = 0, \text{ and } K(x_1, \eta) < 0, \text{ for } x_1 < \eta < \eta_2.$$

Let x be any number such that $x_1 < x < \eta_2$ and choose η_1 so that $x < \eta_1 < \eta_2$. Then $(x_1, x; \eta_1, \eta_2)$ is a circular pair and hence

$$\begin{vmatrix} K(x_1, \eta_1) & K(x_1, \eta_2) \\ K(x, \eta_1) & K(x, \eta_2) \end{vmatrix} \leq 0. \qquad (18)$$

Since

$$K(x_1, \eta_2) = 0, \ K(x, \eta_2) \leq 0, \text{ and } K(x_1, \eta_1) < 0,$$

it follows that

$$K(x, \eta_2) = 0. \qquad (19)$$

This shows that for *all* x, with $x_1 < x < \eta_2$, $K(x, \eta_2) = 0$. Hence we get the contradiction that $\lim_{x\to\eta_2} K(x, \eta_2) = 0$.

The proof for $n > 1$, makes use of the following result in CIM. In we refer to the result as the six-term identity. It was later pointed out to us that Charles Dodgson (Lewis Carroll) proved a version of this identity in (reference). Now state six-term identity. (later)

Let $(x_1, \ldots, x_n; y_1, \ldots, y_n)$ be a circular pair. We assume that the coordinates on $\partial\Omega$ are chosen so that $0 \le x_1 < \cdots < x_n < y_1 < \cdots < y_n < S$. We will use the notation

$$\kappa(x_1, \ldots, x_n; y_1, \ldots, y_n) = \det(K(x_i, y_j)). \tag{20}$$

Assume that $\kappa(x_1, \ldots, x_n; y_1, \ldots, y_n) = 0$ for some circular pair. First we claim that there is no sequence of points z_j such that $x_n < z_j < y_1$, $\lim_{j\to\infty} z_j = x_n$, and $\lim_{j\to\infty} \kappa(x_1, \ldots, x_n; z_j, y_2, \ldots, y_n) = 0$. For this would imply that there are constants c_k (independent of j) so that

$$K(x_n, z_j) = \sum_{k<n} c_k K(x_k, z_j), \tag{21}$$

and hence

$$\lim_{j\to\infty} K(x_n, z_j) = \sum_{k<n} c_k K(x_k, x_n), \tag{22}$$

contradicting (**??**). Thus there is a number η_1 with $x_n < \eta_1 < y_1$ such that $\kappa(x_1, \ldots, x_n; \eta_1, y_2, \ldots, y_n) = 0$ and

$$\kappa(x_1, \ldots, x_n; \eta, y_2, \ldots, y_n) \neq 0, \text{ for } x_n < \eta < \eta_1.$$

Now it follows from the six term identity (I will supply more details) that

$$\kappa(x_1, \ldots, x_{n-1}, x; \eta_1, y_2, \ldots, y_n) = 0, \text{ for } x_n < x < \eta_1. \tag{23}$$

As above, this contradicts(**??**) and proves the theorem.

5. The Hopf Lemma

We now prove the following theorem, which is a statement of the Hopf lemma for the conductivity equation.

Theorem 5.1. *Let u be a non constant solution of $\nabla(\gamma\nabla u) = 0$, and let $p \in \partial\Omega$ be a point where u assumes a minimum. Then*

$$\frac{\partial u}{\partial n}(p) < 0 \tag{24}$$

Proof. We may assume that $u(p) = 0$. Let $f = u|_{\partial\Omega}$. Since u is not constant, $\mathrm{supp}(f)$ is not empty. Thus there is an interval D around p in $\partial\Omega$ such that $\mathrm{supp}(f) - D$ is not empty. Let ψ be a smooth function on $\partial\Omega$ such that $\psi = 1$ on $\mathrm{supp}(f) - D$, $\psi = 0$ on an interval around p, and $0 \le \psi \le 1$. Let $g = \psi f$ and let v be the solution of $\nabla(\gamma\nabla v) = 0$ with $v|_{\partial\Omega} = g$. Since $f \ge g$ it follows that $u \ge v$. Since $p \notin \mathrm{supp}(g)$

$$0 > \int_{\partial\Omega} K(p, y) g(y) dy = \gamma(p) \frac{\partial v}{\partial n}(p) \ge \gamma(p) \frac{\partial u}{\partial n}(p), \tag{25}$$

which proves the theorem. $\square$

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